Communications in Mathematical Analysis ISSN 0973-3841
Volume 1, Number 2, pp. 91–99, 2006

Combinations of Distinguished Subsets and Cesàro Conullity

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(Communicated by Hüseyin Bor)

Abstract

In this paper we give some characterizations in connection (weak) Cesàro wedge and (strongly) Cesàro conull for Montel and reflexive FK-spaces and we show that subspaces σS and σW are closely related to (strong) Cesàro conullity. We also study the combinations of distinguished subsets.

AMS Subject Classification: Primary 46A45, 40G05; Secondary 40H05.

Keywords: FK-Spaces, Conull Spaces, Cesàro Conull Spaces, Distinguished Subpaces.

1 Introduction

Let w denote the space of all real or complex-valued sequence. An FK-space is a locally convex vector subspace of w which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK-space a normed FK-space. The basic properties of such spaces can be found in [9] , [10] and [11] .

By m, c_0 we denote the spaces of all bounded sequences,null sequences, respectively. These are FK-spaces under $\|x\| = \sup_n |x_n|$. By l^p, $(1 \le p < \infty)$ and cs we shall denote the space of all absolutely p-summable sequences and convergent series, respectively. The sequences spaces

$$h = \left\{ x \in w : \lim_j x_j = 0, \text{ and } \sum_{j=1}^{\infty} j \left| \Delta x_j \right| < \infty \right\},$$

$$q = \left\{ x \in w : \sup_j \left| x_j \right| < \infty \text{ and } \sum_{j=1}^{\infty} j \left| \Delta^2 x_j \right| < \infty \right\},$$

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$$\sigma_0 = \left\{ x \in w : \lim_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| = 0 \right\},$$

$$\sigma_c = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_k \text{ exists} \right\},$$

and

$$\sigma_\infty = \left\{ x \in w : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\}$$

are FK-spaces with the norms

$$\|x\|_h = \sum_{j=1}^{\infty} j \left| \Delta x_j \right| + \sup_j \left| x_j \right|,$$

$$\|x\| q = \sum_{j=1}^{\infty} j \left| \Delta^2 x_j \right| + \sup_j \left| x_j \right|,$$

$$\|x\|_{\sigma_\infty} = \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|$$

respectively, where $\Delta x_j = x_j - x_{j+1}$, $\Delta^2 x_j = \Delta x_j - \Delta x_{j+1}$, [1], [2] .

Throughout the paper e denotes the sequence of ones, $(1, 1, \ldots, 1, \ldots)$; δ^j , $(j = 1, 2, \ldots)$, the sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with the one in the $j-$th position. Let $\phi := l.hull \left\{ \delta^k : k \in N \right\}$ and $\phi_1 = \phi \cup \{e\}$. The topological dual of X is denoted by X'. The space is said to have AD if ϕ is dense in X and an FK-space X is said to have AK (respectively σK), if $X \supset \phi$ and for each $x \in X$,$x^{(n)} \to x$ (respectively $\frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x$) in X, where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k = (x_1, x_2, \ldots, x_n, 0, \ldots)$. Every AK-space is a σK-space, [1] .

Let X be an FK-space containing ϕ. Then

$$X^f = \left\{ \left\{ f\left(\delta^k\right) \right\} : f \in X' \right\},$$

$$X^\beta = \left\{ x : \sum_{k=1}^{\infty} x_k y_k \text{ exists for every } y \in X \right\}.$$

In addition, let X be an FK-space and $z \in w$ then $z^{-1}.X = \{x \in w : z.x \in X\}$ is an FK-space, where $z.x = (z_k x_k)$, [9] .

An FK-space containing ϕ is called Cesàro wedge space if $\frac{1}{n} \sum_{k=1}^{n} e^{(k)} \to 0$ in X, and weak Cesàro wedge space if $\frac{1}{n} \sum_{k=1}^{n} e^{(k)} \to 0$ (weakly) in X,where $e^{(k)} = \sum_{j=1}^{k} \delta^j$, [6] .

Also, an FK-space containing ϕ_1 is called strongly Cesàro conull space if $\frac{1}{n} \sum_{k=1}^{n} e^{(k)} \to e$ in X, and Cesàro conull space if $\frac{1}{n} \sum_{k=1}^{n} e^{(k)} \to e$ (weakly) in X, [5] .

We recall some important subspaces of an FK-space introduced by Goes in [3].

Let X be an FK-space containing ϕ. Then

$$\begin{aligned}\sigma W &= \sigma W(X) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x \text{ (weakly) in } X\right\} \\ &= \left\{x : f(x) = \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f(\delta^j), \text{ for all } f \in X'\right\},\end{aligned}$$

$$\begin{aligned}\sigma S &= \sigma S(X) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x \text{ in } X\right\} \\ &= \left\{x : x = \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j \delta^j\right\},\end{aligned}$$

$$\sigma B^{+} = \sigma B^{+}(X) = \left\{x : \left\{\frac{1}{n}\sum_{k=1}^{n} x^{(k)}\right\} \text{ is bounded in } X\right\},$$

$$\sigma F^{+} = \sigma F^{+}(X) = \left\{x : \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f\left(\delta^j\right) \text{ exists for all } f \in X'\right\}.$$

Also $\sigma F = \sigma F^{+} \cap X$, $\sigma B = \sigma B^{+} \cap X$. An FK-space is a σK-space (respectively $S\sigma W$-space, σF-space, σB-space) if $X = \sigma S$ (respectively $X = \sigma W,\ X = \sigma F,\ X = \sigma B$) [1].

It is well known that for an FK-space X

$$\phi \subset \sigma S \subset \sigma W \subset \sigma F \subset \sigma B \subset X.$$

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequences $x = (x_k)$ by the formula $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$, $(i = 1, 2, 3, \ldots)$. The sequence $\left\{a_{ij}\right\}_{j=1}^{\infty}$ is called i-th row of A and is denoted by r^i, $(i = 1, 2, 3, \ldots)$; similary, the j-th column of the matrix A, $\left\{a_{ij}\right\}_{i=1}^{\infty}$ is denoted by k^j, $(j = 1, 2, 3, \ldots)$.

For an FK-space (E, u) we consider the summability domain $E_A := \{x \in w : Ax \in E\}$. Then E_A is an FK- space under the seminorms $p_i = |x_i|$, $(i = 1, 2, \ldots)$, $h_i(x) = \sup_m \left|\sum_{j=1}^{m} a_{ij}x_j\right|$, $(i = 1, 2, \ldots)$ and $(uoA)(x) = u(Ax)$, [9].

2 Distinguished subspace (σS) σW and (Strong) Cesàro Conullity

We show that subspaces σW and σS are closely related to Cesàro conullity and strong Cesàro conullity of the FK-space X.

Theorem 2.1. *Let X be an FK-space containing ϕ and z be a sequence. Then $z \in \sigma W$ if and only if $z^{-1}.X$ is Cesàro conull space.*

Proof. Let $f \in (z^{-1}.X)'$. Then $f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(z.x)$, with $\alpha \in \phi, g \in X'$, ([9] 4.4.10). Thus $f(e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)}) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=k+1}^{\infty} \alpha_j + g(z - \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} z_j\delta^j)$. Since $\alpha \in \phi$, we have $\frac{1}{n}\sum_{k=1}^{n}\sum_{j=k+1}^{\infty} \alpha_j \to 0$. Hence $f(e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)}) \to 0$ for each $f \in (z^{-1}.X)'$ if and only if $g(z - \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} z_j\delta^j) \to 0$. The result now follows immediately. ■

Theorem 2.2. *Let X be an FK-space containing ϕ. Then $e \in \sigma W$ if and only if X is Cesàro conull.*

Proof. Taking $z = e$ in Theorem 2.1., we get the proof. ■

Theorem 2.3. *Let (X, q) be an FK-space containing ϕ and let z be a sequence. Then $z \in \sigma S$ if and only if z^{-1}. X is strongly Cesàro conull space.*

Proof. $z^{-1}.X$ is an FK-space with $p \cup h$ where $p_i(x) = |x_i|$ and $h(x) = q(z.x)$ ([9], 4.3.6). Since $p_i(e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)}) = \frac{i}{n} \to 0$, $(i < n)$ and $h(e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)}) = q(z - \frac{1}{n}\sum_{k=1}^{n} z^{(k)})$, then $z^{-1}.X$ is strongly Cesàro conull if and only if $z \in \sigma S$. ■

Theorem 2.4. *Let X be an FK-space containing ϕ. Then $e \in \sigma S$ if and only if X is strongly Cesàro conull.*

Proof. Taking $z = e$ in Theorem 2.3, we get the result. ■

The following theorems collect some applications to summability domain E_A.

Theorem 2.5. *Let E be an FK-space and A is a matrix such that $E_A \supset \phi_1$. Then E_A is Cesàro conull space if and only if $g(Ae) = \lim_r \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} g(k^j)$ for each $g \in E'$.*

Proof. Necessity: Let $f := g \circ A$ for $g \in E'$. So $f \in E_A'$ by ([9] , 4.4.2). Then $f(e - \frac{1}{r}\sum_{p=1}^{r} e^{(p)}) = (g \circ A)(e - \frac{1}{r}\sum_{p=1}^{r} e^{(p)}) = g(Ae - \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} k^j) = g(Ae) = \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} g(k^j)$. Hence $g(Ae) = \lim_r \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} g(k^j)$ for each $g \in E'$ by hyphothesis.

Sufficiency: Let $f \in E_A'$. Then $f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(Ax)$,for each $\alpha \in w_A^{\beta}, g \in E'$ by ([9] , 4.4.6). So we get $f(e - \frac{1}{r}\sum_{p=1}^{r} e^{(p)}) = \frac{1}{r}\sum_{p=1}^{r}\sum_{j=p+1}^{\infty} \alpha_j + g(Ae - \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} k^j)$. Since $\alpha \in w_A^{\beta} \subset E_A^{\beta} \subset \{e\}^{\beta} = cs$ and by hyphothesis then E_A is Cesàro conull. ■

Theorem 2.6. *Let $z \in w$, E be an FK-space and A be a matrix such that $E_A \supset \phi$ i.e. the columns of A belong to E. Then the following conditions are equivalent:*

1. $z \in \sigma W$,

2. $A.z - \frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \to 0$ *weakly in* E *i.e.* $g\,(\,A.z - \frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\,) \to 0$ *for each* $g \in E'$,

3. $E_{A.z}$ *is Cesàro conull space,*

4. $g(A.z) = \lim_r \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} z_j g(k^j)$ *for each* $g \in E'$,

where $A.z = (a_{nk}z_k)$.

Proof. Since $z^{-1}.E_A = E_{A.z}$ by Theorem 2.1 we have (1) $\Leftrightarrow$ (3). To prove (2) $\Leftrightarrow$ (3); let $B := A.z$. Then E_B is Cesàro conull space if and only if $g(Be) = \lim_r \frac{1}{r}\sum_{p=1}^{r} g(Be^{(p)})$ by Theorem 2.5. Since $Be = (\sum_{j=1}^{\infty} b_{ij}) = (\sum_{j=1}^{\infty} a_{ij}z_j) = A.z$, $(Be^p)_i = (\sum_{j=1}^{p} b_{ij})_i = (\sum_{j=1}^{p} a_{ij}z_j)_i$ and $Be^p = \sum_{j=1}^{p} k^j z_j = A.z^{(p)}$ then the proof is clear. By $Az^{(p)} = \sum_{j=1}^{p} k^j z_j$, one can have (2) $\Leftrightarrow$ (4). ■

Theorem 2.7. *Let* $z \in w$, E *be an* FK*-space and* A *be a matrix such that* $E_A \supset \phi$ *i.e. the columns of* A *belong to* E. *Then the following conditions are equivalent:*

1. $z \in \sigma S$,

2. $Az - \frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \to 0$ *in* E ,

3. $E_{A.z}$ *is strongly Cesàro conull space,*

4. $A.z = \lim_r \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} z_j k^j$.

Proof. Since $z^{-1}.E_A = E_{A.z}$ by Theorem 2.3 we get $(\,i) \Leftrightarrow (iii)$. To prove $(ii) \Leftrightarrow (iv)$, it suffices to observe $Az^{(p)} = \sum_{j=1}^{p} k^j z_j$. $(i) \Longrightarrow (ii)$; Since $z = \lim_r \frac{1}{r}\sum_{p=1}^{r} z^{(p)}$ and $A : E_A \to E$ is continuous by [9] , then $A.z = \lim_r \frac{1}{r}\sum_{p=1}^{r} Az^{(p)}$. $(\,ii) \Longrightarrow (i)$; Since $(w_A, p \cup h)$ is an σK-space with $p_n(x) = |x_n|$, $h_n(x) = \sup_m \left|\sum_{k=1}^{m} a_{nk}x_k\right|$ by ([9] , 4.3.8) then $p_n(z - \frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \to 0$, $h_n(z - \frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \to 0$. Hence $z \in \sigma S$ if $q(A(\,z - \frac{1}{r}\sum_{p=1}^{r} z^{(p)})) \to 0$, where q is typical seminorm of E by ([9] , 4.3.12). But this is $Az - \frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \to 0$ in E. ■

Before giving the following theorems we recall that a Fréchet space E is called (reflexive) Montel whenever the bounded subsets of E are (weakly) compact.

Theorem 2.8. *Let* E *be Montel* FK*-space. Then the following conditions are equivalent:*

1. E_A *is a weak Cesàro wedge (respectively Cesàro conull) space,*

2. *E_A is a Cesàro wedge (respectively strongly Cesàro conull) space,*

3. *$h \subset E_A$ and $r^i \in \sigma_0$, $i = 1, 2, 3, \ldots$(respectively $q \subset E_A$).*

Proof. $(i) \Longrightarrow (ii)$; Since E_A is a weak Cesàro wedge (respectively Cesàro conull) space then $h \subset E_A$, $r^i \in \sigma_0$, $i = 1, 2, 3, \ldots$ (respectively $q \subset E_A$) and $A : h \to E$(respectively $A : q \to E$) is weakly compact by ([6] , Theorem 5.3),(respectively [5] , Theorem 6.2). Let B be bounded subset in h (respectively in q). Then $A(B)$ is weakly relatively compact and then weakly bounded in E. Since E is Montel space and $A(B)$ is bounded in E then $A(B)$ is relatively compact in E. Thus $A : h \to E$ (respectively $A : q \to E$) is compact and so E_A is a Cesàro wedge (respectively strongly Cesàro conull) space by ([6], Theorem 5.1), (respectively [5] , Theorem 6.1).

$(ii) \Longrightarrow (iii)$; By ([6] ,Theorem 5.1), ([5], Theorem 6.1).

$(iii) \Longrightarrow (i)$; Let B be bounded subset in h (respectively q). Since the inclucion mapping $I : h \to E_A$ (respectively $I : q \to E_A$) is continuous by hypothesis then $I(B)$ is bounded in E_A. On the other hand, because of $A : E_A \to E$ is continuous, $A(B)$ is bounded in E. Since E is Montel and every Montel space is reflexive then $A(B)$ is relatively compact in E [7] and then $A : h \to E$ (respectively $A : q \to E$) is weakly compact. Hence by hypothesis and ([6] , Theorem 5.3), (respectively [5] , Theorem 6.2) the proof is obtained. ■

In particular Theorem 2.8 holds when $E = w$.

Theorem 2.9. *A Montel FK-space is a Cesàro wedge space if and only if it contains h; and it is a strongly Cesàro conull space if and only if it contains q.*

Proof. Taking $A = I$ in Theorem 2.8 we get the proof. ■

Theorem 2.10. *Let E be reflexive FK-space. Then the following conditions are equivalent:*

1. *E_A is a weak Cesàro wedge (respectively Cesàro conull) space,*

2. *$h \subset E_A$ and $r^i \in \sigma_0$, $i = 1, 2, 3, \ldots$(respectively $q \subset E_A$).*

Proof. This is just Theorem 2.8. ■

In particular Theorem 2.10 holds when $E = l_p$, $(p > 1)$.

Theorem 2.11. *A reflexive FK-space is a weak Cesàro wedge space if and only if it contains h; and it is a Cesàro conull space if and only if it contains q.*

Proof. Taking $A = I$ in Theorem 2.10 we get the proof. ■

3 Combinations of Distinguished Subsets and Cesàro Conullity

Let X and Y be FK-spaces, X with paranorm p and Y with paranorm q. Then $Z = X + Y$ with the unrestricted inductive limit topology is an FK-space. The paranorm r of Z is given by $r(z) = \inf\{p(x) + q(y) : z = x + y, x \in X, y \in Y\}$, ([8], Section 13.4).

Theorem 3.1. *Let X and Y be FK-spaces, $Z = X + Y$. Then $G(Z) \supseteq G(X) + G(Y)$ for $G = \sigma S, \sigma W, \sigma F$ or σB.*

Proof. First $G = \sigma S$. Let $x \in \sigma S(X)$, $y \in \sigma S(Y)$. Then $p(x - \frac{1}{r}\sum_{p=1}^{r} x^{(p)}) \to 0$, $q(y - \frac{1}{r}\sum_{p=1}^{r} y^{(p)}) \to 0$. Hence $r\left(z - \frac{1}{r}\sum_{p=1}^{r} z^{(p)}\right) = r\left((x+y) - \frac{1}{r}\sum_{p=1}^{r}(x^{(p)} + y^{(p)})\right)$

$$= \inf\left\{p\left(x - \frac{1}{r}\sum_{p=1}^{r} x^{(p)}\right) + q\left(y - \frac{1}{r}\sum_{p=1}^{r} y^{(p)}\right) : z = x + y, x \in X, y \in Y\right\}$$

$$\leq p(x - \frac{1}{r}\sum_{p=1}^{r} x^{(p)}) + q(y - \frac{1}{r}\sum_{p=1}^{r} y^{(p)}) \to 0,$$

and so $z = x + y \in \sigma S$.

$G = \sigma W$. Let $x \in \sigma W(X)$, $y \in \sigma W(Y)$ and $f \in Z'$. Then $f\ |_X \in X'$, $f\ |_Y \in Y'$. Hence $f(z) = f(x) + f(y) = \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} x_j f(\delta^j) + \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} y_j f(\delta^j) = \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p}(x_j + y_j) f(\delta^j) = \frac{1}{r}\sum_{p=1}^{r}\sum_{j=1}^{p} z_j f(\delta^j)$.

The proofs for $G = \sigma F$ or $G = \sigma B$ are similar.

Let $\{X^n\}_{n=1}^{\infty}$ be a sequence of FK-spaces. p_n the paranorm of X^n and $\{q_{nk}\}_{k=1}^{\infty}$ be the seminorms of X^n. Let $Y = \bigcap_{n=1}^{\infty} X^n$. It is well known that Y is an FK-space with paranorm $q = \sum_{n=1}^{\infty} \frac{p_n}{2^n(1+p_n)}$ and seminorms $\{q_{nk}\}_{k=1}^{\infty}$, ([10] , 11.3.3). ■

Theorem 3.2. *Let $\{X^n\}_{n=1}^{\infty}$ be a sequence of FK-spaces and $Y = \bigcap_{n=1}^{\infty} X^n$. Then $G(Y) = \bigcap_{n=1}^{\infty} G(X^n)$ for $G = \sigma S, \sigma W, \sigma F$ or σB.*

Proof. For each n, $G(Y) \subseteq G(X^n)$ by [4] , hence $G(Y) \subseteq \bigcap_{n=1}^{\infty} G(X^n)$ for $G = \sigma S, \sigma W, \sigma F$ or σB. To prove the reverse containment we need to consider each case separately.

Let $z \in \bigcap_{n=1}^{\infty} \sigma S(X^n)$. Then $q_{nk}(z - \frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \to 0$ for each fixed n and k, but this are the seminorms for Y. Hence $\frac{1}{r}\sum_{p=1}^{r} z^{(p)} - z \to 0$,which implies $z \in \sigma S(Y)$.

Let $z \in \bigcap_{n=1}^{\infty} \sigma W(X^n)$ and $f \in Y'$. By ([8] , 7.2.16), $f = \sum_{i=1}^{l} f_i$, where $f_i \in (X^i)'$. Since $f_i(\frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \to f(z)$ for $i = 1, 2, 3, \ldots, l$, $f_i(\frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \to f(z)$. Thus $z \in \sigma W(Y)$.

The proof for $E = \sigma F$ is similar.

Let $z \in \bigcap_{n=1}^{\infty} \sigma B(X^n)$. Then for any fixed l and m, $q_{lm}(\frac{1}{r}\sum_{p=1}^{r} z^{(p)}) \leq K_{lm}$ for all n. Hence $z \in \sigma B(Y)$. ■

Definition 3.3. An FK-space X containing ϕ_1 is said to belong to the class O_σ or to be an O_σ-space if $\sigma W \supset X \cap \sigma_\infty$.

Theorem 3.4. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of O_σ-spaces. Then $Y = \bigcap_{n=1}^{\infty} X_n$ is an O_σ-space.*

Proof. Let $x \in Y \cap \sigma_\infty$. Then $x \in \sigma_\infty$ and $x \in X_n$ for all $n = 1, 2, 3, \ldots$. Since X_n is an O_σ-space , $x \in \sigma W(X_n)$. Hence $x \in \bigcap_n \sigma W(X_n) = \sigma W(\bigcap_n X_n)$ by Theorem 3.2. ■

We get the following theorem by the definition O_σ-space.

Theorem 3.5. *An O_σ-space is Cesàro conull.*

Theorem 3.6. *Let $X \subset Y$ with X closed in Y. Then X is O_σ-space if Y is.*

Proof. X and Y have the same topology. ■

Theorem 3.7. *Let $X \supset \phi_1$ be a weakly sequential complete FK-space. Then X is an O_σ-space.*

Proof. Let $f \in X'$. Since $f\,|_{\sigma_0} \in \sigma_0'$ and $\sigma_0' = h$, [2] , then $\left\{ f\left(\delta^j\right) \right\} \in h$. Let $x \in \sigma_\infty \cap X$. Since $\sum_{j=1}^{k} x_j f(\delta^j) = \sum_{j=1}^{k-1} \frac{s_j}{j} j f(\delta^j) + \frac{s_k}{k} k f(\delta^k)$ by Abel identity and $f\left(\delta^j\right) \in h$ then $f(\frac{1}{n}\sum_{k=1}^{n} e^{(k)}) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f\left(\delta^j\right)$ is convergent. Since X is an FK-space and weakly complete then $\frac{1}{n}\sum_{k=1}^{n} e^{(k)}$ has weak limit $x \in \sigma_\infty \cap X$ i.e $x \in \sigma W(X)$. ■

Theorem 3.8. *Every weakly sequential complete FK-space is Cesáro conull.*

Remark. Let $\sigma_\infty \subset X$. Then X need not be O_σ-space.

Since σ_c is closed in σ_∞ and σ_c is not Cesàro conull then σ_∞ is not Cesàro conull, [5] ;and it is not O_σ-space either by Theorem 3.5.

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