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# A GENERALIZED CHAIN RULE FOR FORMAL POWER SERIES

**XIAO-XIONG GAN***
Department of Mathematics, Morgan State University,
Baltimore, Maryland 21251

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**Abstract**

Given a formal power series $g(x) = \sum_{n=0}^{\infty} b_n\, x^n$ and a *nonunit* $f(x) = \sum_{n=1}^{\infty} a_n\, x^n$, it is well-known that the derivative of the composition of formal power series satisfies the Chain Rule, that is, $(g \circ f)' = g'(f) \cdot f'$. It is also proved that the right distributive law for formal power series exists if the composed series, such as $f$ above, is a nonunit. This paper provides a generalized Chain Rule without the requirement of *nonunitness* for the composed formal power series. A generalized right distributive law for formal power series is proved in this paper too.

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## 1 Introduction

The composition of formal power series or functional composition is always an interesting topic for mathematicians and therefore many developments about it can be seen almost every year. For example, Raney [1] investigated the functional composition patterns and created a Lagrange inversion formula more than forty years ago, Bender [2] provided a lifting theorem in early seventies and then Garsia and Joni [3] brought us a new expression for umbral operators and another Lagrange inversion formula, and Li [4] studied how to commute two formal power series in 1997. More recent applications can be found in [5], [6], and [7].

These results about the composition of formal power series require that the composed formal power series is a nonunit, that is, the constant term of this series is zero.

In 2002, Gan and Knox [8] provided a necessary and sufficient condition for the existence of the composition of formal power series and hence provided a possibility of removing the requirement of nonunitness. In this paper we will use this result as our main tool to generalize the differentiation properties for the composition of formal power series.

*E-mail: gxiaoxio@morgan.edu

For our convenience and for the consistency of the notations, we recall some definitions below.

**Definition 1.1.** Let $S$ be a ring and let $l \in \mathbb{N}$ be given, *a formal power series* on $S$ is defined to be a mapping from $\mathbb{N}^l$ to $S$, where $\mathbb{N}$ represents the set of all natural numbers. We denote the set of all such mappings by $\mathbb{X}(S)$, or $\mathbb{X}$.

In this paper, we only discuss formal power series from $\mathbb{N}$ to $S$. A formal power series $f$ in $x$ from $\mathbb{N}$ to $S$ is usually denoted by

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots, \quad \text{where} \quad \{a_j\}_{j=0}^{\infty} \subset S.$$

In this case, $a_k$ is called the $k$-th coefficient of $f$, $\forall k \in \mathbb{N} \cup \{0\}$. If $a_0 = 0$, $f$ is called a *nonunit*.

Let $f$ and $g$ be formal power series in $x$ with $f(x) = \sum_{n=0}^{\infty} f_n x^n$ and $g(x) = \sum_{n=0}^{\infty} g_n x^n$, and let $r \in S$, then $g + f, rf$, and $g \cdot f$ are defined as

$$(g + f)(x) = g(x) + f(x) = \sum_{n=0}^{\infty} (g_n + f_n)\, x^n,$$

$$(rf)(x) = rf(x) = \sum_{n=0}^{\infty} (rf_n)\, x^n,$$

and

$$(f \cdot g)(x) = g(x) \cdot f(x) = \sum_{n=0}^{\infty} c_n\, x^n, c_n = \sum_{j=0}^{n} g_j f_{n-j}, n = 0, 1, 2, \ldots.$$

It is clear that all those operations are well defined, that is, $g + f,\ rf$, and $g \cdot f$ are all in $\mathbb{X}$.

**Definition 1.2.** Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ be a formal power series over a ring $S$, the derivative of $f(x)$ is a formal power series denoted by

$$f'(x) = a_0' + a_1' x + a_2' x^2 + \cdots = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

that is $a_n' = (n+1) a_{n+1}$ for $n = 0, 1, 2, \ldots$.

It is proved that the derivative of the product of two formal power series satisfies the Product Rule, that is if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two formal power series, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

The Power Rule of differentiation, $(f^n(x))' = nf^{n-1}(x) \cdot f'(x)$, is just an application of the Product Rule.

**Definition 1.3.** Let $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n\, x^n$ be formal power series in $x$ over a ring $S$. Write

$$f^n(x) = [f(x)]^n = \sum_{k=0}^{\infty} a_k^{(n)}\, x^k$$

for $n = 0, 1, 2, \ldots$, where $a_0^{(0)} = 1, a_k^{(0)} = 0 \quad \forall k \in \mathbb{N}$. Then the composition $g \circ f$ in $x$ is defined to be a formal power series

$$\sum_{k=0}^{\infty} c_k\, x^k$$

where $c_k = \sum_{n=0}^{\infty} b_n\, a_k^{(n)}$ for $k = 0, 1, 2, \ldots$ if $c_k$ exists in $S$ for each $k$. It is convenient to write the composition of $g$ with $f$ as $g \circ f(x) = \sum_{n=0}^{\infty} b_n\, (f(x))^n$ if it exists.

If $f$ is a nonunit formal power series, the Chain Rule for the formal power series is true. Henrici provided the following theorem in his book [9].

**Theorem A.** *If $A$ is a formal power series and $Q$ a nonunit, then $(A \circ Q)'$ exists and*

$$(A \circ Q)' = (A' \circ Q) \cdot Q'.$$

*Is it really necessary for $Q$ above to be a nonunit formal power series? If it is not, what can be said?*

Theorem B below gives a necessary and sufficient condition for existence of the composition of some formal power series.

**Theorem B [8].** *Let $S$ be a field with a metric. Let $\mathbb{X}$ be the set of all formal power series from $\mathbb{N}$ to $S$. Let $f, g \in \mathbb{X}$ be given with the forms*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \qquad g(x) = b_0 + b_1 x + \cdots + b_n x^n \cdots,$$

*and deg $(f) \neq 0$. Then the composition $g \circ f$ exists if and only if*

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in S \quad \forall k \in \mathbb{N} \cup \{0\}.$$

*where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$.*

## 2 A Generalized Chain Rule

Theorem B provides a necessary and sufficient condition for the existence of the composition of formal power series. Can we generalize the Chain Rule for formal power series in Theorem A and no longer require the nonunitness for the composed series? In order to answer this question, not only we have to show that the existence of $g(f(x))$ implies the existence of $g'(f(x))$ , but also we need to show that these two compositions are equal, that is,

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

**Lemma 2.1.** *Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be two formal power series in $x$ over $\mathbb{R}$. Then $g^{(m)}(f(x))$ exists if and only $g(f(x))$ exists where $m \in \mathbb{N}$ and $g^{(m)}$ is the $m$th derivative of $g$.*

*Proof.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series over $\mathbb{R}$. Let $n \in \mathbb{N}$ be given and write

$$f^n(x) = \sum_{k=0}^{\infty} a_k^{(n)} x^k,$$

we first show that

$$a_{k+1}^{(n)} = \frac{n}{k+1} \sum_{j=0}^{k} (k+1-j) a_j^{(n-1)} a_{k+1-j} \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{1}$$

By the power rule of derivatives and the product rule for the formal power series, we have

$$\begin{aligned} (f^n(x))' &= n\, f^{n-1}(x) f'(x) \\ &= n\Big(\sum_{j=0}^{\infty} a_j^{(n-1)} x^j\Big)\Big(\sum_{i=0}^{\infty} (i+1) a_{i+1} x^i\Big) \\ &= n \sum_{k=0}^{\infty} \Big[\sum_{j=0}^{k} a_j^{(n-1)} (k+1-j) a_{k+1-j}\Big] x^k. \end{aligned}$$

On the other hand,

$$(f^n(x))' = \sum_{k=0}^{\infty} (k+1) a_{k+1}^{(n)} x^k.$$

Then, $(k+1) a_{k+1}^{(n)} = n\, [\sum_{j=0}^{k} a_j^{(n-1)} (k+1-j) a_{k+1-j}]$.

Thus,

$$a_{k+1}^{(n)} = \frac{n}{k+1} \sum_{j=0}^{k} (k+1-j) a_j^{(n-1)} a_{k+1-j} \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Next, let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be two formal power series in $x$ over $\mathbb{R}$, we show that

$$g(f(x)) \text{ exists if and only } g'(f(x)) \text{ exists.} \tag{2}$$

Suppose $g'(f(x))$ exists. By Definition 1.2 and Theorem B, we have

$$\sum_{n=k}^{\infty} \binom{n}{k} (n+1) b_{n+1} a_0^{n-k} \in \mathbb{R} \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{3}$$

For the existence of $g(f(x))$, by the Theorem B again, we need only show that

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in \mathbb{R} \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{4}$$

If $a_0 = 0$, (4) is trivial. We suppose that $a_0 \neq 0$. Let $k \in \mathbb{N} \cup \{0\}$ be given. Define

$$\phi(x) = \sum_{n=k}^{\infty} \binom{n}{k} (n+1) b_{n+1} a_0^{n-k}\, x^{n-k}.$$

Since (3) is true, $\phi(x)$ converges uniformly on $[0,1]$ by Abel's Limit Theorem. Let $\phi_m(x)$ be the partial sum of this power series for $m = k, k+1, \ldots$. The uniform convergence of the power series $\phi(x)$ on $[0,1]$ yields that

$$\begin{aligned}
\int_0^1 \phi(x)\,dx &= \lim_{m\to\infty} \int_0^1 \sum_{n=k}^{m} \tbinom{n}{k}(n+1)b_{n+1}a_0^{n-k}x^{n-k}\,dx \\
&= \lim_{m\to\infty} \sum_{n=k}^{m} \tbinom{n}{k}(n+1)b_{n+1}a_0^{n-k}\frac{1}{n-k+1} \\
&= \lim_{m\to\infty} \sum_{n=k}^{m} \tbinom{n+1}{k}b_{n+1}a_0^{n-k} \\
&= \lim_{m\to\infty} \sum_{j=k+1}^{m} \tbinom{j}{k}b_j a_0^{j-k-1} \\
&= \frac{1}{a_0} \sum_{j=k+1}^{\infty} \tbinom{j}{k}b_j a_0^{j-k} \\
&= \frac{1}{a_0} \sum_{j=k}^{\infty} \tbinom{j}{k}b_j a_0^{j-k} - \frac{1}{a_0}b_k.
\end{aligned}$$

Then $\sum_{j=k}^{\infty}\binom{j}{k}b_j a_0^{j-k} \in \mathbb{R}$. Since $k$ is arbitrary, (4) is true for all $k$, and hence $g(f(x))$ exists by Theorem B.

Now suppose $g(f(x))$ exists. Then (4) is true by Theorem B. We need only show that (3) is true. Notice that for any $k \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned}
\sum_{n=k}^{\infty}\tbinom{n}{k}(n+1)b_{n+1}a_0^{n-k} &= (k+1)\sum_{n=k}^{\infty}\tbinom{n+1}{k+1}\,b_{n+1}\,a_0^{n+1-(k+1)} \\
&= (k+1)\sum_{m=k+1}^{\infty}\tbinom{m}{k+1}\,b_m\,a_0^{m-(k+1)} \\
&= r\sum_{m=r}^{\infty}\tbinom{m}{r}\,b_m\,a_0^{m-r}
\end{aligned}$$

for every $r \in \mathbb{N}$ if we write $r = k+1$.

Thus, (3) is true.

Applying mathematical induction, we can easily have the conclusion that the $m$th order derivative $g^{(m)}(f(x))$ exists if and only if $g(f(x))$ exists. ■

Notice that the formula (1) applies to any formal power series over a ring $S$.

We now introduce a generalized Chain Rule for formal power series.

**Theorem 2.2.** *Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be two formal power series in $x$ over $\mathbb{R}$. Then $(g\circ f)'$ exists if and only if $g\circ f$ exists and*

$$(g\circ f)'(x) = g'(f(x))\,f'(x). \tag{5}$$

*Proof.* The first part of the conclusion of the theorem is true by Lemma 2.1 above. We need only show that the equality (5) is true, that is, the corresponding coefficients on both sides of (5) are equal.

Since $g \circ f(x)$ exists, by Definition 1.3, we have

$$g \circ f(x) = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} b_n a_k^{(n)} \right) x^k,$$

where $a_k^{(n)}$ is the $k$th coefficient of $f^n(x),\ n \in \mathbb{N} \cup \{0\}$; $a_0^{(0)} = 1, a_k^{(0)} = 0 \quad \forall\ k \in \mathbb{N}$; and $\sum_{n=0}^{\infty} b_n a_k^{(n)} \in \mathbb{R} \forall k \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
(g \circ f)'(x) &= \sum_{k=1}^{\infty} k \left( \sum_{n=0}^{\infty} b_n a_k^{(n)} \right) x^{k-1} \\
&= \sum_{k=1}^{\infty} k \left( \sum_{n=1}^{\infty} b_n a_k^{(n)} \right) x^{k-1} \\
&= \sum_{m=0}^{\infty} (m+1) (\sum_{n=1}^{\infty} b_n a_{m+1}^{(n)}) x^m.
\end{aligned}$$

If we write $(g \circ f)'(x) = \sum_{m=0}^{\infty} r_m\, x^m$, then

$$r_m = (m+1) \sum_{k=1}^{\infty} b_k a_{m+1}^{(k)}, m = 0, 1, 2, \ldots. \tag{6}$$

On the other hand, $g'(x) = \sum_{n=0}^{\infty} (n+1) b_{n+1} x^n$, and then

$$g'(f(x)) = \sum_{k=0}^{\infty} \left[ \sum_{n=0}^{\infty} (n+1) b_{n+1} a_k^{(n)} \right] x^k.$$

If we write $g'(f(x)) \cdot f'(x) = \sum_{m=0}^{\infty} c_m x^m$, then apply the product rule and Formula (1), we have

$$\begin{aligned}
c_m &= \sum_{j=0}^{m} \left[ \sum_{n=0}^{\infty} (n+1) b_{n+1} a_j^{(n)} \right] \cdot (m+1-j) a_{m+1-j} \\
&= \sum_{n=0}^{\infty} (n+1) b_{n+1} \cdot \sum_{j=0}^{m} (m+1-j) a_j^{(n)} a_{m+1-j} \\
&= \sum_{n=0}^{\infty} (n+1) b_{n+1} \cdot a_{m+1}^{(n+1)} \cdot \frac{m+1}{n+1} \\
&= \sum_{n=0}^{\infty} (m+1) b_{n+1} a_{m+1}^{(n+1)}
\end{aligned}$$

for $m = 0, 1, 2, \ldots$. Applying (6), we have

$$c_m = \sum_{n=0}^{\infty} (m+1) b_{n+1} a_{m+1}^{(n+1)} = \sum_{k=1}^{\infty} (m+1) b_k a_{m+1}^{(k)} = r_m, \quad m = 0, 1, 2, \ldots.$$

The proof is completed. ■

## 3 A Generalized Right Distributive Law

The *right distributive law* for formal power series is a very interesting result. This law says

$$(A \cdot B) \circ P = (A \circ P) \cdot (B \circ P)$$

for formal power series $A, B$, and $P$ if the compositions involved exist.

It is proved[9] that the right distributive law is true if the formal power series $P$ above is a nonunit. The following theorem tells us that the right distributive law may also be true even if the composed formal power series is not a nonunit.

**Theorem 3.1.** *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n, B(x) = \sum_{n=0}^{\infty} b_n x^n$ and $P(x) = \sum_{n=0}^{\infty} p_n x^n$ be three formal power series over $\mathbb{R}$. The right distributive law, that is,*

$$(A \circ P)(B \circ P) = (AB) \circ P$$

*holds if both $A \circ P$ and $B \circ P$ exist.*

*Proof.* We suppose that both $A \circ P$ and $B \circ P$ exist. Let $P^n(x) = \sum_{k=0}^{\infty} p_k^{(n)} x^k$ for all $n \in \mathbb{N}$ as in Definition 1.3, and write

$$(A \circ P(x))(B \circ P(x)) = \sum_{m=0}^{\infty} r_m x^m.$$

Since $P^{n+i} = P^n \cdot P^i$, it follows that $p_m^{(n+i)} = \sum_{j=0}^{m} p_j^{(n)} p_{m-j}^{(i)}$ for each $m = 0, 1, 2, \ldots$. Then

$$\begin{aligned} r_m &= \sum_{j=0}^{m} \left( \sum_{n=0}^{\infty} a_n p_j^{(n)} \right) \cdot \left( \sum_{i=0}^{\infty} b_i p_{m-j}^{(i)} \right) \\ &= \sum_{n=0}^{\infty} a_n \sum_{i=0}^{\infty} b_i \left( \sum_{j=0}^{m} p_j^{(n)} \, p_{m-j}^{(i)} \right) \\ &= \sum_{n=0}^{\infty} a_n \sum_{i=0}^{\infty} b_i \, p_m^{(n+i)} \end{aligned}$$

is a real number for each $m$ because both $A \circ P$ and $B \circ P$ exist.

On the other hand, if we write $(AB) \circ P(x) = \sum_{m=0}^{\infty} s_m x^m$, then

$$\begin{aligned} s_m &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} a_j b_{n-j} \right) p_m^{(n)} \\ &= \sum_{j=0}^{\infty} a_j \sum_{n=j}^{\infty} b_{n-j} p_m^{(n)} \\ &= \sum_{j=0}^{\infty} a_j \sum_{n=0}^{\infty} b_n p_m^{(n+j)} \\ &= r_m, \end{aligned}$$

for $m = 0, 1, 2, \ldots$. Then $s_m \in \mathbb{R}$ and $s_m = r_m$ for each $m$. This completes the proof. ∎

The theorems in this paper helps us to establish a method to compute the coefficients of the composition of formal power series no matter the composed formal power series is a nonunit or not. That will alow us to extend many computation formulas in the related fields. We will discuss these in a forthcoming paper.

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