Communications in Mathematical Analysis ISSN 0973-3841
Volume 1, Number 2, pp. 115–119, 2006

A Strengthened Carleman's Inequality

Hu Yue*
College of Mathematics and Informatics, Henan Polytechnic University,
Jaozuo City, Henan 454000, China,
Current address: College of Mathematics and Informatics,
Henan Polytechnic University, Jaozuo City, Henan 454000, China

(Communicated by Gaston M. N'Guerekata)

Abstract

In this paper, it is proved that

$$\sum_{n=1}^{\infty}(a_1a_2\ldots a_n)^{1/n} < e\sum_{n=1}^{\infty}(1-\sum_{k=1}^{\infty} b_k/(n+1)^k)a_n.$$

Where $a_n \geq 0$, $n = 1, 2, \cdots$, $0 < \sum_{n=1}^{\infty} a_n < \infty$ and $b_0 = 1$, $b_n = \frac{1}{n}\sum_{k=1}^{n}\frac{b_{n-k}}{k+1}$, $n = 1, 2, \cdots$.

AMS Subject Classification: Primary 54C40, 14E20; Secondary 46E25, 20C20.

Keywords: Carleman's inequality, Differential geometry, algebraic geometry.

1 Introduction

The following Carleman inequality is well known [1].

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n. \tag{1.1}$$

where $e = \lim_{n\to\infty}(1+1/n)^n$, $a_n \geq 0$, $n = 1, 2, \cdots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

The constant e is sharp in the sence that it cannot be replaced by a smaller one.

Recently, the inequality (1) has also been improved by many authors, for example: Yang Bicheng and L. Debnath [2] with

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+2}\right)a_n, \tag{1.2}$$

*E-mail: huu3y2@163.com

in [3] by Yan Ping and Sun Guozheng with

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1+\frac{1}{n+1/5}\right)^{-1/2}a_n, \tag{1.3}$$

and in [4] by X. Yang with

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2(n+1)}-\frac{1}{24(n+1)^2}-\frac{1}{48(n+1)^3}\right)a_n, \tag{1.4}$$

and

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{6}\frac{b_k}{(n+1)^k}\right)a_n, \tag{1.5}$$

where $b_1 = 1/2, b_2 = 1/24, b_3 = 1/48, b_4 = 73/5760, b_5 = 11/1280,$
$b_6 = 1945/580608$. It is conjectured in [4] that if $(1+1/x)^x = e\left[1-\sum_{k=1}^{\infty} b_k/(x+1)^k\right]$, $x > 0$, then $b_k > 0, k = 1, 2, \cdots$. In this paper, we will prove the following Theorem and the conjecture is also proved to some extent.

Theorem 1.1: *Let $a_n \geq 0, n = 1, 2, \cdots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then we have*

$$\sum_{n=1}^{\infty}(a_1a_2\ldots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{\infty}b_k/(n+1)^k\right)a_n. \tag{1.6}$$

Where $b_n = \frac{1}{n}\sum_{k=1}^{n}\frac{b_{n-k}}{k+1}$, $n = 1, 2, \cdots$. $b_0 = 1$.

2 Lemma

Lemma 2.1: *Let $f(t) = (1-t)^{1-1/t}$, if $0 < |t| < 1$ and $f(0) = \lim_{t\to 0} f(t)$. Then*

$$f(t) = e(b_0 - b_1t - b_2t^2 - \cdots). \tag{2.1}$$

Where $b_0 = 1$, $b_n = \frac{1}{n}\sum_{k=1}^{n}\frac{b_{n-k}}{k+1}$, $n = 1, 2, \cdots$.

Proof. By $f(t) = (1-t)^{1-1/t}$, we have

$$\ln f(t) = \left(1-\frac{1}{t}\right)\ln(1-t). \tag{2.2}$$

Using Maclaurin series, from (8), we obtain

$$\ln f(t) = 1 - \frac{t}{2} - \sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)t^n. \tag{2.3}$$

Differentiating with respect to t, we get

$$\frac{f'(t)}{f(t)} = -\frac{1}{2} - \sum_{n=2}^{\infty}\left(1-\frac{n}{n+1}\right)t^{n-1}, \tag{2.4}$$

and hence $f'(t) = f(t)\varphi(t)$, where $\varphi(t) = -\frac{1}{2} - \sum_{n=2}^{\infty}\left(1 - \frac{n}{n+1}\right)t^{n-1}$. Applying Leibnitz rule leads to

$$f^{(n)}(t) = \sum_{k=0}^{n-1} C_{n-1}^{k} f^{(n-k-1)}(t)\varphi^{k}(t), \quad n = 1, 2, \cdots . \tag{2.5}$$

Easy calculating reveals that

$$\varphi^{(n)}(0) = -\frac{n!}{n+2}, \quad n = 0, 1, 2, \cdots. \tag{2.6}$$

Hence

$$f^{(n)}(0) = -\sum_{k=0}^{n-1} \frac{k!}{k+2} C_{n-1}^{k} f^{(n-k-1)}(0). \tag{2.7}$$

Therefore

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^{n} = f(0) - \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{k+2}\frac{f^{(n-k-1)}(0)}{(n-k-1)!}\right)t^{n}. \tag{2.8}$$

Let

$$b_0 = \frac{f(0)}{e}, b_n = \frac{f^{(n)}(0)}{n!e}, \quad n = 1, 2, \cdots ,$$

we conclude

$$f(t) = e(b_0 - b_1 t - b_2 t^2 - \cdots).$$

The proof of Lemma 2.1 is complete. ■

Lemma 2.2: *For all natural number m, then*

$$\left(1 + \frac{1}{m}\right)^{m} = e\left(b_0 - \frac{b_1}{m+1} - \frac{b_2}{(m+1)^2} - \cdots\right). \tag{2.9}$$

Where $b_0 = 1$, $b_n = \frac{1}{n}\sum_{k=1}^{n} \frac{b_{n-k}}{k+1}$, $n = 1, 2, \cdots$.

Proof. Apply lemma 2.1 with $t = 1/m + 1$. ■

3 Proof of Theorem

Let $c_i > 0 (i = 1, 2, \cdots)$. By geometric and arithmetic means inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \le \frac{1}{n}\sum_{m=1}^{n} c_m a_m. \tag{3.1}$$

Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}\\
&\quad=\sum\left(\frac{c_1a_1c_2a_2\cdots c_na_n}{a_1a_2\cdots a_n}\right)^{1/n}\\
&\quad=\sum_{n=1}^{\infty}(c_1c_2\cdots c_n)^{-1/n}(c_1a_1c_2a_2\cdots c_na_n)^{1/n}\\
&\quad\le\sum_{n=1}^{\infty}(c_1c_2\cdots c_n)^{1/n}\frac{1}{n}\sum_{m=1}^{n}c_ma_m\\
&\quad=\sum_{m=1}^{\infty}c_ma_m\sum_{n=m}^{\infty}\frac{1}{n}(c_1c_2\cdots c_n)^{-1/n}.
\end{aligned}$$

Set $c_m = (m+1)^m/m^{m-1}$, $(m = 1, 2, \cdots)$, then

$$c_1c_2\cdots c_n = (n+1)^n,$$

and

$$\sum_{n=m}^{\infty}\frac{1}{n}(c_1c_2\cdots c_n)^{-1/n} = \sum_{n=m}^{\infty}\frac{1}{n(n+1)} = \frac{1}{m}.$$

Hence

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le \sum_{m=1}^{\infty}\frac{c_m}{m}a_m = \sum_{m=1}^{\infty}\left(1+\frac{1}{m}\right)^m a_m.$$

By Lemma 2.2, we get

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{m=1}^{\infty}\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(m+1)^k}\right)a_m. \tag{3.2}$$

Because of $0 < \sum_{n=1}^{\infty} a_n < \infty$, the sign of equlity in (17) not holds. The proof of Theorem 2.2 is complete. ■

Acknowledgements

The author would like to thank referee for his very valuable advice.

References

[1] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, London, 1952.

[2] Bicheng Yang and L. Debnath, Some inequlities involving the constant e and an application to Carleman's inequlity. *J. Math. Anal. Appl.*, **223** (1998), pp. 347–353.

[3] Yan Ping and Guozheng Sun, A Strengthened Carleman's inequality. *J. Math. Anal. Appl.*, **240** (1999), pp. 290–293.

[4] Xiaojing Yang, On Carleman's inequality, *J. Math. Anal. Appl.*, **253** (2001), pp. 691–694.