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Estimates of Positive Solutions to a Boundary Value Problem for the Beam Equation

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Abstract

We consider a two-point boundary value problem for the fourth order beam equation. New upper and lower estimates of positive solutions of the problem are obtained.

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1 Introduction

This paper discusses the fourth order boundary value problem

$$u''''(t) = f(t, u(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = u'(1) = u(1) = 0. \tag{1.2}$$

The equation (1.1) is often referred to as the beam equation. It models the deflection or bending of an elastic beam under a certain force. The conditions (1.2) mean that the beam is embedded at both ends $t = 0$ and $t = 1$.

Fourth order two point boundary value problems have received a lot of attention in the literature due to their many applications in elasticity. For example, the problem (1.1)-(1.2) was investigated by Korman [3], Kosmatov [4], and Yao [7]. In 2005, Ma and Tisdell [5] studied a special case of the problem (1.1)-(1.2) in which $f(t, u) = p(t)u^\lambda$ with $\lambda \in (0, 1)$. The Green's function $G : [0, 1] \times [0, 1] \to [0, \infty)$ for the problem (1.1)-(1.2) is given by

$$G(t, s) = \begin{cases} (1/6)t^2(1-s)^2(3s - t - 2ts), & \text{if} \quad 0 \le t \le s \le 1, \\ (1/6)s^2(1-t)^2(3t - s - 2st), & \text{if} \quad 0 \le s \le t \le 1. \end{cases}$$

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Then the problem (1.1)–(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \quad 0 \le t \le 1. \tag{1.3}$$

It is easy to see that

$$G(t,s) > 0 \quad \text{for} \quad (t,s) \in (0,1) \times (0,1). \tag{1.4}$$

Throughout the paper we assume that

(H) $f : (0,1) \times [0,\infty) \to [0,\infty)$ is a continuous function.

The purpose of this paper is to prove a type of upper and lower estimates of positive solutions for the problem (1.1)–(1.2). In this paper by a positive solution to the problem (1.1)–(1.2), we mean a function $u \in C^3[0,1] \cap C^4(0,1)$ which solves the equation (1.1) on (0, 1), satisfies (1.2), and is such that $u(t) > 0$ for $0 < t < 1$. Note that $C^3[0,1] \cap C^4(0,1)$ is a wider class than $C^4[0,1]$.

2 Estimates of Positive Solutions

In order to give the new set of upper and lower estimates of positive solutions for the problem (1.1)–(1.2), we define the functions $a : [0,1] \to [0,1]$ and $b : [0,1] \to [0,1]$ by

$$a(t) = \begin{cases} (27/4)\, t^2(1-t), & \text{if } 0 \le t \le 1/2, \\ (27/4)\, t(1-t)^2, & \text{if } 1/2 \le t \le 1, \end{cases}$$

$$b(t) = \begin{cases} (27/4)\, t(1-t)^2, & \text{if } 0 \le t \le 1/3, \\ 1, & \text{if } 1/3 \le t \le 2/3, \\ (27/4)\, t^2(1-t), & \text{if } 2/3 \le t \le 1. \end{cases}$$

It's easy to see that both $a(t)$ and $b(t)$ are continuous functions on [0, 1], and

$$b(t) \ge a(t) \ge \frac{27}{8} \min\{t^2, (1-t)^2\}, \quad 0 \le t \le 1. \tag{2.1}$$

Note that the expressions for $a(t)$ and $b(t)$ contain the fraction 27/4, whose purpose is to ensure that $\max\limits_{0 \le t \le 1} b(t) = 1$. In this paper, we let the Banach space $X = C[0,1]$ be equipped with the supremum norm

$$\|v\| = \max_{0 \le t \le 1} |v(t)| \quad \text{for all} \quad v \in X.$$

We begin with several technical lemmas.

Lemma 2.1. *If $u \in C^3[0,1] \cap C^4(0,1)$ satisfies* (1.2) *and*

$$u''''(t) \ge 0 \quad \textit{for} \quad 0 < t < 1, \tag{2.2}$$

then $u(t) \ge 0$ for $0 \le t \le 1$.

Proof. By mean value theorem, because $u(0) = u(1) = 0$, there exists $t_1 \in (0, 1)$ such that $u'(t_1) = 0$. Since $u'(0) = u'(t_1) = u'(1) = 0$, there exist $t_2 \in (0, t_1)$ and $s_2 \in (t_1, 1)$ such that $u''(t_2) = u''(s_2) = 0$. We see from (2.2) that $u''(t)$ is concave upward on the interval $[0, 1]$. Therefore,

$$u''(t) \geq 0 \text{ on } (0, t_2), \quad u''(t) \leq 0 \text{ on } (t_2, s_2), \quad u''(t) \geq 0 \text{ on } (s_2, 1).$$

Since $u(0) = u'(0) = 0$, and $u(t)$ is concave upward on $(0, t_2)$, we have $u(t) \geq 0$ for $0 \leq t \leq t_2$. Similarly, we have $u(t) \geq 0$ for $s_2 \leq t \leq 1$. Since $u(t_2) \geq 0$, $u(s_2) \geq 0$, and since $u(t)$ is concave downward on (t_2, s_2), we have $u(t) \geq 0$ for $t_2 \leq t \leq s_2$. The proof is complete. ■

Lemma 2.2. *Suppose that $u \in C^3[0, 1] \cap C^4(0, 1)$ satisfies* (1.2) *and* (2.2). *If $u(r) > 0$ for some $r \in (0, 1)$, then*

(i) $u(t) > 0$ *for* $0 < t < 1$*;*

(ii) there is a unique $c \in (0, 1)$ *such that* $u'(c) = 0$*;*

(iii) $u(c) = \|u\|$.

Proof. By Lemma 2.1, we have $u(t) \geq 0$ for $0 \leq t \leq 1$. Note that (2.2) implies that u'' is concave upward on $[0, 1]$.

By the mean value theorem, since $u(0) = 0 < u(r)$ and $u(r) > 0 = u(1)$, there exist $t_1 \in (0, r)$ and $s_1 \in (r, 1)$ such that $u'(t_1) > 0$ and $u'(s_1) < 0$. Since $u'(0) < u'(t_1)$, $u'(t_1) > u'(s_1)$, and $u'(s_1) < u'(1)$, there exist $t_2 \in (0, t_1)$, $\beta \in (t_1, s_1)$, and $s_2 \in (s_1, 1)$ such that $u''(t_2) > 0$, $u''(\beta) < 0$, and $u''(s_2) < 0$.

Because u'' is concave upward, there exist $\alpha_1 \in (t_2, \beta)$ and $\alpha_2 \in (\beta, s_2)$ such that $u''(\alpha_1) = u''(\alpha_2) = 0$, and

$$u''(t) > 0 \quad \text{on} \quad [0, \alpha_1), \quad u''(t) < 0 \quad \text{on} \quad (\alpha_1, \alpha_2), \quad u''(t) > 0 \quad \text{on} \quad (\alpha_2, 1].$$

Since $u'(0) = u'(1) = 0$, there exists $c \in (0, 1)$ such that $u'(c) = 0$. It is easy to see that $c \in (\alpha_1, \alpha_2)$, $u''(c) < 0$, and

$$u'(t) > 0 \quad \text{for} \quad 0 < t < c, \quad u'(t) < 0 \quad \text{for} \quad c < t < 1. \tag{2.3}$$

Thus we proved that c is unique. Combining (2.3) with the fact that $u(0) = u(1) = 0$, we have $u(t) > 0$ for $0 < t < 1$, and $u(c) = \max_{0 \leq t \leq 1} u(t)$. This completes the proof of the lemma. ■

One of the implications of Lemma 2.2 is that if $u \in C^3[0, 1] \cap C^4(0, 1)$ satisfies (1.2) and (2.2), then either $u(t) \equiv 0$ on $[0, 1]$, or $u(t) > 0$ for $0 < t < 1$.

Lemma 2.3. *Let $\gamma > 0$ be a real number. Suppose that $w \in C[0, \gamma]$, $w(\gamma) = 0$, and $w(t)$ is concave downward. If*

$$\int_0^\gamma w(t)\, dt = 0, \tag{2.4}$$

then

$$\int_0^\gamma t w(t)\, dt \geq 0.$$

Proof. In view of (2.4), there exists $\theta \in (0, \gamma)$ such that $w(\theta) = 0$. Because $w(t)$ is concave downward, and because $u(\theta) = u(\gamma) = 0$, we have $w(t) \leq 0$ on $(0, \theta)$ and $w(t) \geq 0$ on (θ, γ). It follows that $(t - \theta)w(t) \geq 0$ for $0 \leq t \leq \gamma$, which implies that

$$\int_0^\gamma (t - \theta)w(t)dt \geq 0.$$

Therefore,

$$\int_0^\gamma tw(t)dt \geq \int_0^\gamma \theta w(t)dt = 0.$$

The proof is complete. ∎

Lemma 2.4. *Suppose that* $u \in C^3[0, 1] \cap C^4(0, 1)$ *satisfies* (1.2) *and* (2.2)*, and*

$$u(t) > 0 \quad \textit{for} \quad 0 < t < 1. \tag{2.5}$$

Let c *be the unique zero of* u' *in* $(0, 1)$*, and let* $p(t) = -u''(t)$*,* $0 \leq t \leq 1$*. Then*

$$\int_0^c sp(s)ds = \int_c^1 (1 - s)p(s)ds. \tag{2.6}$$

Proof. It is obvious that

$$u(c) = u(c) - u(0) = \int_0^c u'(t)dt = \int_0^c \int_t^c (-u''(s))dsdt$$

$$= \int_0^c \int_t^c p(s)dsdt = \int_0^c sp(s)ds,$$

and

$$u(c) = u(c) - u(1) = -\int_c^1 u'(t)dt = -\int_c^1 \int_c^t u''(s)\,dsdt$$
$$= \int_c^1 \int_c^t p(s)dsdt = \int_c^1 (1 - s)p(s)ds,$$

from which (2.6) follows immediately. The proof is complete. ∎

Lemma 2.5. *Suppose that* $u \in C^3[0, 1] \cap C^4(0, 1)$ *satisfies* (1.2)*,* (2.2)*, and* (2.5)*. Let* c *be the unique zero of* u' *in* $(0, 1)$*. Then* $1/3 < c < 2/3$*.*

Proof. We let $p(t) = -u''(t)$, $0 \leq t \leq 1$. From the proof of Lemma 2.2, we see that $p(c) > 0$ and $p(1) < 0$.

Claim. $p'(c) \leq 2p(c)/c$.

Proof of the claim. Assume the contrary that $p'(c) > 2p(c)/c$. Then, because $p(t)$ is concave downward, we have

$$p'(t) \geq p'(c) > 2p(c)/c \text{ for } t \in [0, c).$$

Therefore, for each $t \in [0, c)$, we have

$$p(t) = p(c) - \int_t^c p'(s)\,ds < p(c) - 2(c-t)p(c)/c = (2t/c - 1)p(c),$$

which implies that

$$\int_0^c p(t)dt < \int_0^c (2t/c - 1)p(c)dt = 0.$$

On the other hand, we have

$$\int_0^c p(t)dt = -\int_0^c u''(t)dt = -u'(c) + u'(0) = 0, \tag{2.7}$$

which is a contradiction. The proof of the claim is complete.

Let $w(t) = p(t) - (2t/c - 1)p(c)$, $0 \le t \le c$. Then $w(t)$ is concave downward, $w(c) = 0$, and

$$\int_0^c w(t)\,dt = 0.$$

By Lemma 2.3, the last equation implies that

$$\int_0^c w(t)t\,dt \ge 0.$$

Therefore,

$$\int_0^c tp(t)dt \ge \int_0^c t \cdot (2t/c - 1)p(c)\,dt = c^2 p(c)/6. \tag{2.8}$$

On the other hand, because $p'(c) \le 2p(c)/c$, and because $p(t)$ is concave downward, we have

$$p'(t) \le p'(c) \le 2p(c)/c \;\text{ for }\; t \in (c, 1],$$

which implies that

$$p(t) \le p(c) + \int_c^t 2p(c)c^{-1}ds = (2t/c - 1)p(c) \;\text{ for }\; t \in (c, 1]. \tag{2.9}$$

We see from the proof of Lemma 2.2 that $p(1) < 0$. By continuity, because $p(1) < 0 < p(c)$, there exists $\beta \in (c, 1)$ such that

$$p(t) < (2t/c - 1)p(c) \;\text{ for }\; \beta \le t \le 1. \tag{2.10}$$

In view of (2.9) and (2.10), we have

$$\int_c^1 (1-t)p(t)\,dt < \int_c^1 (1-t)(2t/c - 1)p(c)dt = (1/(3c) - 1/2 + c^2/6)p(c). \tag{2.11}$$

We see from (2.6), (2.8), and (2.11) that

$$1/(3c) - 1/2 + c^2/6 > c^2/6.$$

Simplifying, we get $c < 2/3$.

To show that $c > 1/3$, we let $v(t) = u(1-t)$, $0 \le t \le 1$. It is obvious that $v(0) = v'(0) = v'(1) = v(1) = 0$, $v(t) > 0$ for $0 < t < 1$, and

$$v''''(t) = u''''(1-t) \ge 0, \quad 0 < t < 1.$$

Since c is the unique zero of u' in $(0,1)$, $1-c$ is the unique zero of v' in $(0,1)$. From the early portion of the proof, we see that $1-c < 2/3$. It follows immediately that $c > 1/3$. The proof is complete. ■

Lemma 2.5 is an interesting result in its own right.

Lemma 2.6. *Suppose that $u \in C^3[0,1] \cap C^4(0,1)$ satisfies* (1.2), (2.2), *and* (2.5). *Let c be the unique zero of u' in $(0,1)$. Then*

$$u(t) \ge (27/4)t^2(1-t)u(c) \quad \textit{for} \quad 0 \le t \le c.$$

Proof. If we define $h(t) = u(t) - (27/4)t^2(1-t)u(c)$, $0 \le t \le c$, then

$$h'(t) = u'(t) - (27/4)(2t - 3t^2)u(c), \quad h''(t) = u''(t) - (27/2)(1-3t)u(c),$$

$$h'''(t) = u'''(t) + (81/2)u(c),$$

$$h''''(t) = u''''(t) \ge 0, \quad 0 < t \le c. \tag{2.12}$$

From the above equations it is easy to verify that

$$h(0) = 0, \quad h(c) > 0, \quad h'(0) = 0, \quad h'(c) < 0.$$

In verifying $h(c) > 0$ and $h'(c) < 0$, we used the fact that $1/3 < c < 2/3$. To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le c$.

By Mean Value Theorem, because $h(0) = 0 < h(c)$, there exists $t_1 \in (0,c)$ such that $h'(t_1) > 0$. Because $h'(0) = 0 < h'(t_1)$ and $h'(t_1) > 0 > h'(c)$, there exist $t_2 \in (0,t_1)$ and $r_2 \in (t_1,c)$ such that $h''(t_2) > 0$ and $h''(r_2) < 0$. In view of (2.12), h'' is concave upward on $[0,c]$.

At this point, there are two possible cases:

Case I. $h''(c) \le 0$. In this case, there exists $s_2 \in (t_2, r_2)$ such that

$$h''(t) > 0 \quad \text{for} \quad 0 \le t < s_2, \quad \text{and} \quad h''(t) \le 0 \quad \text{for} \quad s_2 \le t \le c,$$

Because $h'(0) = 0 > h'(c)$, there exists $s_1 \in (t_1, c)$ such that

$$h'(t) > 0 \quad \text{for} \quad 0 \le t < s_1, \quad \text{and} \quad h'(t) \le 0 \quad \text{for} \quad s_1 \le t \le c. \tag{2.13}$$

Since $u(0) = 0$ and $u(c) > 0$, we have $h(t) \ge 0$ for $0 \le t \le c$.

Case II. $h''(c) > 0$. In this case, there exist $s_2 \in (t_2, r_2)$ and $p_2 \in (r_2, c)$ such that

$$h''(t) > 0 \text{ on } (0, s_2), \quad h''(t) < 0 \text{ on } (s_2, p_2), \quad \text{and} \quad h''(t) > 0 \text{ on } (p_2, c).$$

Because $h'(0) = 0 > h'(c)$, there exists $s_1 \in (t_1, c)$ such that (2.13) is true. Therefore, $h(t) \ge 0$ for $0 \le t \le c$.

In either case, we have $h(t) \ge 0$ for $0 \le t \le c$. The proof is complete. ■

Lemma 2.7. *If* $u \in C^3[0,1] \cap C^4(0,1)$ *satisfies* (1.2), (2.2), *and* (2.5), *then*

$$u(t) \le b(t)\|u\| \quad \textit{for} \quad 0 \le t \le 1/3.$$

Proof. Let c be the unique zero of u' in $(0,1)$. If we define

$$h(t) = b(t)\|u\| - u(t) = (27/4)t(1-t)^2 u(c) - u(t), \quad 0 \le t \le 1/3,$$

then

$$h'(t) = (27/4)(1 - 4t + 3t^2)u(c) - u'(t), \quad h''(t) = (27/2)(-2 + 3t)u(c) - u''(t),$$

$$h'''(t) = (81/2)u(c) - u'''(t),$$

$$h''''(t) = -u''''(t) \le 0, \quad 0 < t \le 1/3. \tag{2.14}$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1/3$. Note that $1/3 < c < 2/3$. We see from the proof of Lemma 2.2 that $u(1/3) < u(c)$, $u''(0) > 0$, and $u'(1/3) > 0$. It is easy to check that

$$h(0) = 0, \quad h(1/3) > 0, \quad h'(0) > 0, \quad h'(1/3) < 0, \quad h''(0) < 0.$$

Claim. If $h''(c) \le 0$, then $h''(t) \le 0$ for $0 \le t \le c$.

Proof of Claim. Let $p(t) = -u''(t)$, $0 \le t \le 1$. Assume the contrary that $h''(\beta) > 0$ for some $\beta \in (0,c)$. Because $h''(\beta) > 0$ and $h''(c) \le 0$, we have

$$p(c) \le (27/2)(2 - 3c)u(c), \quad p(\beta) > (27/2)(2 - 3\beta)u(c).$$

In view of (2.2), $p(t)$ is concave downward. Therefore, for $c < t < 1$, we have

$$p(t) \le p(c) + \frac{p(c) - p(\beta)}{c - \beta}(t - c) < p(c) - (81/2)(t - c)u(c) \le (27/2)(2 - 3t)u(c).$$

It follows that

$$\int_c^1 p(t)\,dt < (27/2)\int_c^1 (2 - 3t)u(c)\,dt = (27/4)(1 - 3c)(1 - c)u(c) < 0.$$

On the other hand,

$$\int_c^1 p(t)\,dt = -\int_c^1 u''(t)\,dt = u'(c) - u'(1) = 0,$$

a contradiction. The proof of the Claim is complete.

At this point, there are two possible cases:

Case I. $h''(t) \le 0$ on $[0,1/3]$. In this case, h is concave downward on $[0,1/3]$. Since $h(0) = 0$ and $h(1/3) > 0$, we have $h(t) \ge 0$ for $0 \le t \le 1/3$.

Case II. $h''(\gamma) > 0$ for some $\gamma \in (0,1/3)$. In this case, by the Claim, it must be true that $h''(c) > 0$. Because $h''(0) < 0$, and h'' is concave downward on $(0,1)$, there exists $\alpha \in (0,1/3)$ such that

$$h''(t) \le 0 \quad \text{on} \quad [0,\alpha], \quad h''(t) \ge 0 \quad \text{on} \quad [\alpha, 1/3].$$

Since $u'(0) > 0$ and $u'(1/3) < 0$, there exists $\delta \in (0, 1/3)$ such that

$$h'(t) \geq 0 \quad \text{on} \quad [0, \delta], \quad h'(t) \leq 0 \quad \text{on} \quad [\delta, 1/3].$$

Since $h(0) = 0$ and $h(1/3) > 0$, we have $h(t) \geq 0$ for $0 \leq t \leq c$.

In either case, we have $h(t) \geq 0$ for $0 \leq t \leq c$. The proof is complete. ■

Theorem 2.8. *If $u \in C^3[0, 1] \cap C^4(0, 1)$ satisfies* (1.2)*,* (2.2)*, and* (2.5)*, then*

$$a(t)\|u\| \leq u(t) \leq b(t)\|u\| \quad \text{for} \quad 0 \leq t \leq 1. \tag{2.15}$$

In particular, if $u(t)$ is a positive solution to the problem (1.1)–(1.2)*, then it satisfies* (2.15)*.*

Proof. Let c be the unique zero of u' in $(0, 1)$. First, we note that

$$a(t) = \min\left\{(27/4)t^2(1-t), (27/4)t(1-t)^2\right\}, \quad 0 \leq t \leq 1.$$

From Lemma 2.6 we see that $u(t) \geq a(t)u(c)$ for $0 \leq t \leq c$. If we let $v(t) = u(1-t)$, $0 \leq t \leq 1$, then $1 - c$ is the unique zero of v' in $(0, 1)$. By applying Lemma 2.6 to $v(t)$, we get $v(t) \geq a(t)v(1-c)$ for $0 \leq t \leq 1-c$, which implies that $u(t) \geq a(t)u(c)$ for $c \leq t \leq 1$. Thus we proved the left half of (2.15).

From Lemma 2.7 we see that $u(t) \leq b(t)u(c)$ for $0 \leq t \leq 1/3$. By applying Lemma 2.7 to $v(t) = u(1-t)$ we get $u(t) \leq b(t)\, u(c)$ for $2/3 \leq t \leq 1$. And it is obvious that $u(t) \leq \|u\| = b(t)\|u\|$ for $1/3 \leq t \leq 2/3$. Thus we proved the right half of (2.15). The proof is complete. ■

Remark 2.9. Theorem 2.8 provides not only a new upper estimate but also a new lower estimate of positive solutions for the problem (1.1)–(1.2).

In 2004, Yao [7] studied the problem (1.1)–(1.2) and used the following lower estimate of positive solutions to the problem (1.1)–(1.2) to define a positive cone K of X:

$$u(t) \geq \frac{2}{3}\min\{t^2, (1-t)^2\}\|u\| \text{ for } 0 \leq t \leq 1. \tag{2.16}$$

Now this estimate (2.16) is improved and implemented by Theorem 2.8. Actually, in view of (2.1), we have

$$\frac{a(t)}{(2/3)\min\{t^2, (1-t)^2\}} \geq \frac{81}{16} > 5.$$

Therefore, our lower estimate $a(t)\|u\|$ improves the estimate (2.16) significantly. No upper estimate of positive solutions for the problem (1.1)–(1.2) has been obtained in the literature.

Remark 2.10. Note that it is possible that the problem (1.1)–(1.2) does not have a positive solution. Actually the upper and lower estimates given in Theorem 2.8 can help us find sufficient conditions for existence and nonexistence of positive solutions of the boundary value problem. The reader is referred to [1, 6] for some works in this line.

Example 2.11. Consider the boundary value problem

$$u''''(t) = \frac{24}{t(1-t)}\sqrt{u(t)}, \quad 0 < t < 1, \tag{2.17}$$

$$u(0) = u'(0) = u'(1) = u(1) = 0. \tag{2.18}$$

By Theorem 1.1 of [5], the problem (2.17)–(2.18) has at least one positive solution in $C^3[0, 1] \cap C^4(0, 1)$. If we let

$$u^*(t) = t^2(1 - t)^2, \quad 0 \le t \le 1,$$

then $u^* \in C^3[0, 1] \cap C^4(0, 1)$ and $u^*(t)$ is a positive solution to the problem (2.17)–(2.18). According to Theorem 2.8, it holds that

$$a(t) \le u^*(t)/\|u^*\| \le b(t), \quad 0 \le t \le 1.$$

It is easy to see that $\|u^*\| = 1/16$. If we draw the graphs of $a(t)$, $b(t)$, and $u^*(t)/\|u^*\|$, we will see that the graph of $u^*(t)/\|u^*\|$ does lie between those of $a(t)$ and $b(t)$.

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