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Hahn-Banach Theorem in Generalized 2-Normed Spaces

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Abstract

In this paper we prove an extension Hahn-Banach theorem in the context of generalized 2-normed spaces.

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1 Introduction

In [4] Z. Lewandowska introduced a generalization of Gähler 2-normed space (see [2]) as follows.

Definition 1.1. Let X and Y be real linear spaces. Denote by D a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets $D_x = \{y \in Y; (x, y) \in D\}$ and $D^y = \{x \in X; (x, y) \in D\}$ are linear subspaces of the spaces Y and X, respectively.

A function $\|.,.\|\colon D \to [0, \infty)$ will be called a generalized 2-norm on D if it satisfies the following conditions:

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(1) $\|x, \alpha y\| = |\alpha| \cdot \|x, y\| = \|\alpha x, y\|$ for any real number α and all $(x, y) \in D$;

(2) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$ for $x \in X$, $y, z \in Y$ with $(x, y), (x, z) \in D$;

(3) $\|x+y, z\| \leq \|x, z\| + \|y, z\|$ for $x, y \in X$, $z \in Y$ with $(x, z), (y, z) \in D$.

The set D is called a 2-normed set. In particular, if $D = X \times Y$, the function $\|., .\|$ is said to be a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, \|., .\|)$ is called a generalized 2-normed space. If $X = Y$, then the generalized 2-normed space $(X \times X, \|., .\|)$ is denoted by $(X, \|., .\|)$. In the case that $X = Y$, $D = D^{-1}$, where $D^{-1} = \{(y, x) : (x, y) \in D\}$, and $\|x, y\| = \|y, x\|$ for all $(x, y) \in D$, we call $\|., .\|$ a generalized symmetric 2-norm and D a symmetric 2-normed set.

Recall that in Gähler definition of a 2-norm $\|x, y\| = 0$ if and only if x and y are linearly dependent, and this is a crucial difference between Gähler's approach and Lewandowska's one.

Example 1.2. Let X be a real linear space having two seminorms $\|.\|_1$ and $\|.\|_2$. Then $(X, \|., .\|)$ is a generalized 2-normed space with the 2-norm defined by

$$\|x, y\| = \|x\|_1 \cdot \|y\|_2;\ x, y \in X.$$

Example 1.3. Let X be a real inner product space. Then X is a symmetric generalized 2-normed space under the 2-norm $\|x, y\| = | < x, y > |$; for all $x, y \in X$.

Example 1.4. Suppose that s be the linear space of all sequences of real numbers. Put $\|x, y\| = \sum_{n=1}^{\infty} |x_n||y_n|$, where $x = \{x_n\}$, $y = \{y_n\} \in s$. Then $D = \{(x, y) \in s \times s : \|x, y\| < \infty\}$ is a symmetric 2-normed set and the function $\|., .\| : D \to [0, \infty)$ is a generalized symmetric 2-norm on D.

A. G. White [7] defined and investigated the concept of bounded 2-linear functional from $X \to X$, where X denote a 2-normed space in the sense of Gähler. In [1], a 2-normed space $(L, \|., .\|)$ was said to possess the property (p) if, for any subspace M of L with $\dim M > 1$ and any bounded bilinear functional f defined on M, there is a bounded bilinear functional F on L which is an extension of f to L and has the same norm. In that paper, X. X. Cui proved that not all Gähler 2-normed spaces possess the property (p) and provided a necessary and sufficient condition for a bounded bilinear functional f defined on a subspace M of $(L, \|., .\|)$ to be extendable to a norm preserving bounded bilinear functional on L. Later, authors of [3] introduced the notion of 2-normed complex linear spaces and established a Hahn-Banach extension theorem for complex linear 2-functionals.

In [5] Z. Lewandowska introduced the following definition of a bounded 2-linear operator (see also [6]):

Definition 1.5. Let X be a real linear space, $D \subseteq X \times X$ be a 2-normed set, Y a normed space. An operator $F : D \to Y$ is said to be 2-linear if it satisfies the following conditions:

(i) $F(a+c, b+d) = F(a, b) + F(a, d) + F(c, b) + F(c, d)$ for all $a, b, c, d \in X$ such that $a, c \in D^b \cap D^d$.

(ii) $F(\alpha a, \beta b) = \alpha \cdot \beta \cdot F(a, b)$ for all $\alpha, \beta \in \mathbb{R}$ and all $(a, b) \in D$.

Definition 1.6. Let X be a real linear space, $D \subseteq X \times X$ be a 2-normed set, Y a normed space. A 2-linear operator $F\colon D \to Y$ is said to be bounded if there is a positive number K such that $\|F(a, b)\| \le K\|a, b\|$; $(a, b) \in D$. Then the number $\|F\| = \inf\{K > 0 : \|F(a, b)\| \le K \cdot \|a, b\|; (a, b) \in D\}$ is called the norm of the 2-linear operator F.

Example 1.7. Consider $(X, \|., .\|)$ in the previous example and define $F : X \times X \to \mathbb{R}$ by $F(x, y) =< x, y >$. Then F is a bounded 2-linear operator and $\|F\| = 1$.

In this paper we aim to establish an extension Hahn-Banach theorem in the context of generalized 2-normed spaces.

2 Main Results

We are ready to proved our main result.

Theorem 2.1. *Let $(X, \|., .\|)$ be a generalized 2-normed space and M be a linear subspace of $X \times X$. If F_0 is a real bounded 2-linear functional on M, then there exists a real bounded 2-linear extension F on $X \times X$ such that $F(a, b) = F_0(a, b)$ for all $(a, b) \in M$ and $\|F\| = \|F_0\|$.*

Proof. If $M = X \times X$ or $\|F_0\| = 0$ then take $F = F_0$, otherwise without lose the generality assume that $\|F_0\| = 1$. Consider the family $\mathcal{A}$ of all possible extensions of F_0 of norm one, i.e. the set of all pairs (G, L) where L is a linear subspace of $X \times X$ containing M and G is a bounded 2-linear operator $G\colon L \to \mathbb{R}$ such that $G(a, b) = F_0(a, b)$ for all $(a, b) \in M$ and $\|G\| = 1$. Partially order $\mathcal{A}$ with $\le$ as follows: given $(G_1, L_1), (G_2, L_2) \in \mathcal{A}$, put $(G_1, L_1) \le (G_2, L_2)$ if and only if G_2 is an extension of G_1 that is $L_1 \subseteq L_2$, $G_2(a, b) = G_1(a, b)$ for all $(a, b) \in L_1$ and $\|G_2\| = \|G_1\|$.

The family $\mathcal{A}$ is non-empty, because $(F_0, M) \in \mathcal{A}$. Let $\mathcal{T}$ be a linearly ordered subset of $\mathcal{A}$. Define $\widetilde{L} = \bigcup_{(G, L)\in\mathcal{T}} L$. Clearly $\widetilde{L}$ is a real linear subspace of $X \times X$ and contains M. Define $\widetilde{G}\colon \widetilde{L} \to \mathbb{R}$ by $\widetilde{G}(a, b) = G(a, b)$ where G is associated with some L, $(G, L) \in \mathcal{T}$, which contains (a, b). The operator $\widetilde{G}$ is well-defined because, if both L_i and L_j contain (a, b) then $G_i(a, b) = G_j(a, b)$, since either $(G_i, L_i) \le (G_j, L_j)$ or $(G_j, L_j) \le (G_i, L_i)$. Notice that $\widetilde{G}$ is a bounded 2-linear operator on $\widetilde{L}$ that is an extension of every G and $\|\widetilde{G}\| = 1$. So the constructed pair $(\widetilde{G}, \widetilde{L})$ is hence an upper bound for the chain $\mathcal{T}$. By using Zorn's Lemma there exists a maximal element $(F, L_m) \in \mathcal{A}$. To complete the proof it is enough to show that $L_m = X \times X$. Suppose by contrary that there exists (a_0, b_0) in $X \times X \setminus L_m$. Then consider the linear space $L' = L_m + \mathbb{R}(a_0, b_0) = \{(a + ta_0, b + t'b_0); (a, b) \in L_m \text{ and } t, t' \in \mathbb{R}\}$. Define $F'\colon L' \to \mathbb{R}$ by

$$F'(a + ta_0, b + t'b_0) = F(a, b) + tt'\gamma,$$

where $(a, b) \in L_m$ and $\gamma \in \mathbb{R}$ will be chosen in such a way that $\|F'\| = 1$.

But $\|F'\| = 1$ provided that

$$|F(a, b) + tt'\gamma| \le \|a + ta_0, b + t'b_0\| \tag{2.1}$$

for all $(a, b) \in M$ and $\gamma \in \mathbb{R}$. Replace (a, b) by $(-ta, -t'b)$; where $t, t' \in \mathbb{R}$, and divide both sides of (1) by $|tt'|$. Then the requirement is that

$$|F(a, b) - \gamma| \le \|a - a_0, b - b_0\| \tag{2.2}$$

for all $(a, b) \in M$ and $\gamma \in \mathbb{R}$. Since F is 2-linear, by choosing γ in such a way that $F(a, b) - \|a - a_0, b - b_0\| \le \gamma \le F(a, b) + \|a - a_0, b - b_0\|$, (2) and therefore (1) holds. Note that γ exists, since for each $(a, b) \in M$ we have:

$$F_0(a, b) - \|a - a_0, b - b_0\| \le F_0(a, b) + \|a - a_0, b - b_0\|.$$

So we have proved that $(F', L') \in \mathcal{A}$, $(F, L_m) \neq (F', L')$ and $(F, L_m) \le (F', L')$ which is a contradiction. ∎

Our next result also deals with the existence of a bounded 2-linear functional having certain properties.

Theorem 2.2. *Let x_0, y_0 be vectors in the generalized 2-normed space $(X, \|., .\|)$such that $\|x_0, y_0\| \neq 0$. Then there exists a real bounded 2-linear functional F, defined on the whole space, such that $F(x_0, y_0) = \|x_0, y_0\|$ and $\|F\| = 1$.*

Proof. Consider the linear space $M = \mathbb{R}(x_0, y_0) = \{(tx_0, t'y_0);\ t, t' \in \mathbb{R}\}$, and consider the functional F_0, defined on M as follows:

$$F_0 \colon M \to \mathbb{R},$$

$$F_0(tx_0, t'y_0) = tt'\|x_0, y_0\|.$$

Clearly, F_0 is a 2-linear functional with the property that $F_0(x_0, y_0) = \|x_0, y_0\|$.

Further, since for any $(x, y) \in M$,

$$| F_0(x, y) |=| tt' | \|x_0, y_0\| = \|tx_0, t'y_0\| = \|x, y\|,$$

we see that F_0 is a bounded 2-linear functional. Moreover $\|F_0\| = 1$.

It now remains only to apply last theorem to assert the existence of a bounded 2-linear functional F, defined on the whole space, extending F_0, and having the same norm as F_0, that is, $\|F\| = 1$. ∎

Note the immediate consequence of the above theorem.

Corollary 2.3. *If X is not the trivial space (the vector space consisting solely of the zero vector), then nonzero bounded 2-linear functionals must exist on this space.*

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