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COMMON FIXED POINTS AND INVARIANT APPROXIMATIONS FOR GENERALIZED $(f,g)-$NONEXPANSIVE MAPPINGS

YISHENG SONG*
College of Mathematics and Information Science,
Henan Normal University, P.R. China, 453007.

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Abstract

In this paper, we prove that there is the unique common fixed point of T,f,g if T is generalized $(f,g)-$contractive and both (T,f) and (T,g) are weakly compatible in a metric space (E,d). We also establish several common fixed point theorems for generalized $(f,g)-$nonexpansive mappings in a linear norm space E. We apply these theorems to derive some results on the existence of common points from the set of best approximations. Our results develop and complement the various known results in the existing literatures.

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1 Introduction and Preliminaries

Let K be a nonempty subset of a metric space (E,d) and T a mapping from K to E. We shall denote the closure of K by $\overline{K}$, the boundary of K by ∂K, and all positive integer by $\mathbb{N}$, and the set of fixed points of T, $\{x\in K; x=Tx\}$, by $F(T)$. When $\{x_n\}$ is a sequence in E, then $x_n\longrightarrow x$(respectively, $x_n \rightharpoonup x$)will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

A mapping $T:K\to E$ is called an $(f,g)-$*contraction* if there exists $0\le k<1$ such that $d(Tx,Ty)\le kd(fx,gy)$ for all $x,y\in K$. If $k=1$, then T is called $(f,g)-$*nonexpansive*. If $g=f$, in the above inequality, T is said to be an $f-$contraction (respectively, $f-$nonexpansive mapping). A point $x\in K$ is a coincidence point (respectively, common fixed point) of f and T if $f(x)=Tx$ (respectively, $x=f(x)=Tx$). The set of coincidence points of f and T is denoted by $C(f,T)$. The pair (f,T) is called to be

*Email: songyisheng123@yahoo.com.cn or songyisheng123@163.com

(i) *compatible* [6] if $fx_n, Tx_n \in K$ and $\lim_{n\to\infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$ for some t in K;

(ii) *weakly compatible* if $T(C(f,T)) \subset K$ and $f(C(f,T)) \subset K$ such that $fTx = Tfx$ whenever $x \in C(f,T)$. Suppose that E is compact metric space and both T and f are continuous self-mapping, then (f,T) compatible equivalent to (f,T) weakly compatible [6, Theorem2.2,Corollary 2.3].

Let K be a nonempty closed convex subset of a normed space E. A mapping $f : K \to K$ is *affine* if K is convex and $f(kx+(1-k)y) = kfx+(1-k)fy$ for all $x,y \in K$ and all $k \in [0,1]$. A subset K of a norm space E is called $q-$*starshaped* with $q \in K$ if $kx+(1-k)q \in K$ for all $x \in K$ and all $k \in [0,1]$. Let T be a mapping form a $q-$starshaped subset K of a normed space E into itself. T is called $q-$*affine* if $T(kx+(1-k)q) = kTx+(1-k)q$ for all $x \in K$ and all $k \in [0,1]$. It is easy to see that if T is $q-$affine, then $Tq = q$.

Let K be a $q-$starshaped subset of a normed space E and T, f two mappings from K to itself. Then (T, f) is called C_q-*commuting* [1] if $fTx = Tfx$ for all $x \in C_q(f,T)$, where $C_q(f,T) = \bigcup\{C(f,T_k); 0 \le k \le 1\}$ and $T_kx = (1-k)q+kTx$. Clearly, C_q-commuting maps are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general. For more detail, see [1].

During the last four decades, several invariant approximation results have been obtained by many mathematicians[1,3-8,14-16]. Recently, in particular, with the introduction of noncommuting maps to this area, Shahzad [16] and Hussain and Khan [4] further proved several invariant approximation results in more general space.

The main aims of this paper is to prove that there is the unique common fixed point of T, f, g if T is generalized $(f,g)-$contractive and both (T, f) and (T,g) are weakly compatible in a metric space (E,d). As application, we will prove the common fixed point theorems for the generalized $(f,g)-$nonexpansive weakly compatible mappings. We apply these theorems to derive some results on the existence of common fixed points from the set of best approximations. Our results, on the one hand, extend and develop the work of Hussain and Jungck [3], Al-Thagafi and Shahzad [1] and Jungck [7], on the other hand, provide generalizations and complementarities of the recent work of Jungck and Sessa [5] and Shahzad [14-16].

2 Common fixed point theorems

Let K be a nonempty subset of a metric space (E,d) and T, f, g be three mappings on K. In this section, we will study the common fixed point theorems of a generalized $(f,g)-$contraction and a generalized $(f,g)-$onexpansive mapping. Now, we introduce the concepts of the generalized $(f,g)-$contraction.

A mapping $T : K \to E$ is called a *generalized* $(f,g)-$*contraction* if there exists a constant $r \in [0,1)$ such that

$$\begin{aligned} d(Tx,Ty) \le & r\max\{(d(fx,gy)), d(Tx,fx), d(Ty,gy), \\ & \frac{1}{2}[d(fx,Ty)+d(Tx,gy)]\} \text{ for all } x,y \in K. \end{aligned} \tag{2.1}$$

It is obvious that the generalized $(f,g)-$contraction contains the $(f,g)-$contraction and the Kannan's mapping (a mapping T is a Kannan's mapping if $d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx)+d(Ty,y)$ for all $x,y \in K$)(see Refs. Reich [11, 12]) as the special cases. Furthermore, the contraction is its main subclass also (when $f = g = I$ in $(f,g)-$contraction).

Example 1. Let $E = (-\infty,+\infty)$ be endowed with the Euclidean metric $d(x,y) = |x-y|$. Assumed that $K = [0,1]$ and $f,g : K \to K$ are given by $f(x) = g(x) = \frac{1}{3}x^2$ for all $x \in [0,1]$. Let $T : K \to K$ be defined by $Tx = \frac{2}{9}x^2,\ x \in K$. Then T is a generalized $(f,g)-$contraction with a constant $r = \frac{2}{3}$ and $C(f,T) = F(g,T) = \{0\} = F(T) \cap F(f) \cap F(g)$. In fact,

$$d(Tx,Ty) = \frac{2}{9}|x^2-y^2| = \frac{2}{3}(\frac{1}{3}|x^2-y^2|) = \frac{2}{3}d(fx,gy).$$

Example 2. Assumed that E is as Example 1 and $K = [0,1]$. Let $f,g : K \to E$ be respectively given by $f(x) = x-1,\ g(x) = x^2-1$. If $T : K \to E$ is defined by $Tx = \frac{1}{2}(x^2+1)$, then T is a generalized $(f,g)-$contraction with a constant $r = \frac{1}{2}$. Indeed, if $y \leq x$, then $d(Tx,Ty) = \frac{1}{2}(x^2-y^2) \leq \frac{1}{2}(x-y^2) = \frac{1}{2}d(fx,gy)$ since $x^2 \leq x$. If $x < y$, then

$$d(Tx,Ty) = \frac{1}{2}(y^2-x^2) \leq \frac{1}{2} \leq \frac{3}{4}+\frac{1}{4}x^2-\frac{1}{2}x = \frac{1}{2}|x-1-\frac{1}{2}(x^2+1)| = \frac{1}{2}d(fx,Tx)$$

since the function $\varphi(x) = \frac{3}{4}+\frac{1}{4}x^2-\frac{1}{2}x$ is nonincreasing in $[0,1]$ and $\min\limits_{x\in[0,1]} \varphi(x) = \frac{1}{2}$. Clearly, $C(f,T) = \emptyset$, $C(g,T) = \emptyset$ and $F(T) \cap F(f) \cap F(g) = \emptyset$.

Example 3. Assumed that E,K are as Example 2 and $f,g : K \to E$ are given by $f(x) = 2x,\ x \in K$ and by $g(x) = 2x^2$ for $x \in K$, respectively. Let $Tx = \frac{1}{2}x^2,\ x \in K$. Then T is a generalized $(f,g)-$contraction with a constant $r = \frac{1}{2}$. In fact, if $y \leq x$, then $d(Tx,Ty) = \frac{1}{2}(x^2-y^2) \leq \frac{1}{2}(x-y^2) = \frac{1}{4}d(fx,gy)$; if $x < y$, then $d(Tx,Ty) = \frac{1}{2}(y^2-x^2) \leq \frac{1}{2}y^2 \leq \frac{3}{4}y^2 = \frac{1}{2}|2y^2-\frac{1}{2}y^2| = \frac{1}{2}d(gy,Ty)$. Clearly, $\overline{T(K)} = [0,\frac{1}{2}]$, $g(K) = [0,2]$ and $f(K) = [0,2]$ and $C(f,T) = C(g,T) = \{0\} = F(T) \cap F(f) \cap F(g)$.

Next, we give our main results which assures $C(f,T) = C(g,T) \neq \emptyset$ and $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Theorem 2.1 *Let K be a subset of a metric space (E,d), and $T,f,g : K \to E$ three mappings with $\overline{T(K)} \subset f(K) \bigcap g(K)$. Suppose that $\overline{T(K)}$ is complete, and T is a generalized $(f,g)-$contraction with a constant $r \in [0,1)$. Then neither $C(T,f)$ nor $C(T,g)$ is empty. Moreover, if, in addition, both (T,f) and (T,g) are weakly compatible, then $F(T) \cap F(f) \cap F(g)$ is singleton.*

Proof. Let x_0 be any fixed element in K. Since $\overline{T(K)} \subset f(K) \cap g(K)$, there is a sequence $\{x_n\}$ in K such that

$$Tx_{2n} = fx_{2n+1} \text{ and } Tx_{2n+1} = gx_{2n+2} \text{ for all } n \geq 0.$$

It follows from Eq.(2.1) that

$$\begin{aligned}d(Tx_{2n+1},Tx_{2n}) \le & r\max\{d(fx_{2n+1},gx_{2n}),d(Tx_{2n+1},fx_{2n+1}),d(Tx_{2n},gx_{2n}),\\ & \frac{1}{2}[d(fx_{2n+1},Tx_{2n})+d(Tx_{2n+1},gx_{2n})]\}\\ = & r\max\{d(Tx_{2n},Tx_{2n-1}),d(Tx_{2n+1},Tx_{2n}),d(Tx_{2n},Tx_{2n-1}),\\ & \frac{1}{2}[d(Tx_{2n},Tx_{2n})+d(Tx_{2n+1},Tx_{2n-1})]\}\\ \le & r\max\{d(Tx_{2n},Tx_{2n-1}),\frac{1}{2}[d(Tx_{2n+1},Tx_{2n})+d(Tx_{2n},Tx_{2n-1})]\}.\end{aligned}$$

And

$$\begin{aligned}d(Tx_{2n-1},Tx_{2n}) \le & r\max\{d(fx_{2n-1},gx_{2n}),d(Tx_{2n-1},fx_{2n-1}),d(Tx_{2n},gx_{2n}),\\ & \frac{1}{2}[d(fx_{2n-1},Tx_{2n})+d(Tx_{2n-1},gx_{2n})]\}\\ = & r\max\{d(Tx_{2n-2},Tx_{2n-1}),d(Tx_{2n-1},Tx_{2n-2}),d(Tx_{2n},Tx_{2n-1}),\\ & \frac{1}{2}[d(Tx_{2n-2},Tx_{2n})+d(Tx_{2n-1},Tx_{2n-1})]\}\\ \le & r\max\{d(Tx_{2n-2},Tx_{2n-1}),\frac{1}{2}[d(Tx_{2n-2},Tx_{2n-1})+d(Tx_{2n-1},Tx_{2n})]\}.\end{aligned}$$

Thus we have proved that for all $n \ge 0$,

$$d(Tx_{n+1},Tx_n) \le rd(Tx_{n-1},Tx_n) \le r^n d(Tx_1,Tx_0).$$

Hence for all $m \ge n \ge 0$, noting $r \in [0,1)$, a constant,

$$d(Tx_m,Tx_n) \le \sum_{i=n}^{m-1} d(Tx_i,Tx_{i+1}) \le \sum_{i=n}^{m-1} r^i d(Tx_1,Tx_0) \le \frac{r^n}{1-r}d(Tx_1,Tx_0).$$

Then

$$d(Tx_m,Tx_n) \to 0, \text{ as } m,n \to \infty.$$

That is, $\{Tx_n\}$ is a Cauchy sequence. Since $\overline{T(K)}$ is complete, then $\{Tx_n\}$ converges to some $z \in \overline{T(K)}$, and by the definition of $\{Tx_n\}$, we obtain that

$$\lim_{n\to\infty} gx_{2n} = z = \lim_{n\to\infty} fx_{2n+1}.$$

Hence there exists $u,v \in K$ such that $fu = z = gv$ (since $\overline{T(K)} \subset f(K)\bigcap g(K)$).

Let ε be any positive number and N a large enough natural number such that for any $n > N$,

$$d(z,gx_{2n}) < \varepsilon, d(Tx_n,z) < \varepsilon, d(fx_{2n+1},z) < \varepsilon.$$

It follows from Eq.(2.1)that

$$\begin{aligned}
d(Tu,z)-\varepsilon \leq & d(Tu,Tx_{2n}) \\
\leq & r\max\{d(fu,gx_{2n}),d(Tu,fu),d(Tx_{2n},gx_{2n}), \\
& \frac{1}{2}[d(fu,Tx_{2n})+d(Tu,gx_{2n})]\} \\
\leq & r\max\{d(z,gx_{2n}),d(Tu,z),d(Tx_{2n},z)+d(z,gx_{2n}), \\
& \frac{1}{2}[d(z,Tx_{2n})+d(Tu,z)+d(z,gx_{2n})]\} \\
\leq & r\max\{\varepsilon,d(Tu,z),2\varepsilon,\frac{1}{2}[2\varepsilon+d(Tu,z)]\}.
\end{aligned}$$

Case 1. $2r\varepsilon \geq d(Tu,z)-\varepsilon$. Then $3\varepsilon \geq d(Tu,z)$.

Case 2. $rd(Tu,z) \geq d(Tu,z)-\varepsilon$. Then $\dfrac{\varepsilon}{1-r} \geq d(Tu,z)$.

Case 3. $r(\varepsilon+\dfrac{d(Tu,z))}{2}) \geq d(Tu,z)-\varepsilon$. Then $4\varepsilon \geq d(Tu,z)$.

Since ε is a arbitrary positive number, $Tu = z$.

We have proved that $u \in C(T,f)$. Similarly, we also have $v \in C(T,g) \neq \emptyset$. (i) is proved.

Next we prove (ii). As (T,f) and (T,g) are weakly compatible and $Tu = fu = z = Tv = gv$, then

$$gz = gTv = Tgv = Tz = Tfu = fTu = fz.$$

We claim that z is a common fixed point of T, f, g. Suppose $z \neq Tz$, then

$$\begin{aligned}
d(z,Tz) = & d(Tu,Tz) \leq r\max\{(d(fu,gz)),d(Tu,fu),d(Tz,gz), \\
& \frac{1}{2}[d(fu,Tz)+d(Tu,gz)]\} \\
\leq & r\max\{(d(z,Tz)),0,0,\frac{1}{2}[d(z,Tz)+d(z,Tz)]\} \\
\leq & rd(z,Tz).
\end{aligned}$$

Which is a contradiction. Therefore $z \in F(T)\bigcap F(f)\bigcap F(g)$. If there exists another point $v \in K$ such that $v = Tv = gv = fv$, then using similar to above argumentation we get

$$\begin{aligned}
d(z,v) = d(Tz,Tv) \leq & r\max\{(d(fz,gv)),d(Tz,fz),d(Tv,gv), \\
& \frac{1}{2}[d(fz,Tv)+d(Tz,gv)]\} \\
\leq & rd(z,v).
\end{aligned}$$

Hence $z = v$. The proof is complete. □

Corollary 2.2 *Let K be a subset of a metric space (E,d), and $T,f,g : K \to K$ three mappings with $\overline{T(K)} \subset f(K)\bigcap g(K)$. Suppose that $\overline{T(K)}$ is complete, and T is a generalized $(f,g)-$contraction with a constant $r \in [0,1)$. Then neither $C(T,f)$ nor $C(T,g)$ is empty. Moreover, if, in addition, both (T,f) and (T,g) are weakly compatible, then $F(T)\cap F(f)\cap F(g)$ is singleton.*

Corollary 2.3 *Let K be a subset of a metric space (E,d), and $T,f,g:K\to K$ three mappings. Assumed that T is a (f,g)-contractive mapping with a constant $r\in(0,1)$. Suppose that $\overline{T(K)}\subset f(K)\cap g(K)$ and $\overline{T(K)}$ is complete, then $C(T,f)\neq\emptyset$ and $C(T,g)\neq\emptyset$. Moreover, if both (T,f) and (T,g) are weakly compatible, then $F(T)\cap F(f)\cap F(g)$ is singleton.*

Theorem 2.1 and Corollary 2.2 and 2.3 contains the Banach Contraction Principle as a special case($f=g=I$, an identic operator). It generalizes Hussain and Jungck [3, Theorem 2.1], Al-Thagafi and Shahzad [1, Theorem 2.1]. It also extends Shahzad [15, Lemma 2.1] and Pant [8, Theorem 1].

Let K be a nonempty $q-$starshaped subset of a normed space E. A mapping $T:K\to K$ is called to be *generalized $(f,g)-$onexpansive* if $\forall x,y\in K$,

$$\begin{aligned}\|Tx-Ty\|\leq\max\{&\|fx-gy\|,d(fx,[Tx,q]),d(gy,[Ty,q]),\\&\frac{1}{2}[d(fx,[Ty,q])+d(gy,[Tx,q])]\}.\end{aligned}\tag{2.2}$$

As an application of the above results of the generalized $(f,g)-$contraction, we obtain the following results in a normed space E.

Theorem 2.4 *Let K be a nonempty $q-$starshaped subset of a normed space E, and $T,f,g:K\to K$ be three continuous mappings and T be a generalized $(f,g)-$onexpansive mapping. Suppose that both (T,f) and (T,g) are C_q-commuting, and both f and g are $q-$affine. If $\overline{T(K)}$ is a compact subset of $f(K)\cap g(K)$, then $F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

Proof. Let $\{k_n\}$ be a strictly decreasing sequence in $(0,1)$ such that $\lim_{n\to\infty}k_n=1$. For each n, let T_n be a mapping defined by

$$T_nx=(1-k_n)q+k_nTx,\ \forall x\in K.$$

Then, for all n, $\overline{T_n(K)}\subset f(K)\cap g(K)$ by $q-$starshapedness of K and $q-$affiness of f and g. Thus for all $x,y\in K$,

$$\begin{aligned}\|T_nx-T_ny\|\leq&k_n\|Tx-Ty\|\\\leq&k_n\max\{\|fx-gy\|,d(fx,[Tx,q]),d(gy,[Ty,q]),\\&\frac{1}{2}[d(fx,[Ty,q])+d(gy,[Tx,q])]\}\\\leq&k_n\max\{\|fx-gy\|,\|fx-T_nx\|,\|gy-T_ny\|,\\&\frac{1}{2}[\|fx-T_ny\|+\|gy-T_nx\|]\},\end{aligned}$$

then T_n,f,g satisfy Eq.(2.2) with a coefficient $r=k_n\in(0,1)$. Note that (T,f) and (T,g) are C_q-commuting, and f and g are $q-$affine, then $q\in F(f)\cap F(g)$ [1]. If $T_nx=fx=gx$, we have

$$T_nfx=(1-k_n)q+k_nTfx=(1-k_n)fq+k_nfTx=f((1-k_n)q+k_nTx)=fT_nx.$$

Namely, (T_n,f) is weakly compatible. Similarly, (T_n,g) is weakly compatible also. As $\overline{T(K)}$ is compact, then $\overline{T(K)}$ is complete [9, 13]. It follows from Corollary 2.2 that for each n, there exists the unique $x_n\in K$ such that

$$x_n=fx_n=gx_n=k_nTx_n+(1-k_n)q.\tag{2.3}$$

It follows from the compactness of $\overline{T(K)}$ that there exists $\{x_{n_i}\} \subset \{x_n\}$ and $z \in K$ such that

$$Tx_{n_i} \to z \in \overline{T(K)}.$$

Thus, noticing Eq.(2.3),

$$x_{n_i} = fx_{n_i} = gx_{n_i} = k_{n_i} Tx_{n_i} + (1 - k_{n_i})q \to z (i \to \infty). \tag{2.4}$$

The continuity of T and f and g imply $Tx_{n_i} \to Tz$ and $fx_{n_i} \to fz$ and $gx_{n_i} \to gz$, respectively. Hence, noting Eq.(2.4), we get

$$z = Tz = fz = gz.$$

This finishes the proof. □

Corollary 2.5 *Let K be a nonempty $q-$starshaped subset of a normed space E, and $T : K \to K$ a nonexpansive mapping and $\overline{T(K)} \subset K$. If $\overline{T(K)}$ is compact subset of K, then $F(T) \neq \emptyset$.*

Theorem 2.4 generalizes and develops Hussain and Jungck [3, Theorem 2.2(i)], Al-Thagafi and Shahzad [1, Theorem 2.2], Jungck [7, Theorem 3.1] and Shahzad [15, Lemma 2.2].

Theorem 2.6 *Let K be a nonempty $q-$starshaped subset of a Banach space E, and $T, f, g : K \to K$ three weakly continuous mappings and and T be a generalized $(f,g)-$onexpansive mapping. Assumed that $\overline{T(K)}$ is weakly compact subset of $f(K) \cap g(K)$. If both (T, f) and (T,g) are C_q-commuting, and f,g are $q-$affine, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. Let $\{k_n\}$ be a strictly decreasing sequence in $(0,1)$ such that $\lim_{n\to\infty} k_n = 1$. By the proof of Theorem 2.4, there is a common approximate fixed sequence $\{x_n\} \in \overline{T(K)}$ of f,g,T. Since f,g,T are weakly continuous and $\overline{T(K)}$ is weakly compact, then the weak cluster z of $\{x_n\}$ is a common fixed point of f,g,T. The proof is completed. □

Theorem 2.6 extends, develops and complements Hussain and Jungck [3, Theorem 2.2(ii),2.3], Al-Thagafi and Shahzad [1, Theorem 2.4] and Shahzad [14, Theorem 3].

Corollary 2.7 *Let K be a nonempty weakly compact and $q-$starshaped subset of a Banach space E, $T, f, g : K \to K$ three weakly continuous mappings such that $\overline{T(K)} \subset f(K) \cap g(K)$. Assumed that (T, f) and (T,g) are C_q-commuting, and f and g are $q-$affine. If T is $(f,g)-$nonexpansive mapping, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

3 Invariant approximations results

In this section, several invariant approximations results, a further application of the main results in section 2, are obtained. Recall that the set $P_K(u) = \{x \in K; d(x,u) = d(u,K)\}$ is called the set of best approximations to $u \in E$ out of K, where $d(u,K) = \inf_{y\in K} d(y,u)$ [13].

Theorem 3.1 *Let K be a subset of a normed space E, $u \in E$, and $T, f, g : K \to K$ three continuous mappings. Assumed that $P_K(u)$ is nonempty $q-$starshaped, $\overline{T(P_K(u))} \subset P_K(u)$*

is compact, $f(P_K(u))\cap g(P_K(u))=P_K(u)$*,* f *and* g *are* $q-$*affine on* $P_K(u)$*. Suppose* (T,f) *and* (T,g) *are* C_q-*commuting and satisfy for all* $x\in P_K(u)\cup\{u\}$*,*

$$\|Tx-Ty\|\leq\begin{cases}\|fx-gu\| & if\, y=u,\\ \max\{\|fx-gy\|,d(fx,[Tx,q]),d(gy,[Ty,q]),\\ \frac{1}{2}[d(fx,[Ty,q])+d(gy,[Tx,q])]\} & if\, y\in P_K(u).\end{cases}\tag{3.1}$$

Then $P_K(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$*.*

Proof. Since $\overline{T(P_K(u))}\subset P_K(u)=f(P_K(u))\cap g(P_K(u))$ is compact, the results follows from Theorem 2.4 ($K=P_K(u)$). □

Theorem 3.2 *Let* K *be a subset of a Banach space* E*,* $u\in E$*, and* $T,f,g:K\to K$ *three weakly continuous mappings. Assume that* $P_K(u)$ *is nonempty* $q-$*starshaped,* $f(P_K(u))\cap g(P_K(u))=P_K(u)$*,* $\overline{T(P_K(u))}\subset P_K(u)$ *is weakly compact,* f *and* g *are* $q-$*affine on* $P_K(u)$*. Suppose* (T,f) *and* (T,g) *are* C_q-*commuting and satisfy Eq.(3.1) for all* $x\in P_K(u)\cup\{u\}$*. Then* $P_K(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$*.*

Proof. Since $\overline{T(P_K(u))}\subset P_K(u)=f(P_K(u))\cap g(P_K(u))$ is weakly compact, the results follows from Theorem 2.6 with $K=P_K(u)$. □

The following Theorem 3.3 develops, improves and complements Hussain and Jungck [3, Theorem 2.8-2.11], Al-Thagafi and Shahzad [1, Theorem 4.3, 3.3]. We further note that in our results the following two assumptions are not used:

(a) $f-T$ is demiclosed;

(b) E satisfies Opial's condition.

Theorem 3.3 *Let* K *be a nonempty subset of a Banach space* E *with* $T(\partial K)\subset K$ *and* $u\in F(T)\cap F(f)\cap F(g)$*, where* $T,f,g:K\to K$ *are three weakly continuous mappings. Assume that* $P_K(u)$ *is nonempty* $q-$*starshaped and weakly compact,* $f(P_K(u))\cap g(P_K(u))=P_K(u)$*,* f *and* g *are* $q-$*affine. Suppose that* (T,f) *and* (T,g) *are* C_q-*commuting on* $P_K(u)$ *and satisfy Eq.(3.1) for all* $x\in P_K(u)\cup\{u\}$*. Then* $P_K(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$*.*

Proof. Let $x\in P_K(u)$. Then $\|x-u\|=d(u,K)$ and for all $k\in(0,1)$,

$$\|kx+(1-k)u-u\|=k\|x-u\|<d(u,K).$$

Thus $\{kx+(1-k)u;k\in(0,1)\}\cap K=\emptyset$, and so $x\in\partial K\cap K$. Since $T(\partial K)\subset K$, it follows that $Tx\in K$. Since $fx,gx\in P_k(u)$ and T,f,g satisfy Eq.(3.1) on $P_K(u)\cup\{u\}$, we have

$$\|Tx-u\|=\|Tx-Tu\|\leq\|fx-fu\|=\|fx-u\|=d(u,K)$$

and hence $Tx\in P_K(u)$. Therefore, $T(P_K(u))\subset P_K(u)$. Since $P_K(u)$ is weakly compact, then $P_K(u)$ is closed [13, 2]. Thus $\overline{T(P_K(u))}\subset P_K(u)=f(P_K(u))\cap g(P_K(u))$. Now the result follows from Theorem 2.6 with $K=P_K(u)$. □

Theorem 3.4 *Let* K *be a nonempty subset of a normed space* E *with* $T(\partial K\cap K)\subset K$ *and* $u\in F(T)\cap F(f)\cap F(g)$*, where* $T,f,g:K\to K$ *are three continuous mappings. Assume that* $P_K(u)$ *is nonempty* $q-$*starshaped and compact,* $f(P_K(u))\cap g(P_K(u))=P_K(u)$*,* f *and* g *are* $q-$*affine on* $P_K(u)$*. Suppose that* (T,f) *and* (T,g) *are* C_q-*commuting on* $P_K(u)$ *and satisfy Eq.(3.1) for all* $x\in P_K(u)\cup\{u\}$*. Then* $P_K(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$*.*

Proof. To obtain the result, use an argument similar to that in Theorem 3.3 and apply Theorem 2.6 instead of Theorem 2.4. □

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References

[1] M.A. Al-Thagafi and Naseer Shahzad, *Noncommuting selfmaps and invariant approximations,* Nonlinear Analysis, 64(2006), 2778 - 2786.

[2] D. W. Boyd and T. S. W. Wong, *On nonlinear Contractions,* Proc. Amer. Math. Soc., **20**(1969), 458-464.

[3] N. Hussain and G. Jungck, *Common fixed point and invariant approximation results for noncommuting generalized (f, g)-nonexpansive maps,* J. Math. Anal. Appl., 321(2006), 851-861.

[4] N. Hussain and A. R. Khan, *Common fixed point results in best approximation theory,* Applied Math. Lett., 16 (2003), 575-580.

[5] G. Jungck and S. Sessa, *Fixed point theorems in best approximation theory,* Math. Japon., 42 (1995), 249-252.

[6] G. Jungck, *Common fixed points for commuting and compatible maps on compacta,* Proc. Amer. Math. Soc., 103 (1988), 977-983.

[7] G. Jungck, *Coincidence and fixed points for compatible and relatively nonexpansive maps,* Internat. J. Math. Math. Sci., 16(1) (1993), 95-100.

[8] R.P. Pant, *Common fixed points of noncommuting mappings,* J. Math. Anal. Appl., 188 (1994), 436-440.

[9] Kelly and Namioka, *Linear topological spaces,* 1963, p.61.

[10] Robert E. Megginson, *An introduction to Banach space theory* , 1998 Springer-Verlag New Tork, Inc.

[11] S. Reich, *Kannan's fixed point theorem*, Boll Un. Mat. Ital. 4 (1971) 1-11.

[12] S. Reich, *Some remark's concerning contraction mapping*, Canad. Math. Bull. 14 (1971), 121-124.

[13] S.P. Singh, B.Watson and P. Srivastava, *Fixed Point Theory and Best Approximation: The KKM-map Principle,* Kluwer Academic Publishers, Dordrecht, 1997, p.2-6,73-78.

[14] Naseer Shahzad, *On R-subcommuting maps and best approximations in Banach spaces,* Tamkang J. Math., 32 (2001) 51-53.

[15] Naseer Shahzad, *Invariant approximation and R-subweakly commuting maps,* J. Math. Anal. Appl., 257 (2001), 39-45.

[16] Naseer Shahzad, *Invariant approximations, generalized I-contractions, and R-subweakly commuting maps,* Fixed Point Theory and Applications, 2005:1 (2005), 79-86.