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A NOTE ON LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA FUNCTIONS

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Abstract

The function $\frac{(x/e)^x}{\Gamma(x+1/2)}$ strictly logarithmically completely monotonic on $(0,\infty)$ is proved and an alternative proof of the function $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ strictly logarithmically completely monotonic on $(0,\infty)$ is given.

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1 Introduction

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0,1,2,\ldots). \tag{1.1}$$

If the inequality (1.1) is strict, then f is said to be strictly completely monotonic on I. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [3], probability theory [5, 8, 10], physics [7], numerical and asymptotic analysis [9, 16], and combinatorics [2]. A detailed collection of the most important properties of completely monotonic functions can be found in [15, Chapter IV], and in an abstract in [4].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \quad (x \in I; n = 1,2,\ldots). \tag{1.2}$$

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If the inequality (1.2) is strict, then f is said to be strictly logarithmically completely monotonic. This definition was introduced in [11]. Moreover, the authors showed that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

Euler's gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (Rez > 0) \tag{1.3}$$

is one of the most important functions in analysis and its applications. The logarithmic derivative of the gamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ is known in literature as psi function or digamma function.

From asymptotic formula [1, p. 257]

$$\Gamma(az+b) \sim \sqrt{2\pi}e^{-az}(az)^{az+b-1/2} \quad (|argz| < \pi, a > 0), \tag{1.4}$$

we conclude that

$$\lim_{x\to\infty} \frac{(x/e)^x}{\Gamma(x+1/2)} = \frac{1}{\sqrt{2\pi}}. \tag{1.5}$$

This result leads to the following question: is the function

$$x \longmapsto \frac{(x/e)^x}{\Gamma(x+1/2)} \tag{1.6}$$

logarithmically completely monotonic on $(0,\infty)$? Theorem 1 in Section 2 answers this question.

In [6], the authors proved that for $x \in (0,1)$,

$$\frac{x}{[\Gamma(x+1)]^{1/x}} < \left(1+\frac{1}{x}\right)^x < \frac{x+1}{[\Gamma(x+1)]^{1/x}}. \tag{1.7}$$

For $x \geq 1$,

$$\left(1+\frac{1}{x}\right)^x \geq \frac{x+1}{[\Gamma(x+1)]^{1/x}}, \tag{1.8}$$

and equality occurs for $x = 1$.

It is easy to see that

$$\lim_{x\to\infty} \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x = 1. \tag{1.9}$$

In [12], it has been shown that the function $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ is strictly logarithmically completely monotonic on $(0,\infty)$, but this proof is quite intricate. Theorem 2 in Section 2 provides a simple alternative proof.

2 Main Results

Theorem 1. *The function $f(x) = \frac{(x/e)^x}{\Gamma(x+1/2)}$ is strictly logarithmically completely monotonic on $(0,\infty)$.*

Proof. Using the representations [13, p. 153]

$$\psi(x) = -\frac{1}{2x} + \ln x - \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} - \frac{1}{2}\right) e^{-xt} dt \quad (x>0), \tag{2.1}$$

$$\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t} dt \quad (x>0), \tag{2.2}$$

$$\frac{1}{x+s} = \int_0^\infty e^{-(x+s)t} dt \quad (x>0; s>0), \tag{2.3}$$

we imply

$$\begin{aligned}(\ln f(x))' &= \ln x - \psi(x+\frac{1}{2}) \\ &= \int_0^\infty \left(-\frac{e^{t/2}}{t} + \frac{1}{e^t-1} + 1\right) e^{-(x+1/2)t} dt \\ &= -\int_0^\infty \frac{e^t-1-te^{t/2}}{t(e^t-1)} e^{-(x+1/2)t} dt\end{aligned} \tag{2.4}$$

and therefore

$$(-1)^n (\ln f(x))^{(n)} = \int_0^\infty \frac{e^t-1-te^{t/2}}{e^t-1} t^{n-2} e^{-(x+1/2)t} dt > 0 \tag{2.5}$$

for $x>0$ and $n=1,2,\ldots$. At the last step, by applying the following result

$$e^t - 1 - te^{t/2} = \sum_{n=3}^\infty \frac{2^{n-1}-n}{2^{n-1}\cdot n!} t^n > 0. \tag{2.6}$$

The proof of Theorem 1 is complete. □

Theorem 2. *The function* $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ *is strictly logarithmically completely monotonic on* $(0,\infty)$.

Proof. It has been shown [14] that the function $1+\frac{1}{x}\ln\Gamma(x+1) - \ln(x+1)$ is strictly completely monotonic on $(-1,\infty)$. Hence, the function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly logarithmically completely monotonic on $(-1,\infty)$.

Using the representations (2.2) and (2.3), we imply

$$\left((x+1)\ln(1+\frac{1}{x})\right)' = \ln(1+\frac{1}{x}) - \frac{1}{x} = -\int_0^\infty \frac{te^t-e^t+1}{t} e^{-(x+1)t} dt \tag{2.7}$$

and therefore

$$(-1)^n \left((x+1)\ln(1+\frac{1}{x})\right)^{(n)} = \int_0^\infty (te^t - e^t + 1) t^{n-2} e^{-(x+1)t} dt > 0 \tag{2.8}$$

for $x>0$ and $n=1,2,\ldots$. At the last step, by applying the following result

$$te^t - e^t + 1 = \sum_{n=2}^\infty \frac{n-1}{n!} t^n > 0. \tag{2.9}$$

This shows that the function $(1+1/x)^{x+1}$ is strictly logarithmically completely monotonic on $(0,\infty)$.

It is easy to see that the product of (strictly) logarithmically completely monotonic functions is also (strictly) logarithmically completely monotonic. Write $g(x)$ as

$$g(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}\left(1+\frac{1}{x}\right)^{x+1}. \tag{2.10}$$

Clearly, the function g is strictly logarithmically completely monotonic on $(0,\infty)$. The proof of Theorem 2 is complete. □

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