Communications in Mathematical Analysis ISSN 0973-3841
Volume 2, Number 1, pp. 1–12, 2007

THE EXISTENCE AND NONEXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR NEUTRAL DELAY COMPETITION MODEL

XIAO WANG*
Department of Mathematics and System Science, College of Science,
National University of Defense Technology, Changsha, Hunan,
People's Republic of China 410073

ZHIXIANG LI†
Department of Mathematics and System Science, College of Science,
National University of Defense Technology,Changsha, Hunan,
People's Republic of China 410073

(Communicated by Xiao-Xiong Gan)

Abstract

By using the theory of coincidence degree, sufficient conditions for the existence and nonexistence of positive periodic solutions of a class of neutral delay competition model are obtained.

AMS Subject Classification: Primary 35C25; Secondary 34K13.

Keywords: Coincidence degree, neutral functional differential equation, positive periodic solution.

1 Introduction

As pointed out by Kuang[3], it would be of interest to investigate the existence of periodic solutions of neutral delay interacting population models.

In 1993, Kuang [3] proposed an open problem (open problem 9.2) to obtain sufficient conditions for the existence of positive periodic solutions of the following equation:

$$x'(t) = r(t)x(t)[a(t) - \beta(t)x(t) - b(t)x(t - \tau(t)) - c(t)x'(t - \tau(t))],$$

*E-mail: wxiao_-98@yahoo.com.cn
†E-mail: zhxli02@yahoo.com.cn

where $r(t), a(t), \beta(t), b(t), \tau(t)$ and $c(t)$ are nonnegative continuous periodic functions. Many authors have studied this problem by various tools, such as abstract continuous theorem of k-set contractive operator and the theory of coincidence degree, see for examples [6–9].

By using an abstract continuous theorem of k-set contractive operator and some analysis techniques, Lu and Ge studied the system in [6,7] as follows:

$$x'(t) = x(t)[a(t) - \beta(t)x(t) - \Sigma_{j=1}^{n} b_j(t)x(t - \tau_j(t)) - \Sigma_{i=1}^{m} c_i(t)x'(t - \gamma_i(t))],$$

where all the functions $a(t), \beta(t), b_j(t), \tau_j(t), c_i(t)$ and $\gamma_i(t)$ are continuous ω-periodic functions with $\tau_j(t) \geq 0, \gamma_i(t) \geq 0, \forall t \in [0,\omega]$,$\omega > 0$ is a constant. Furthermore, $\tau_j, c_i \in C^1(R, R)$ with $\tau_j'(t) < 1, \forall t \in [0, \omega]$ and $\gamma_i \in C^2(R, R)$ with $\gamma_i'(t) < 1, \forall t \in [0,\omega]$,$\forall i \in \{1, 2, \ldots, m\}, \forall j \in \{1, 2, \ldots, n\}$.

In 1991, Kuang [4] first introduced the following neutral delay competition model:

$$\begin{cases} x'(t) = r_1 x(t)[1 - k_1 x(t) - ax(t - \tau_1) - bx'(t - \tau_2) - c_1 y(t - \tau_3)], \\ y'(t) = r_2 y(t)[1 - c_2 x(t - \tau_4) - k_2 y(t - \tau_5)], \end{cases} \tag{1.1}$$

where all parameters except b are assumed to be positive constants. Moreover, he also studied the local stability, oscillation of solution of system (1.1) in [4] and obtained some sufficient conditions for bounded solutions of system (1.1) in [5].

Motivated by these works, in this paper, we consider the existence of positive periodic solutions for the following neutral delay competition model:

$$\begin{cases} x'(t) = x(t)[r_1(t) - k_1(t)x(t) - a(t)x(t - \tau_1(t)) - b(t)x'(t - \tau_2(t)) \\ \qquad - c_1(t)y(t - \tau_3(t))], \\ y'(t) = y(t)[r_2(t) - c_2(t)x(t - \tau_4(t)) - k_2(t)y(t - \tau_5(t))], \end{cases} \tag{1.2}$$

where $r_i(t), a(t), b(t), k_i(t), \tau_j(t)$ and $c_i(t)$ are continuous ω-periodic functions with $\tau_j(t) \geq 0, \forall t \in [0, \omega], j = 1, 2, \ldots, 5$,$k_i(t) \geq 0, c_i(t) \geq 0, \forall t \in [0, \omega], i = 1, 2.\omega > 0$ is a constant.Furthermore, $\tau_1, b \in C^1(R, R)$ with $\tau_1'(t) < 1, \forall t \in [0, \omega]$ and $\tau_2 \in C^2(R, R)$ with $\tau_2'(t) < 1, \forall t \in [0, \omega]$.

Obviously, the system (1.2) contains the system (1.1). As far as we know, there are few results of positive ω-periodic solutions of the system (1.2). In the present paper, we establish the existence and nonexisting results for the system (1.2) and our methods are based on an application of the continuation theorem of the coincidence degree theory which was proposed in [1] by Gaines and Mawhin.

In Section 2, we introduce some notations and lemmas to study the existence of positive periodic solutions of the system (1.2). In Section 3, we establish and prove our main results by continuation theorem. Finally, we give a concrete example to show our main Theorem 3.1 in Section 4.

2 Notations and Lemmas

In this section, we shall introduce some notations and lemmas. Let X, Z be real Banach spaces, $L : DomL \subset X \to Z$ be a linear Fredholm mapping of index 0, and $N : X \to Z$ be continuous. Let $P : X \to X, Q : Z \to Z$ be continuous projectors such that $ImP = KerL, KerQ = ImL$

and $X = KerL \bigoplus KerP, Z = ImL \bigoplus ImQ$. Obviously, $L : DomL \cap KerP \to ImL$ is one to one, so its inverse $K_P : ImL \to DomL \cap KerP$. $J : ImQ \to KerL$ is an isomorphism of ImQ onto $KerL$.

In the sequel, we introduce the continuation theorem in [1] as follows:

Lemma 2.1. *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Z$ be continuous operator which is L-compact on $\bar{\Omega}$(i.e,QN and $K_P(I-Q)N$ are relatively compact on $\bar{\Omega}$). Assume that*

1. *for each $\lambda \in (0,1)$,$x \in DomL \cap \partial\Omega$, $Lx \neq \lambda Nx$;*
2. *for each $\lambda \in (0,1)$,$x \in KerL \cap \partial\Omega$,$QNx \neq 0$ and $deg\{JQN, KerL \cap \Omega, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $DomL \cap \bar{\Omega}$.

For convenience, we use the notations: $C_\omega = \{x : x(t) = (x_1(t), x_2(t))^T \in C(R; R^2), x(t) \equiv x(t+\omega), \forall t \in R\}$ with norm defined by $|x|_0 = \max\limits_{t\in[0,\omega]} \{|x_1(t)|,$ $|x_2(t)|\}$, and $C^1_\omega = \{x : x(t) = (x_1(t), x_2(t))^T \in C^1(R; R^2), x(t) \equiv x(t+\omega), \forall t \in R\}$ with the norm defined by $|x|_1 = \max\limits_{t\in[0,\omega]} \{|x|_0, |x'|_0\}$. Then C_ω and C^1_ω are Banach spaces with the norm $|\cdot|_0$ and $|\cdot|_1$, respectively. We denote $\bar{f} = \dfrac{1}{\omega}\displaystyle\int_0^\omega f(t)dt$ and $\bar{f}^* = \dfrac{1}{\omega}\displaystyle\int_0^\omega |f(t)|dt$, wherever f is a continuous ω-periodic function.

Take the transformation $x(t) = e^{x_1(t)}$ and $y(t) = e^{x_2(t)}$, then Eqns (1.2) can be rewritten as follows:

$$\begin{cases} x_1'(t) = r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} \\ \qquad\qquad - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} - c_1(t)e^{x_2(t-\tau_3(t))}, \\ x_2'(t) = r_2(t) - c_2(t)e^{x_1(t-\tau_4(t))} - k_2(t)e^{x_2(t-\tau_5(t))}. \end{cases} \tag{2.1}$$

Remark 2.2. Obviously, if Eqns (2.1) has a ω-periodic solution $x(t) = (x_1(t), x_2(t))^T$, then $(e^{x_1(t)}, e^{x_2(t)})^T$ is a positive ω-periodic solution of Eqns (1.2). So we need only to show that Eqns (2.1) has at least one ω-periodic solution.

Let

$$f_1(t, x_1(t), x_2(t)) = r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} - c_1(t)e^{x_2(t-\tau_3(t))}$$

and

$$f_2(t, x_1(t), x_2(t)) = r_2(t) - c_2(t)e^{x_1(t-\tau_4(t))} - k_2(t)e^{x_2(t-\tau_5(t))}.$$

Then Eqns (2.1) can be rewritten in the following form:

$$\begin{cases} x_1'(t) = f_1(t, x_1(t), x_2(t)) - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))}, \\ x_2'(t) = f_2(t, x_1(t), x_2(t)). \end{cases}$$

In order to apply Lemma 2.1 to study Eqns (1.2), we set $X = C^1_\omega, Z = C_\omega$. Let $L : C^1_\omega \to C_\omega$ defined by

$$Lx = \frac{dx}{dt} = \begin{pmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{pmatrix}$$

and $N : C^1_\omega \to C_\omega$ defined by

$$Nx = \begin{pmatrix} f_1(t, x_1, x_2) - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} \\ f_2(t, x_1, x_2) \end{pmatrix}.$$

Denote continuous projective operators P and Q as

$$Px = \frac{1}{\omega}\int_0^\omega x(t)dt = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega x_1(t)dt \\ \frac{1}{\omega}\int_0^\omega x_2(t)dt \end{pmatrix}, x \in X,$$

and

$$Qz = \frac{1}{\omega}\int_0^\omega z(t)dt = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega z_1(t)dt \\ \frac{1}{\omega}\int_0^\omega z_2(t)dt \end{pmatrix}, z \in Z.$$

So we have

$$KerL = \{x : x \in X, x = c, c \in R^2\}, ImL = \left\{z : z \in Z, \int_0^\omega z(t)dt = 0\right\}$$

and L is a Fredholm mapping of index 0. It is not difficult to see that P and Q satisfy

$$ImP = KerL, KerQ = ImL = Im(I - Q)$$

and there is an inverse $K_P : ImL \to DomL \cap KerP$ of L defined as

$$K_P(z)(t) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)dsdt.$$

Let $b_1(t) = \dfrac{b(t)}{1-\tau'(t)}$, it follows that

$$\begin{aligned} QN(x)(t) &= \begin{pmatrix} \frac{1}{\omega}\int_0^\omega [f_1(t, x_1(t), x_2(t)) - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))}]dt \\ \frac{1}{\omega}\int_0^\omega f_2(t, x_1(t), x_2(t))dt \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \left[f_1(t, x_1(t), x_2(t)) + b_1'(t)e^{x_1(t-\tau_2(t))}\right]dt \\ \frac{1}{\omega}\int_0^\omega f_2(t, x_1(t), x_2(t))dt \end{pmatrix}. \end{aligned} \tag{2.2}$$

Moreover, by direct computation,we obtain

$$
\begin{aligned}
K_P(I-Q)N(x)(t) = & \begin{pmatrix} \int_0^t \left[f_1(s,x_1(s),x_2(s)) + b_1'(s)e^{x_1(s-\tau_2(s))} \right] ds \\ \int_0^t f_2(s,x_1(s),x_2(s))ds \end{pmatrix} \\
& - \begin{pmatrix} b_1(t)e^{x_1(t-\tau_2(t))} - b_1(0)e^{x_1(0-\tau_2(0))} \\ 0 \end{pmatrix} \\
& - \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \int_0^t f_1(s,x_1(s),x_2(s))dsdt \\ \frac{1}{\omega}\int_0^\omega \int_0^t f_2(s,x_1(s),x_2(s))dsdt \end{pmatrix} \\
& + \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \int_0^t b(s)x_1'(s-\tau_2(s))e^{x_1(s-\tau_2(s))}dsdt \\ 0 \end{pmatrix} \\
& - \begin{pmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\frac{1}{\omega}\int_0^\omega \int_0^t f_1(s,x_1(s),x_2(s))dsdt \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\frac{1}{\omega}\int_0^\omega \int_0^t f_2(s,x_1(s),x_2(s))dsdt \end{pmatrix} \\
& + \begin{pmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\frac{1}{\omega}\int_0^\omega \int_0^t b(s)x_1'(s-\tau_2(s))e^{x_1(s-\tau_2(s))}dsdt \\ 0 \end{pmatrix}.
\end{aligned}
\tag{2.3}
$$

Lemma 2.3. *Let R_1, R_2 be two positive constants and $\Omega = \{x : x \in C_\omega^1, |x|_0 < R_1, |x'|_0 < R_2\}$, then $N : \Omega \to C_\omega$ is L-compact on $\bar{\Omega}$.*

Proof. We need only prove that QN and $K_P(I-Q)N$ are relatively compact on $\bar{\Omega}$. By (2.2), we conclude that QN is relatively compact on $\bar{\Omega}$.

Next, we shall prove $K_P(I-Q)N$ is relatively compact on $\bar{\Omega}$ by using Ascoli-Arzela theorem. It is not difficult to see that $K_P(I-Q)N$ is uniformly bounded on $\bar{\Omega}$, so we need only to show that function family $K_P(I-Q)N(\bar{\Omega})$ is equi-continuous.

Considering $\frac{d}{dt}[K_P(I-Q)N(x)(t)], \forall x \in \bar{\Omega}$. From (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}[K_P(I-Q)N(x)(t)] \\
&= \begin{pmatrix} f_1(t, x_1(t), x_2(t)) + b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} \\ f_2(t, x_1(t), x_2(t)) \end{pmatrix} \\
&- \begin{pmatrix} \frac{1}{\omega^2}\int_0^\omega \int_0^t f_1(s, x_1(s), x_2(s)) - b(s)x_1'(s-\tau_2(s))e^{x_1(s-\tau_2(s))}dsdt \\ \frac{1}{\omega^2}\int_0^\omega \int_0^t f_2(s, x_1(s), x_2(s))dsdt \end{pmatrix}.
\end{aligned}$$

Obviously, there is a positive constant M such that $\left|\frac{d}{dt}[K_P(I-Q)N(x)(t)]\right|_0 \le M, \forall x \in \bar{\Omega}$. This implies that function family $K_P(I-Q)N(\bar{\Omega})$ is equi-continuous. So $K_P(I-Q)N$ is relatively compact on $\bar{\Omega}$ and N is L-compact on $\bar{\Omega}$. The proof of Lemma 2.3. is complete. ■

The following Lemma and Remark will be used in the sequel.

Lemma 2.4. *([6,7]). Suppose* $\tau \in C^1(R,R), \tau(t+\omega) = \tau(t), \forall t \in R$ *and* $\tau'(t) < 1, \forall t \in [0,\omega]$. *Then the function* $t-\tau(t)$ *has a unique inverse* $\mu(t)$ *satisfying* $\mu \in C(R,R)$ *with* $\mu(t) = \mu(t+\omega), \forall t \in R$.

Remark 2.5. ([6,7]). By using Lemma 2.4, we see that if $g \in C_\omega$, $\tau \in C^1(R,R), \tau(t+\omega) = \tau(t), \forall t \in R$ and $\tau'(t) < 1, \forall t \in [0,\omega]$, then $g(\mu(t+\omega)) = g(\mu(t)+\omega) = g(\mu(t)), \forall t \in R$, where $\mu(t)$ is the inverse function of $t-\tau(t)$, which together with $\mu \in C(R,R)$ implies that $g(\mu(t)) \in C_\omega$.

3 Existence of Positive Periodic Solutions

In this section, we will give and prove our main results.

Let $\mu(t)$ and $\gamma(t)$ are inverses of $t-\tau_1(t), t-\tau_2(t)$, respectively.$b_1(t) = \frac{b(t)}{1-\tau_2'(t)}$ and $\Gamma(t) = k_1(t) + \frac{a(\mu(t))}{1-\tau_1'(t)} - b_1'(\gamma(t))$.

We propose the following two assumptions:

(H1) $\bar{r}_i > 0, i = 1,2$ and $\bar{\Gamma} \ge 0$.

(H2) $\frac{\bar{r}_1}{\bar{c}_1} > \frac{\bar{r}_2}{\bar{k}_2}e^{(\bar{r_2}+\bar{r_2}^*)\omega}$ and $\frac{\bar{r}_2}{\bar{c}_2} > \max\left\{\frac{\bar{r}_1}{\bar{\Gamma}}e^{(\bar{r_1}+\bar{r_1}^*)\omega}, \frac{\bar{r}_1}{\bar{k}_1+\bar{a}}\right\}$.

Suppose (H2) holds, let $B_1 = \left|\ln\left(\frac{\bar{r}_1}{\bar{\Gamma}}\right)\right| + (\bar{r_1}+\bar{r_1}^*)\omega, B_2 = \left|\ln\left(\frac{\bar{r}_2}{\bar{k}_2}\right)\right| + (\bar{r_2}+\bar{r_2}^*)\omega$, $B_3 = \left|\ln\frac{\bar{r}_1 - \bar{c}_1\frac{\bar{r}_2}{\bar{k}_2}e^{(\bar{r_2}+\bar{r_2}^*\omega)}}{\bar{\Gamma}}\right| + (\bar{r_1}+\bar{r_1}^*)\omega$ and $B_4 = \left|\ln\frac{\bar{r}_2 - \bar{c}_2\frac{\bar{r}_1}{\bar{\Gamma}}e^{(\bar{r_1}+\bar{r_1}^*\omega)}}{\bar{k}_2}\right| + (\bar{r_2}+\bar{r_2}^*)\omega$.

Theorem 3.1. *For Eqns (1.2), suppose (H1) and (H2) hold, in addition, we assume that: (H3) there exists a constant* $R_1 > \max\{B_i + 1, i = 1, 2, 3, 4.\}$ *such that* $|b|_0 e^{R_1} < 1$ *and* $|x|_0 < R_1$ *for the unique solution* $x = (x_1, x_2)^T$ *of the system*

$$\begin{cases} \bar{k}_1 e^{x_1} + \bar{a} e^{x_1} + \bar{c}_1 e^{x_2} = \bar{r}_1, \\ \bar{c}_2 e^{x_1} + \bar{k}_2 e^{x_2} = \bar{r}_2. \end{cases} \tag{3.1}$$

Then Eqns (1.2) has at least one positive ω*-periodic solution.*

Remark 3.2. It is easy to verify that if (H2) holds, then Eqns (3.1) has a unique solution $x = (x_1, x_2)^T$.

Proof. Let $x(t)$ be an arbitrary ω-periodic solution of the operator equation as follows

$$Lx = \lambda N x, \lambda \in (0, 1),$$

where L and N defined as in Section 2, respectively. So we have

$$\begin{aligned} x_1'(t) =& \lambda[r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} \\ & - c_1(t)e^{x_2(t-\tau_3(t))}], \end{aligned} \tag{3.2}$$

and

$$x_2'(t) = \lambda[r_2(t) - c_2(t)e^{x_1(t-\tau_4(t))} - k_2(t)e^{x_2(t-\tau_5(t))}]. \tag{3.3}$$

Integrating both sides of (3.2) and (3.3) over $[0, \omega]$, respectively, we obtain

$$\bar{r}_1\omega = \int_0^\omega [k_1(t)e^{x_1(t)} + a(t)e^{x_1(t-\tau_1(t))} - b_1'(t)e^{x_1(t-\tau_2(t))} + c_1(t)e^{x_1(t-\tau_2(t))}]dt, \tag{3.4}$$

and

$$\bar{r}_2\omega = \int_0^\omega [c_2(t)e^{x_1(t-\tau_4(t))} + k_2(t)e^{x_2(t-\tau_5(t))}]dt. \tag{3.5}$$

Let $t - \tau_1(t) = s$, i.e., $t = \mu(s)$, then we get

$$\int_0^\omega a(t)e^{x_1(t-\tau_1(t))}dt = \int_{-\mu(0))}^{\omega-\mu(0))} \frac{a(\mu(s))}{1-\tau_1'(\mu(s)))} e^{x_1(s)}ds. \tag{3.6}$$

According to Lemma 2.4 and Remark 2.5, we have

$$\int_0^\omega a(t)e^{x_1(t-\tau_1(t))}dt = \int_0^\omega \frac{a(\mu(s))}{1-\tau_1'(\mu(s))} e^{x_1(s)}ds. \tag{3.7}$$

Similarly, we achieve

$$\int_0^\omega b_1'(t)e^{x_1(t-\tau_2(t))}dt = \int_0^\omega \frac{b_1'(\gamma(s))}{1-\tau_2'(\gamma(s))} e^{x_1(s)}ds.$$

So, from (3.4), (3.6), (3.7) and (H1), we have

$$\bar{r}_1\omega = \int_0^\omega [\Gamma(t)e^{x_1(t)} + c_1(t)e^{x_2(t-\tau_3(t))}]dt. \tag{3.8}$$

Since $x(t) = (x_1(t), x_2(t))^T \in C_\omega$, there are $t_1, t_2 \in [0, \omega]$ and $s_1, s_2 \in [0, \omega]$ such that

$$\begin{aligned} x_1(t_1) &= \min_{t\in[0,\omega]} x_1(t), x_1(t_2) = \max_{t\in[0,\omega]} x_1(t); \\ x_2(s_1) &= \min_{t\in[0,\omega]} x_2(t), x_2(s_2) = \max_{t\in[0,\omega]} x_2(t). \end{aligned} \tag{3.9}$$

According to (3.5), (3.8), and (3.9), we have

$$\bar{\Gamma}\omega e^{x_1(t_1)} \le \bar{r}_1\omega \quad \text{and} \quad \bar{k}_2\omega e^{x_2(s_1)} \le \bar{r}_2\omega,$$

i.e.

$$x_1(t_1) \le \ln\left(\frac{\bar{r}_1}{\bar{\Gamma}}\right) \quad \text{and} \quad x_2(s_1) \le \ln\left(\frac{\bar{r}_2}{\bar{k}_2}\right). \tag{3.10}$$

By (3.2) and (3.8), we have

$$\begin{aligned} \int_0^\omega |x_1'(t)|dt &= \lambda \int_0^\omega |r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} \\ &\quad - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} - c_1(t)e^{x_2(t-\tau_3(t))}|dt \\ &\le \int_0^\omega |r_1(t)|dt + \int_0^\omega [k_1(t)e^{x_1(t)} + a(t)e^{x_1(t-\tau_1(t))} \\ &\quad - b_1'(t)e^{x_1(t-\tau_2(t))} + c_1(t)e^{x_1(t-\tau_2(t))}]dt \\ &= (\bar{r_1} + \bar{r_1}^*)\omega. \end{aligned} \tag{3.11}$$

Similarly, from (3.3) and (3.5), we have

$$\int_0^\omega |x_2'(t)|dt \le (\bar{r_2} + \bar{r_2}^*)\omega. \tag{3.12}$$

By (3.10) and (3.11), we have

$$x_1(t) \le x_1(t_1) + \int_0^\omega |x_1'(t)|dt \le \ln\left(\frac{\bar{r}_1}{\bar{\Gamma}}\right) + (\bar{r_1} + \bar{r_1}^*)\omega. \tag{3.13}$$

Similarly, we can get

$$x_2(t) \le x_2(s_1) + \int_0^\omega |x_2'(t)|dt \le \ln\left(\frac{\bar{r}_2}{\bar{k}_2}\right) + (\bar{r_2} + \bar{r_2}^*)\omega. \tag{3.14}$$

From (3.8) (3.9) and (3.14), we have

$$\bar{r}_1\omega \le \bar{\Gamma}\omega e^{x_1(t_2)} + \bar{c}_1\omega e^{x_2(s_2)} \le \bar{\Gamma}\omega e^{x_1(t_2)} + \bar{c}_1\omega\frac{\bar{r}_2}{\bar{k}_2}e^{(\bar{r_2}+\bar{r_2}^*)\omega},$$

i.e.,

$$x_1(t_2) \ge \ln\frac{\bar{r}_1 - \bar{c}_1\frac{\bar{r}_2}{\bar{k}_2}e^{(\bar{r_2}+\bar{r_2}^*)\omega}}{\bar{\Gamma}}.$$

Similarly, we have

$$x_2(s_2) \ge \ln\frac{\bar{r}_2 - \bar{c}_2\frac{\bar{r}_1}{\bar{\Gamma}}e^{(\bar{r_1}+\bar{r_1}^*)\omega}}{\bar{k}_2}.$$

So, we have

$$x_1(t) \geq x_1(t_2) - \int_0^\omega |x_1'(t)|dt \geq \ln \frac{\bar{r_1} - \bar{c}_1 \frac{\bar{r_2}}{\bar{k}_2} e^{(\bar{r_2}+\bar{r_2}^*)\omega}}{\bar{\Gamma}} - (\bar{r_1} + \bar{r_1}^*)\omega$$

and

$$x_2(t) \geq x_2(s_2) - \int_0^\omega |x_2'(t)|dt \geq \ln \frac{\bar{r_2} - \bar{c}_2 \frac{\bar{r_1}}{\bar{\Gamma}} e^{(\bar{r_1}+\bar{r_1}^*)\omega}}{\bar{k}_2} - (\bar{r_2} + \bar{r_2}^*)\omega.$$

Let $\Omega = \{x : x \in C_\omega^1, |x|_0 < R_1, |x'|_0 < R_2\}$, where R_1 is given by (H3) and

$$R_2 > \max\left\{ \frac{|r_1|_0 + |k_1|_0 e^{R_1} + |a|_0 e^{R_1} + |c_1|_0 e^{R_1}}{1 - |b|_0 e^{R_1}} + 1, |r_2|_0 + |c_2|_0 e^{R_1} + |k_2|_0 e^{R_1} + 1 \right\}.$$

From Lemma 2.3, we know that $N : \Omega \to C_\omega$ is L-compact on $\bar{\Omega}$.

In what follows, we will prove that

$$Lx \neq \lambda N x, \tag{3.15}$$

for any $\lambda \in (0,1)$ and $x \in \partial\Omega$. In view of $x \in \partial\Omega$, we see either $|x|_0 = R_1, |x'|_0 \leq R_2$ or $|x|_0 \leq R_1, |x'|_0 = R_2$. If $|x|_0 = R_1, |x'|_0 \leq R_2$, it is not difficult see that (3.15) is true, since R_1 is independent of $\lambda \in (0,1)$.

If $|x|_0 \leq R_1, |x'|_0 = R_2$, (3.15) is also true. Suppose the contrary, then there must be a $\lambda \in (0,1)$ and an $x \in \partial\Omega$ such that

$$Lx = \lambda N x,$$

i.e.,

$$\begin{aligned} x_1'(t) =& \lambda[r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} \\ & - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} - c_1(t)e^{x_2(t-\tau_3(t))}], \end{aligned} \tag{3.16}$$

and

$$x_2'(t) = \lambda[r_2(t) - c_2(t)e^{x_1(t-\tau_4(t))} - k_2(t)e^{x_2(t-\tau_5(t))}].$$

From (3.16), we have

$$\begin{aligned} & |x_1'(t)| \\ & = \lambda|r_1(t) - k_1(t)e^{x_1(t)} - a(t)e^{x_1(t-\tau_1(t))} \\ & \quad -b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))} - c_1(t)e^{x_2(t-\tau_3(t))}| \\ & \leq |r_1|_0 + |k_1|_0 e^{R_1} + |a|_0 e^{R_1} + |c_1|_0 e^{R_1} + |b|_0 |x_1'(t)| e^{R_1}, \end{aligned}$$

i.e.,

$$\begin{aligned} |x_1'(t)| \leq & \frac{|r_1|_0 + |k_1|_0 e^{R_1} + |a|_0 e^{R_1} + |c_1|_0 e^{R_1}}{1 - |b|_0 e^{R_1}} \\ < & \frac{|r_1|_0 + |k_1|_0 e^{R_1} + |a|_0 e^{R_1} + |c_1|_0 e^{R_1}}{1 - |b|_0 e^{R_1}} + 1 \leq R_2. \end{aligned}$$

This implies that $|x_1'| < R_2$. Similarly, we have $|x_2'| < R_2$, so $R_2 = |x'|_0 = \max\{|x_1'|, |x_2'|\} < R_2$, which is a contradiction.

For $x \in KerL \cap \partial\Omega$, then $x = (x_1, x_2)^T \in R^2$ and $|x|_0 = R_1$, so by (H2) and (H3), we have

$$QNx = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega [f_1(t,x_1(t),x_2(t)) - b(t)x_1'(t-\tau_2(t))e^{x_1(t-\tau_2(t))}]dt \\ \frac{1}{\omega}\int_0^\omega f_2(t,x_1(t),x_2(t))dt \end{pmatrix}$$
$$= \begin{pmatrix} \bar{k}_1 e^{x_1} + \bar{a}e^{x_1} + \bar{c}_1 e^{x_2} - \bar{r}_1 \\ \bar{c}_2 e^{x_1} + \bar{k}_2 e^{x_2} - \bar{r}_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$deg\{JQN, KerL \cap \Omega, 0\} = \text{sign}(\det(a_{ij}) \neq 0,$$

where

$$(a_{ij}) = \begin{pmatrix} \bar{k}_1 + \bar{a} & \bar{c}_1 \\ \bar{c}_2 & \bar{k}_2 \end{pmatrix}.$$

So, Eqns (2.1) has at least one ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ by Lemma 2.1, that is, Eqns (1.2) has at least one positive ω-periodic solution $(x(t), y(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)})^T$. The proof of Theorem 3.1 is complete. ■

Theorem 3.3. *If $\bar{r}_1 > 0, \Gamma(t) \leq 0$ and (H2) holds, then Eqns (1.2) does not exist any positive ω-periodic solution.*

Proof. We need only to prove that Eqns (2.1) does not exist ω-periodic solution. If Eqns (2.1) has a ω-periodic solution $x(t) = (x_1(t), x_2(t))^T$, then by integrating both sides of the first formula of Eqns (2.1) over $[0, \omega]$, we get that

$$\bar{r}_1\omega = \int_0^\omega [k_1(t)e^{x_1(t)} + a(t)e^{x_1(t-\tau_1(t))} - b_1'(t)e^{x_1(t-\tau_2(t))} + c_1(t)e^{x_1(t-\tau_2(t))}]dt,$$

i.e.,

$$\bar{r}_1\omega = \int_0^\omega [\Gamma(t)e^{x_1(t)} + c_1(t)e^{x_2(t-\tau_3(t))}]dt.$$

From (3.14) and (H2), we have

$$\int_0^\omega \Gamma(t)e^{x_1(t)}dt = \bar{r}_1\omega - \int_0^\omega c_1(t)e^{x_2(t-\tau_3(t))}dt > \omega\left[\bar{r}_1 - \bar{c}_1\left(\frac{\bar{r}_2}{\bar{k}_2} + e^{(\bar{r_2}+\bar{r_2}^*)\omega}\right)\right] > 0.$$

So there is a $\xi \in [0, \omega]$ such that

$$\Gamma(\xi)\int_0^\omega e^{x_1(t)}dt > 0,$$

which implies that $\Gamma(\xi) > 0$. It is impossible. This contradiction implies that Eqns (2.1) does not have any ω-periodic solution. The proof of Theorem 3.3 is complete. ■

4 Example

Considering the following system:

$$\begin{cases} x'(t) = x(t)\left[1 - x(t) - x\left(t - \frac{1}{2}\sin t\right) - bx'(t-1) - \frac{1}{2e^{4\pi}}y(t)\right], \\ y'(t) = y(t)\left[1 - \frac{1}{2e^{4\pi}}x(t-\sin t) - y(t-\cos t)\right], \end{cases} \tag{4.1}$$

where b is a parameter and $\omega = 2\pi$.Then we may choose a suitable b such that the Eqns (4.1) has at least one positive 2π-periodic solution.

In order to apply Theorem 3.1, we only to show that the conditions (H1)–(H3) hold.

For Eqns (4.1), we have $\Gamma(t) = 1 + \dfrac{1}{1-\frac{1}{2}\cos t}, t \in [0, 2\pi]$ and $3 \geq \Gamma(t) \geq 2 > 0, \dfrac{1}{3} \leq \bar{\Gamma} \leq \dfrac{1}{2}$, therefore, condition (H1) holds.

Moreover,

$$\frac{\bar{r}_1}{\bar{c_1}} = 2e^{4\pi} > \frac{\bar{r}_2}{\bar{k}_2}e^{(\bar{r_2}+\bar{r_2}^*)\omega} = e^{4\pi}, \frac{\bar{r}_2}{\bar{c_2}} = 2e^{4\pi} > \max\left\{\frac{\bar{r}_2}{\bar{\Gamma}}e^{(\bar{r_2}+\bar{r_2}^*)\omega} < e^{4\pi}, \frac{\bar{r}_1}{\bar{k}_1+\bar{a}} = \frac{1}{2}\right\}.$$

So, condition (H2) is true. Solving the equation

$$\begin{cases} 2e^{x_1} + \frac{1}{2e^{4\pi}}e^{x_2} = 1, \\ \frac{1}{2e^{4\pi}}e^{x_1} + e^{x_2} = 1, \end{cases}$$

we can get

$$\begin{cases} x_1 = \ln\frac{1-c_1}{2-c_1c_2}, \\ x_2 = \ln\frac{2-c_2}{2-c_1c_2}, \end{cases}$$

where $c_1 = c_2 = \dfrac{1}{2e^{4\pi}}$. Let $M_1 = |x|_0 = \max\{|x_1|_0, |x_2|_0\}$. By direct computation, we have

$$B_1 < \ln 2 + 4\pi, B_2 = 4\pi, B_3 < \ln 6 + 4\pi \quad \text{and} \quad B_4 < \ln 6 + 4\pi.$$

Let $R_1 = \max\{M_1 + 1, \ln 6 + 4\pi + 2\}$, if $|b| < \dfrac{1}{e^{R_1}}$, then condition (H3) is correct. So by Theorem 3.1, Eqns (4.1) has at least one positive 2π-periodic solution.

Acknowledgment

The authors thank the referee for careful reading of the manuscript and useful suggestions that helped to improve the paper.

References

[1] R.K. Gaines and J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations. Berlin: Springer, 1977.

[2] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Press, Boston, 1992.

[3] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[4] Y. Kuang, On neutral delay two-species Lotka-Volterra competitive systems, *J. Austral. Math. Soc.*, **Ser. B32**, pp. 311–326, 1991.

[5] Y. Kuang, On neutral delay losgistic Gause-type predator-prey systems, *Dyn. Stab. Systems*, **6**, pp. 173–189, 1991.

[6] S.P. Lu, On the existence of positive periodic solutions for neutral functional differential equation with multiple deviating arguments, *J. Math. Anal. Appl.*, **280**, pp. 321–333, 2003.

[7] S.P. Lu and W.G. Ge, Existence of positive periodic solutions for neutral functional differential equation with multiple delays, *Applied Mathematics and Computation*, **153**, pp. 885–902, 2004.

[8] X.Y. Song and L.S. Chen, Periodic solution of a delay differential equation of plankton allelopathy, *Acta Mathematica Scientia*, **23A(1)**, pp. 8–13, 2003.

[9] Z.H. Yang and J.D. Cao, Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models, *Applied Mathematics and Computation*, **142**, pp. 123–142, 2003.