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A Differential Game of Multiperson Pursuit in the Hilbert Space

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Abstract

A differential game with finite number of pursuers and one evader is considered. The game is described by simple differential equations in Hilbert space. Integral constraints are imposed on the controls of all players. The game is deemed to be completed if the geometric position of a pursuer coincides with that of the evader. It is shown that if the total resources for controls of all pursuers are greater than that of the evader, then the completion of game is possible.

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1 Introduction

Differential games are investigated by many authors and fundamental results are given by Isaacs [3], Pontryagin [5], Krasovskii and Subbotin [4]. In particular, differential pursuit games involving several players in R^n are discussed by authors [6], [1], [2].

In the present paper, we solve a differential games with integral constraints on controls of players; namely a pursuit of one player by finite number of dynamical players in the space of l_2.

Consider the l_2 space of elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots)$, $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, with inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$ and norm $\|\alpha\| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}$.

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Let $H(x_0, r) = \{x \in l_2 : \|x - x_0\| \le r\}$ be a ball in the l_2 space with center x_0 and radius r.

The motions of the finite number of pursuers P_i and the evader E are described by the equations

$$\begin{aligned} &P_i : \dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = \overline{1, m} \\ &E : \dot{y} = v, \quad y(0) = y_0, \end{aligned} \tag{1.1}$$

where $x_i, x_0, u_i, y, y_0, v \in l_2$, $u_i = (u_{i1}, u_{i2}, \ldots, u_{ik}, \ldots)$ is the control parameter of the ith pursuer P_i, and $v = (v_1, v_2, \ldots, v_k, \ldots)$ is the control parameter of the evader E.

It is essential to give the following definitions.

Definition 1.1. A Borel measurable function $u_i(\cdot)$; $u_i : [0, \theta] \to H(0, \rho_i)$, such that

$$\int_0^\infty \| u_i(t) \|^2 \, dt \le \rho_i^2, \tag{1.2}$$

where ρ_i and θ are some given positive numbers, is called an admissible control of the ith pursuer, P_i.

Definition 1.2. A Borel measurable function $v(\cdot)$; $v : [0, \theta] \to H(0, \sigma)$, such that

$$\int_0^\infty \| v(t) \|^2 \, dt \le \sigma^2, \tag{1.3}$$

where σ and θ are some given positive numbers, is called an admissible control of the evader, E.

Once the players' admissible controls $u_i(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players can be obtained easily as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{ik}(t), \ldots), \quad x_{ik}(t) = x_i^0 + \int_0^t u_{ik}(s) ds, \tag{1.4}$$

and

$$y = (y_1(t), y_2(t), \ldots, y_k(t), \ldots), \quad y_k(t) = y^0 + \int_0^t v_k(s) ds. \tag{1.5}$$

One can verify that for a positive number θ, $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$, where $C(0, \theta; l_2)$ is the space of functions

$$f(t) = (f_1(t), f_2(t), \ldots, f_k(t), \ldots) \in l_2, t \ge 0,$$

such that:

1. $f_k(t),\ 0 \le t \le \theta$, are absolutely continuous functions;

2. $f(t), 0 \le t \le \theta$, is a continuous function in the norm of l_2.

Definition 1.3. A function $U_i(x_i, y, v),\ U_i : l_2 \times l_2 \times l_2 \times H(0,\sigma) \to H(0,\rho_i)$ such that the system of

$$\dot{x}_i = U_i(x_i, y, v), \quad x_i(0) = x_{i0},$$
$$\dot{y} = v, \quad y(0) = y_0$$

has a unique solution $(x_i(\cdot), y(\cdot))$, where $x_i(\cdot), y(\cdot) \in C(0,\theta; l_2)$, for any admissible control $v = v(t),\ 0 \le t \le \theta$, of the evader E is called a strategy of the pursuer P_i. A strategy U_i is deemed to be admissible if each control that is used to form this strategy is admissible.

Definition 1.4. It is said that the differential game (1.1) with the initial position $\{y_0, x_{10}, x_{20}, \dots, x_{m0}\}$, $x_{i0} \ne y_0$, $i = \overline{1,m}$ can be completed, if there exists strategies U_i, $i = \overline{1,m}$ of pursuers such that for any admissible control $v(\cdot)$ of evader E, $x_k(t) = y(t)$ for some $t \ge 0$ and an index k.

2 A Pursuit Problem and Its Solution

We begin by considering the one-pursuer and one-evader game:

$$\begin{aligned} P &: \dot{x} = u, \quad x(0) = x_0, \\ E &: \dot{y} = v, \quad y(0) = y_0, \end{aligned} \tag{2.1}$$

where $\int_0^\infty \|u(t)\|^2 dt \le \rho^2$ and $\int_0^\infty \|v(t)\|^2 dt \le \sigma^2$.

Let T be a finite and sufficiently large constant denotes the prefix terminal time of the game. We define the pursuer's strategy as follows:
If $x_0 = y_0$, then we set

$$u(t) = v(t), \quad 0 \le t \le \theta;$$

if $x_0 \ne y_0$, then we choose

$$u(t) = \frac{y_0 - x_0}{\theta} + v(t),$$

where $\theta = \left(\dfrac{\| y_0 - x_0 \|}{\rho - \sigma} \right)^2 \le T$.

Lemma 2.1. *If $\rho > \sigma$ then for any initial position $\{x_0, y_0\}$, pursuer's strategy (2.2)–(2.2) ensures that $x(\theta) = y(\theta)$ in the game (2.1).*

Proof. If $x_0 = y_0$, then it readily follows from (2.2) that $x(t) = y(t)$, $0 \le t \le \theta$, in particular $x(\theta) = y(\theta)$. Now assume that $x_0 \ne y_0$. By (1.4), (1.5) and (2.2), we have

$$\begin{aligned} x(\theta) &= x_0 + \int_0^\theta u(t)dt \\ &= x_0 + \int_0^\theta \left(\frac{y_0 - x_0}{\theta} + v(t)\right)dt \\ &= x_0 + \frac{y_0 - x_0}{\theta}\theta + \int_0^\theta v(t)dt \\ &= y_0 + \int_0^\theta v(t)dt \\ &= y(\theta). \end{aligned}$$

Furthermore for any admissible $v(t)$, one can verify the admissiblity of the pursuer's strategy:

$$\begin{aligned}\left(\int_0^\theta \|u(t)\|^2 dt\right)^{1/2} &= \left(\int_0^\theta \left(\frac{y_0 - x_0}{\theta} + v(t)\right)^2 dt\right)^{1/2} \\ &\leq \left(\int_0^\theta \left(\frac{y_0 - x_0}{\theta}\right)^2 dt\right)^{1/2} + \left(\int_0^\theta v(t)^2 dt\right)^{1/2} \\ &\leq \frac{\| y_0 - x_0 \|}{\theta^{1/2}} + \sigma \\ &= (\rho - \sigma) + \sigma = \rho\end{aligned}$$

■

3 Main Result

Let us consider the game (1.1) with admissible controls of the pursuers and evader as (1.2) and (1.3), respectively such that

$$\rho_i < \sigma, \quad \text{for all } \ i = 1, \dots, m. \tag{3.1}$$

In other words, we are considering a game where the evader E has advantage in resources over each pursuer of the game. However, we prove that the completion of the game is still possible.

Theorem 3.1. *Consider the game (1.1) with admissible controls of the pursuers and evader as (1.2) and (1.3), respectively such that (3.1) holds. If*

$$\sum_{i=1}^{m} \rho_i > \sigma, \tag{3.2}$$

then for any initial position $\{y_0, x_{10}, x_{20}, \dots, x_{m0}\}$*, the game (1.1) will be completed.*

Proof. Without loss of generality, we assume that $y_0 \neq x_{i0}$, for all $i = \overline{1, m}$. Let $\rho = \sum_{i=1}^{m} \rho_i$ and let $\sigma_i = \frac{\sigma}{\rho}\rho_i$, for $i = \overline{1, m}$. It is clearly that $\sigma_i < \rho_i$ and $\sum_{i=1}^{m} \sigma_i = \sigma$.

Denotes T the finite terminal time of the game.

We shall construct the strategies as follows.

Define $\Phi_m = \sum_{i=1}^{m} \theta_i$. First we devide the game into m phases.

At the first phase, we define the pursuers' strategy as follows:

$$u_1(t) = \frac{y_0 - x_{10}}{\theta_1} + v(t), \quad 0 \leq t \leq \theta_1; \quad \text{and } u_i(t) = 0, \quad \forall i = \overline{2, m}, \tag{3.3}$$

where $\theta_1 = \left(\dfrac{\| y_0 - x_{10} \|}{\rho_1 - \sigma_1} \right)^2$. If $\int_0^{\theta_1} \| u_1(t) \|^2 \, dt \le \rho_1^2$ and $\int_0^{\theta_1} \| v(t) \|^2 \, dt \le \sigma_1^2$, then the strategy defined is admissible, since

$$\begin{aligned}
\left(\int_0^{\theta_1} \| u_1(t) \|^2 \, dt \right)^{1/2} &= \left(\int_0^{\theta_1} \| \frac{y_0 - x_{10}}{\theta_1} + v(t) \|^2 \, dt \right)^{1/2} \\
&\le \left(\int_0^{\theta_1} \| \frac{y_0 - x_{10}}{\theta_1} \|^2 \, dt \right)^{1/2} + \left(\int_0^{\theta_1} \| v(t) \|^2 \, dt \right)^{1/2} \\
&\le \rho_1 - \sigma_1 + \sigma_1 = \rho_1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
x_1(\theta_1) &= x_{10} + \int_0^{\theta_1} u(t) dt \\
&= x_{10} + \int_0^{\theta_1} \left(\frac{y_0 - x_{10}}{\theta_1} + v(t) \right) dt \\
&= x_{10} + y_0 - x_{10} + \int_0^{\theta} v(t) dt = y(\theta_1)
\end{aligned}$$

and hence, completes the proof. Therefore, if $x_1(\tau_1) \neq y(\tau_1)$ for all $\tau_1 \in [0, \theta_1]$, then we will have either $\int_0^{\theta_1} \| v(t) \|^2 \, dt > \rho_1$ or $> \sigma_1$. Since $\rho_1 > \sigma_1$, the inequality $\int_0^{\theta_1} \| v(t) \|^2 \, dt > \sigma_1$ holds if no contact between the first pursuer and the evader during the time interval $[0, \theta_1]$. In such case, the game will continue to next phase.

In a general phase, says ith phase, for any $i = \overline{2, m}$, we construct the pursuers' strategy as follow:

$$\begin{aligned}
u_i(t) &= \frac{y_0 - x_{i0}}{\theta_i} + v(t), \quad \Phi_{i-1} \le t \le \Phi_i; \\
u_k(t) &= 0, \quad \forall k = \overline{1, m} \setminus \{i\}, \qquad (3.4)
\end{aligned}$$

where $\theta_i = \left(\dfrac{\| y_0 - x_{i0} \|}{\rho_i - \sigma_i} \right)^2$. Again, if $\int_{\Phi_{i-1}}^{\Phi_i} \| v(t) \|^2 \, dt \le \sigma_i^2$, then the strategy defined is admissible and it guarantees the completion of the game. Thus if the contact between the $i-$pursuer and the evader is not occured, we must have that $\int_{\Phi_{i-1}}^{\Phi_i} \| v(t) \|^2 \, dt > \sigma_i^2$.

Assume that in contrary the theorem is false, in which there is no contact occured between the evader with the pursuers until the terminal time. Since $x_i(\tau) \neq y(\tau)$ for all $i = \overline{1, m}$ and $\tau \in [0, \Phi_m]$, we must have

$$\begin{aligned}
&\int_0^{\theta_1} \| v(t) \|^2 \, dt > \sigma_1^2 \\
&\int_{\theta_1}^{\Phi_2} \| v(t) \|^2 \, dt > \sigma_2^2 \\
&\qquad \vdots \\
&\int_{\Phi_{m-1}}^{\Phi_m} \| v(t) \|^2 \, dt > \sigma_m^2
\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^\infty \| v(t) \|^2 \, dt &\geq \int_0^{\theta_1} \| v(t) \|^2 \, dt + \int_{\theta_1}^{\Phi_2} \| v(t) \|^2 \, dt \\ &+ \ldots \int_{\Phi_{m-1}}^{\Phi_m} \| v(t) \|^2 \, dt \\ &> \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_m^2 = \sigma^2,\end{aligned}$$

which is a contradiction with (1.3). Obviously there must exist an i such that $\int_{\Phi_{i-1}}^{\Phi_i} \| v(t) \|^2 dt \leq \sigma_i^2$. At that particular time interval, contact is ensured and thus end the proof. ■

4 Conclusion

We show that under the integral constraints, an evading player, cannot avoid an exact contact with any finite number of pursuing players whose individual resources are less than of this evading player. In the case where the pursuers are of countably many, an anologue proof should be possible.

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