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FURTHER RESULTS OF INTEGRABILITY FOR THE DUNKL TRANSFORM

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Abstract

In this paper, we establish further results concerning integrability of the Dunkl transform of function f on $\mathbb{R}$ and in radial case on $\mathbb{R}^d$, when f satisfies a suitable condition.

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1 Introduction

On the real line, Dunkl operators are differential-difference operators introduced in 1989, by C. Dunkl in [3] and are denoted by Λ_α where α is a real parameter $> -\frac{1}{2}$. These operators, are associated with the reflexion group $\mathbb{Z}_2$ on $\mathbb{R}$. Dunkl kernel E_α is used to define Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [5]. Rösler in [7] shows that Dunkl kernel verify a product formula. This allows us to define Dunkl translation τ_x , $x \in \mathbb{R}$. As a result, we have a Dunkl convolution.

On $\mathbb{R}^d$, we consider the differential-difference operators $T_j, 1 \leq j \leq d$, associated with a positive root system R_+ and a non negative multiplicity function k, introduced by C. Dunkl in [3] and called Dunkl operators. These operators can be regarded as a generalization of partial

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derivatives and lead to generalizations of various analytic structure, like the exponential function, the Fourier transform, the translation operators and the convolution (see [2, 4, 5, 12, 14, 15]). K. Trimèche has introduced in [15] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$. At the moment an explicit formula for the Dunkl translation operator of function $\tau_x(f)$ is unknown in general. In fact τ_x may not even be a positive operator. However, a such formula is known when f is a radial function and the L^p-boundedness of τ_x for radial functions is established (see next section).

In this paper, we establish further results concerning integrability of the Dunkl transform of function f on $\mathbb{R}$ and in radial case on $\mathbb{R}^d$, when f satisfies a suitable condition. Analogous results of integrability have been obtained in Lipshitz-Hankel spaces for Hankel transform on $(0, +\infty)$ (see[1]).

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

In section 3, we establish further results of integrability for the Dunkl transform.

In the sequel c represents a suitable positive constant which is not necessarily the same in each occurence. Furthermore, we denote by

- $\star$ $\mathcal{E}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$.
- $\star$ $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\star$ $\mathcal{D}(\mathbb{R}^d)$ the space of functions in $\mathcal{E}(\mathbb{R}^d)$ with compact support.
- $\star$ $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives.

2 Preliminaries

- For a real parameter $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\alpha(\lambda\ .)$ on $\mathbb{R}$ has been introduced by C. Dunkl in [3] and is given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R}, \tag{2.1}$$

where j_α is the normalized Bessel function of the first kind and order α (see [16]). The Dunkl kernel $E_\alpha(\lambda\ .)$ is the unique solution on $\mathbb{R}$ of initial problem for the Dunkl operator (see [3]).

Let μ_α the weighted Lebesgue measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx.$$

For every $1 \le p \le +\infty$, we denote by $L^p(\mu_\alpha)$ the space $L^p(\mathbb{R}, d\mu_\alpha)$ and we use $\|\ \ \|_{p,\alpha}$ as a shorthand for $\|\ \ \|_{L^p(\mu_\alpha)}$.

The Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [5], is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy) f(y) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

According to [7], for $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z)$$

where $\gamma_{x,y}$ is a signed measure on $\mathbb{R}$.

For $x \in \mathbb{R}$, τ_x is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself . According to [11], the operator τ_x can be extended to $L^p(\mu_\alpha)$, $1 \le p \le 2$ and for $f \in L^p(\mu_\alpha)$ we have

$$\|\tau_x(f)\|_{p,\alpha} \le 4\|f\|_{p,\alpha}. \tag{2.2}$$

- Let W be a finite reflexion group on $\mathbb{R}^d$, associated with a root system R and R_+ the positive subsystem of R (see [2, 4, 5, 8, 12]). We denote by k a nonnegative multiplicity function defined on $\mathbb{R}$ with the property that k is W-invariant. We associate with k the index

$$\gamma = \gamma(R) = \sum_{\xi \in R_+} k(\xi) \ge 0,$$

and the weight function w_k defined by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d,$$

where $\langle \ , \ \rangle$ denotes the usual Euclidean inner product. Further, we introduce the Mehta-type constant c_k by

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x) dx \right)^{-1}.$$

For every $1 \le p \le +\infty$, we denote by $L^p_k(\mathbb{R}^d)$ the space $L^p(\mathbb{R}^d, w_k(x)dx)$, $\quad L^p_k(\mathbb{R}^d)^{rad}$ the subspace of those $f \in L^p_k(\mathbb{R}^d)$ that are radial and we use $\| \ \|_{p,k}$ as a shorthand for $\| \ \|_{L^p_k(\mathbb{R}^d)}$.

By using the homogeneity of w_k, it is shown in [8] that for $f \in L^1_k(\mathbb{R}^d)^{rad}$, there exists a function F on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function F is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d/2-1}\Gamma(\gamma + \frac{d}{2})}\ , \tag{2.3}$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) w_k(x) dx &= \int_0^{+\infty} \Big(\int_{S^{d-1}} w_k(ry) d\sigma(y) \Big) F(r) r^{d-1} dr \\ &= \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})} \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr. \end{aligned} \tag{2.4}$$

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C. Dunkl in [4]. For $y \in \mathbb{R}^d$, the function $x \mapsto E_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of initial problem for Dunkl operators (see [3]). This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. Below, let us collect some properties of Dunkl kernel which we can find in [4, 8, 9, 10, 12, 14].

Properties 2.1.

1. For all $x, y \in \mathbb{C}^d,\ E_k(x, y) = E_k(y, x)$.
2. For all $x, y \in \mathbb{C}^d$ and $\lambda \in \mathbb{C},\ E_k(\lambda x, y) = E_k(x, \lambda y)$.

The Dunkl transform $\mathcal{F}_k$ which was introduced and studied by C. Dunkl in [5] (see also [2]) is defined for $f \in D(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = \int_{\mathbb{R}^d} f(y)E_k(-ix, y)w_k(y)dy, \quad x \in \mathbb{R}^d.$$

According to ([8], Proposition 2.4), we have the following results:

$$\int_{S^{d-1}} E_k(ix, y)w_k(y)d\sigma(y) = \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})}\, j_{\gamma+\frac{d}{2}-1}(\|x\|), \quad x \in \mathbb{R}^d \tag{2.5}$$

and for f in $L^1_k(\mathbb{R}^d)^{rad}$,

$$\mathcal{F}_k(f)(x) = c_k^{-1} H_{\gamma+\frac{d}{2}-1}(F)(\|x\|), \quad x \in \mathbb{R}^d, \tag{2.6}$$

where F is the function defined on $[0, +\infty)$ by $F(\|x\|) = f(x),\ x \in \mathbb{R}^d$ and $H_{\gamma+\frac{d}{2}-1}$ is the Hankel transform of order $\gamma + \dfrac{d}{2} - 1$.

K. Trimèche has introduced in [15] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$ on $\mathcal{E}(\mathbb{R}^d)$. As un operator on $L^2_k(\mathbb{R}^d)$, τ_x is bounded. A priori it is not at all clear whether the translation operator can be defined for L^p- functions for p different from 2. However, according to [12] the operator τ_x can be extended to $L^p_k(\mathbb{R}^d)^{rad}$, $1 \leq p \leq 2$ and for $f \in L^p_k(\mathbb{R}^d)^{rad}$ we have

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}. \tag{2.7}$$

3 Integrability of the Dunkl Transform of Function

In this section, we establish further results concerning integrability of the Dunkl transform of function f on $\mathbb{R}$ and in radial case on $\mathbb{R}^d$, when f satisfies a suitable condition.

Put $q = \dfrac{p}{p-1}$ the conjugate of p for $1 < p \leq 2$.

Theorem 3.1. *Let $\beta > 0$, $A > 0$, $1 < p \leq 2$ and $f \in L^p(\mu_\alpha)$. If f satisfies*

$$\sup_{x\in(0,+\infty)} \frac{\|\tau_x(f) - f\|_{p,\alpha}}{x^\beta} < A\ , \tag{3.1}$$

then

1. *For* $0 < \beta \leq \dfrac{2(\alpha+1)}{p}$*, we have*

$$\mathcal{F}_\alpha(f) \in L^{p'}(\mu_\alpha) \text{ provided that } \frac{2(\alpha+1)p}{\beta p + 2(\alpha+1)(p-1)} < p' \leq q\ .$$

2. *For* $\beta > \dfrac{2(\alpha+1)}{p}$*, we have* $\mathcal{F}_\alpha(f) \in L^1(\mu_\alpha)$*.*

Proof. Since $f \in L^p(\mu_\alpha)$, by [11] and (2.2) we have

$$\mathcal{F}_\alpha(\tau_x(f) - f)(y) = (E_\alpha(ixy) - 1)\mathcal{F}_\alpha(f)(y),$$

$x \in (0, +\infty)$ and a.e $y \in \mathbb{R}$. Then according to [2] and (2.2) we can assert by the Marcinkiewicz interpolation theorem (see [13]), that

$$\begin{aligned} \|\mathcal{F}_\alpha(\tau_x(f) - f)\|_{q,\alpha} &= \Big(\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)|^q |E_\alpha(ixy) - 1|^q d\mu_\alpha(y)\Big)^{1/q} \\ &\leq c\ \|\tau_x(f) - f\|_{p,\alpha}. \end{aligned}$$

Moreover there exists $a, b \in (0, +\infty)$, such that

$$|j_\alpha(xy) - 1| \geq a(xy)^2, \quad \text{for each } \ 0 < |xy| < b. \tag{3.2}$$

Hence by (2.1), we can write

$$\begin{aligned} |E_\alpha(ixy) - 1| &\geq |j_\alpha(xy) - 1| \\ &\geq a(xy)^2\ , \quad 0 < |xy| < b, \end{aligned}$$

so using (3.1), it follows that for $x \in (0, +\infty)$,

$$x^2\Big(\int_{|y| \leq \frac{b}{x}} |\mathcal{F}_\alpha(f)(y)|^q |y|^{2q} d\mu_\alpha(y)\Big)^{1/q} \leq c\ \|\tau_x(f) - f\|_{p,\alpha} \leq c\ x^\beta. \tag{3.3}$$

Let $p' \in \Big]\dfrac{2(\alpha+1)p}{\beta p + 2(\alpha+1)(p-1)},\ q\Big]$, we define the function

$$g(t) = \int_{1 \leq |y| \leq t} |\mathcal{F}_\alpha(f)(g)|^{p'} |y|^{2p'} d\mu_\alpha(y), \quad t > 1.$$

By using Hölder inequality and (3.3), we have

$$\begin{aligned} g(t) &\leq \Big(\int_{1 \leq |y| \leq t} |\mathcal{F}_\alpha(f)(y)|^q |y|^{2q} d\mu_\alpha(y)\Big)^{p'/q} \Big(\int_{1 \leq |y| \leq t} d\mu_\alpha(y)\Big)^{1 - \frac{p'}{q}} \\ &\leq c\ t^{(2-\beta)p' + 2(\alpha+1)(1 - \frac{p'}{q})}, \quad t > 1. \end{aligned}$$

Then we get

$$\begin{aligned}\int_{1\leq|y|\leq t}|\mathcal{F}_\alpha(f)(y)|^{p'}d\mu_\alpha(y) &= \int_1^t y^{-2p'}g'(y)dy\\ &= t^{-2p'}g(t)+2p'\int_1^t y^{-2p'-1}g(y)dy\\ &\leq c\left(t^{-\beta p'+2(\alpha+1)(1-\frac{p'}{q})}+1\right),\quad t>1,\end{aligned}$$

since $\mathcal{F}_\alpha(f)\in L^q(\mu_\alpha)$ and $L^q(]-1,1[,d\mu_\alpha)\subset L^{p'}(]-1,1[,d\mu_\alpha)$, we obtain the desired result (1).

By proceeding in the same manner as in the proof on (1) when $\beta>\dfrac{2(\alpha+1)}{p}$ and $p'=1$, we deduce (2). ■

Theorem 3.2. *Let $\beta>0$, $A>0$, $1<p\leq 2$ and $f\in L_k^p(\mathbb{R}^d)^{rad}$. If f satisfies*

$$\sup_{t\in(0,+\infty)}\frac{w_p(f)(t)}{t^\beta}<A \tag{3.4}$$

where

$$w_p(f)(t)=\int_{S^{d-1}}\|\tau_{tu}(f)-f\|_{p,k}d\sigma(u),$$

then

1. *For $0<\beta\leq\dfrac{2(\gamma+\frac{d}{2})}{p}$, we have*

$$\mathcal{F}_k(f)\in L_k^{p'}(\mathbb{R}^d)\ \textit{provided that}\ \frac{2(\gamma+\frac{d}{2})p}{\beta p+2(\gamma+\frac{d}{2})(p-1)}<p'\leq q.$$

2. *For $\beta>\dfrac{2(\gamma+\frac{d}{2})}{p}$, we have $\mathcal{F}_k(f)\in L_k^1(\mathbb{R}^d)$.*

Proof. Let $f\in L_k^p(\mathbb{R}^d)^{rad}$, by [15] and (2.7) we can write

$$\mathcal{F}_k(\tau_{tu}(f)-f)(x)=(E_k(itu,x)-1)\mathcal{F}_k(f)(x),$$

$u\in S^{d-1}$, $t\in(0,\infty)$ and a.e $x\in\mathbb{R}^d$, then according to [2, 5, 8] and (2.7), we can assert by Marcinkiewicz interpolation theorem (see [13]), that

$$\left(\int_{\mathbb{R}^d}|\mathcal{F}_k(f)(x)|^q|E_k(itu,x)-1|^q w_k(x)dx\right)^{\frac{1}{q}}\leq c\|\tau_{tu}(f)-f\|_{p,k}. \tag{3.5}$$

On the other hand, from (2.3), (2.4), (Properties 2.1. (1), (2)) and (2.6) we have

$$\int_{\mathbb{R}^d}|\mathcal{F}_k(f)(x)|^q|E_k(itu,x)-1|^q w_k(x)dx \tag{3.6}$$

$$= h_k \int_0^{+\infty} |H_{\gamma+\frac{d}{2}-1}(F)(r)|^q \Big(\int_{S^{d-1}} |E_k(itru, z) - 1|^q w_k(z) d\sigma(z) \Big) \times r^{2\gamma+d-1} dr,$$

where $h_k = \dfrac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}$ and F is the function defined on $(0, +\infty)$ by $F(\|x\|) = f(x)$, for all $x \in \mathbb{R}^d$.

By (2.5) and Hölder's inequality, we get

$$h_k |j_{\gamma+\frac{d}{2}-1}(rt) - 1| = |\int_{S^{d-1}} [E_k(itru, z) - 1] w_k(z) d\sigma(z)|$$

$$\leq \Big(\int_{S^{d-1}} w_k(z) d\sigma(z) \Big)^{1/p} \Big(\int_{S^{d-1}} |E_k(itru, z) - 1|^q w_k(z) d\sigma(z) \Big)^{1/q},$$

so using (2.3) we obtain

$$|j_{\gamma+\frac{d}{2}-1}(rt) - 1|^q \leq c \int_{S^{d-1}} |E_k(itru, z) - 1|^q w_k(z) d\sigma(z).$$

Then according to (3.5) and (3.6) it follows that

$$\int_0^{+\infty} |H_{\gamma+\frac{d}{2}-1}(F)(r)|^q |j_{\gamma+\frac{d}{2}-1}(rt) - 1|^q r^{2\gamma+d-1} dr \leq c\, \|\tau_{tu}(f) - f\|_{p,k}.$$

From (3.2) and (3.4), we can write

$$t^2 \left(\int_0^{b/t} |H_{\gamma+\frac{d}{2}-1}(F)(r)|^q r^{2q} r^{2\gamma+d-1} dr \right)^{1/q} \leq c\, w_p(f)(t) \leq c\, t^\beta.$$

Let $p' \in \left] \dfrac{2(\gamma+\frac{d}{2})p}{\beta p + 2(\gamma+\frac{d}{2})(p-1)}\ ,\ q \right]$, we define the function

$$g(\delta) = \int_1^\delta |H_{\gamma+\frac{d}{2}-1}(F)(r)|^{p'} r^{2p'} r^{2\gamma+d-1} dr, \quad \delta > 1.$$

By reasoning as in the proof on Theorem 3.1, we can assert that

$$\int_1^\delta |H_{\gamma+\frac{d}{2}-1}(F)(r)|^{p'} r^{2\gamma+d-1} dr \quad \leq \quad c \left(\delta^{-\beta p' + 2(\gamma+\frac{d}{2})(1-\frac{p'}{q})} + 1 \right), \qquad \delta > 1.$$

Since

$$H_{\gamma+\frac{d}{2}-1}(F) \in L^q\big((0, +\infty),\ r^{2\gamma+d-1} dr\big),$$

we obtain

$$\int_0^{+\infty} |H_{\gamma+\frac{d}{2}-1}(F)(r)|^{p'} r^{2\gamma+d-1} dr < +\infty,$$

thus using (2.4) and (2.6), we conclude that

$$\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^{p'} w_k(x) dx < +\infty.$$

By proceeding in the same manner as in the proof on (1) when $\beta > \dfrac{2(\gamma+\frac{d}{2})}{p}$ and $p' = 1$, we deduce (2). ■

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