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BOUNDARY LOCALIZATION OF DOMAINS ON TAUT MANIFOLDS

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Abstract

Domain characterizations on two taut manifolds are obtained by some local biholomorphism properties.

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1 Introduction

Characterizations of complex manifolds by means of holomorphic automorphism groups have been studied in many papers [1, 10]. Characterizations of domains on different complex manifolds were shown in [7]. Here we present a different version of domain characterization. The main result of this paper is:

THEOREM 1. Let D_1 and D_2 be domains on two taut manifolds X_1 and X_2 respectively. Suppose there are co-compact discrete subgroups $\Gamma_1 \subset Aut(D_1)$, $\Gamma_2 \subset Aut(D_2)$ and D_1 is locally biholomorphic to D_2 at two h-convex points $p_1 \in \partial D_1$, $p_2 \in \partial D_2$. Then D_1 is biholomorphic to D_2.

There are some important examples [4]. The method is essentially based on Wong's techniques on boundary localization [10]. A full understanding of the general type of boundary point or local biholomorphism will lead to some generalization of the theorem. Basic definitions and some backgrounds are presented in section 2. We prove the main theorem in section 3.

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2 Basic Definitions and Remarks

2.1. Normal family of holomorphic mappings

Let M and N be two metric spaces. A subset F of C(M,N)= set of continuous mappings between M and N, is called normal if every sequence of F contains a subsequence which is either relatively compact in C(M,N) or compactly divergent. A sequence $\{f_i\} \subset$ C(M,N) is called compactly divergent if for any compact sets K $\subset$ M and $K' \subset$ N there exists n_0 such that $f_i(K) \cap K' = \emptyset$ for all $i \geq n_0$.

Definition: A complex manifold N is said to be *taut* if for every complex manifold M, the set of all holomorphic mappings from M to N, denoted by Hol(M,N), is a normal family.

It is known, for instance, that a C^1 pseudoconvex domain is taut.

A subset $F \subseteq$ C(M,N) is called an equicontinuous family if for any $\varepsilon > 0$ and any point $x \in$ M there is a neighborhood U of x such that if $x' \in U$, then $d_N(f(x), f(x')) < \varepsilon$ for all $f \in F$. Here N is a metric space equipped with a metric d_N inducing its underlying topology.

Definition: Let N be a complex manifold equipped with a metric d inducing its underlying topology. (N,d) is called a *tight* manifold if for every complex manifold M, Hol(M,N) is equicontinuous.

2.2. Kobayashi metrics

Definition: Let M be a complex manifold of dimension n, $X \in$ M, k an integer between 1 and n. The Eisenman differential k-measure on M is a function $E_M^k : \wedge^k T_x(M) \to R$ such that for all $(x,v) \in \wedge^k T(M)$, $E_M^k(x,v)$ = inf $\{ r^{-2k} \mid$ there exists a $f \in$ Hol(B_k(r),M) such that f(0) = x, $df\left(\left(\frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_k}\right)(0)\right) = v\}$, where $B_k(r) = \{w = (w_1, w_2, \ldots, w_k) \in C^k \mid\mid w \mid < r\}$

When $k = n$, it associates with the Eisenman-Kobayashi volume form denoted by the same symbol $E_M^n = \mid E_M^n \mid dz_1 \wedge d\bar{z_1} \wedge \cdots \wedge dz_n \wedge d\bar{z_n}$, where $\mid E_M^n \mid$ is a local function on M. Here we identify $E_M^n(x,v) = E_M^n(x, v \wedge \bar{v})$, $\bar{v}$ = complex conjugate of the vector $v \in \wedge^n T_n(M)$. When k=1, it corresponds to the Kobayashi-Royden differential metric, denoted by $K_M = \sqrt{E_M^1}$. Its integrated form is called the Kobayashi distance function on M, denoted by d_M^K [3, 6, 9]. Both E_M^K and d_M^K are decreasing under holomorphic mappings between complex manifolds. As a consequence, they are invariant under biholomorphisms. If d_M^K is a metric on M, then M is called a hyperbolic manifold. If d_M^K is Cauchy complete, then M is called a completely hyperbolic manifold. Cauchy completeness of d_M^K on a hyperbolic manifold M is equivalent to compact completeness, i.e., for every $r > 0$ the level set $M_r = \{y \in M \mid d_M^K(x,y) < r\}$ is relatively compact on M, where x is a fixed point in M. It follows from the definitions that taut manifolds are always tight. Tight manifolds are equivalent to hyperbolic manifolds. Completely hyperbolic manifolds are always taut, but the converse is not necessarily true. It is a well-known fact that a tight manifold does not admit any non-trivial holomorphic curve [6].

2.3. Automorphism groups and local biholomorphism

A domain D on a complex manifold is said to be admitting a compact quotient if A/Aut(D) is compact, where Aut(D) = group of biholomorphisms of D. A subgroup $\Gamma \subset Aut(D)$ is *discrete* if it has no accumulation points in Aut(D), and $co-compact$ if it has a compact fundamental domain in D. One should notice that D/Aut(D) is compact if D covers a compact complex manifold. When D is either a taut manifold or a relatively compact set of a tight manifold, Aut(D) is a Lie group [11]. Furthermore, Aut(D) acts properly on D if D is taut.

Let D be a domain on a complex manifold M and $p \in \partial D$ a fixed boundary point. A boundary neighborhood $\hat{D}$ of D at p means an open set $\hat{D}$ = U ∩ D, where U is an open set in M containing p.

Definition: Let D_1 and D_2 be two domains on two complex manifolds respectively. D_1 is said to be locally biholomorphic to D_2 at two boundary points $p_1 \in \partial D_1$ and $p_2 \in \partial D_2$ if

(i) there exist boundary neighborhoods $\hat{D}_1$ of p_1 and $\hat{D}_2$ of p_2 with a biholomorphism $f : \hat{D}_1 \to \hat{D}_2$.

(ii) there is a sequence $\{x_i\} \subset \hat{D}_1$ converging to p_1 such that $\{f(x_i) \subset \hat{D}_2\}$ will converge to p_2.

2.4. H-convexity

Definition: $p \in \partial D$ is h-convex if there exists a boundary neighborhood $\hat{D}$ such that

(i) $S = u(\partial D \cap \hat{D})$ is convex in some holomorphic chart $u : \hat{D} \to C^n$

(ii) for all complex affine linear $H \subset C^{\,n}$ such that u(p) is an interior point of $S \cap H$ in the topology of H, $S \cap H \subset\subset S$.

Remark 2.4. Using some standard arguments [4], one can prove " If $K \subset\subset D$ is compact, $m_i \in Aut(D)$ and $z \in K$, such that $m_i(z) \to p$ then $m_i(K) \subset\subset \hat{D}, \forall i >> 0$, for any boundary neighborhood $\hat{D}$ of p ".

3 Remarks on The Normal Family of Holomorphic Mappings and Proof of Theorem 1

Remark 3.1. Let D be a domain on a taut manifold such that $\Gamma \subset Aut(D)$ is co-compact in D. Then there exists a compact fundamental domain K in D. If D admits compact quotient then it implies the same conclusion.

Lemma 3.2. Let X and Y be complex manifolds and let Y be taut. Suppose (1) $f_i : X_i \to Y$ is holomorphic and X_i is an increasing family of open sets in X with $X_i \subset\subset X_{i+1}$ and $X = \cup_{i=1}^{\infty} X_i$

(2) there exist compact subsets $K \subset X$ and $L \subset Y$ such that $f_i(K) \cap L \neq \emptyset$ for all sufficiently large i.

Then there is a subsequence of f_i converging uniformly on compact sets to a holomorphic mapping $f : X \to Y$.

Proof: Consider only those j so large that $K \subset X_j$. Fix such a j, then a subsequence of f_i converges uniformly on compact sets on X_j to a holomorphic map $X_j \to Y$ because of the tautness of Y and the assumption that $f_i(K) \cap L \neq \emptyset$ for sufficiently large i. Now let $j \to \infty$. By passing to subsequences repeatedly and by the usual diagonal process, we arrive at the desired holomorphic map $f : X \to Y$.

Lemma 3.3. Let D be a domain on a taut manifold with co-compact group $\Gamma \subset Aut(D)$. Then D is taut. (We shall actually prove that D is completely hyperbolic.)

Proof: By Remark 3.1, there is a compact set $K \subset D$ such that for any point $x \in D$ there exist $y \in K$ and $g \in Aut(D)$ with $g(y) = x$. Let $\varepsilon > 0$ be a sufficiently small number such that $L = \{Z \in D \mid d_D^K(W,Z) \leq \varepsilon, W \in K\}$ is a compact set. Let x_i be a Cauchy sequence in D with respect to d_D^K. We can find a positive integer m such that for all $i \geq m, d_D^K(x_m, x_i) < \varepsilon$.

Moreover there is a $g \in \Gamma$ such that $g(x_m) \in K$. Clearly, $g(x_i) \in L$ for all $i \geq m$ because d_D^K is invariant under Aut(D). Passing through a subsequence if necessary, $g(x_i)$ will converge to a point $q \in L$ because L is compact. It is easy to see x_i must converge to $g^{-1}(q)$, for the same reason that g is an isometry with respect to d_D^K.

Lemma 3.4. Let D be a domain on a taut manifold X with co-compact $Aut(D)$. Let x_i be a sequence of points in D converging to a boundary point $p \in \partial D$. Then there exists m_i $\subset$ Aut(D) such that $\bar{x_i} = m_i^{-1}(x_i)$, through a subsequence if necessary, converges to a point $x \in D$. Furthermore, $\{z_i = M_i(x)\}$ will also converge to p.

Proof: Let K be a compact subset of D as in Remark 3.1 let m_i be an element in Aut(D) such $\bar{x_i} = M_i^{-1}(x_i) \in K$. Through a subsequence, $\{\bar{x_i}\}$ will converge to a point $x \in K \subset D$. To prove $\{z_i = m_i(x)\}$ is convergent to p, we consider the distance with respect to d_D^K as follows:

$$d_D^K(z_i,x_i) = d_D^K(m_i(x),x_i) = d_D^K(x,m_i^{-1}(x_i)) = d_D^K(x,\bar{x_i})$$

The following inequality is clear by distance decreasing property;

(*) $d_D^K \geq d$ on D, where d = Kobayashi metric on X.

We observe that $d_D^K(z_i,x_i) \to 0$ as $i \to \infty$ because $d_D^K(x,\bar{x_i}) \to 0$ as $\{\bar{x}\} \to x$. By (*), $d(z_i,x_i) \to 0$ as $i \to \infty$. Since d is finite around an open set of $p \in \partial D$, hence $d(x_i,p) \to 0$, as $\{x_i\} \to p$. By triangle inequality of d, one has $d(z_i,p) \to 0$ as $i \to \infty$. Thus $\{z_i\} \to p$ as a limit.

Lemma 3.5. Let D be a domain of a taut manifold X. Suppose there is a h-convex point $p \in \partial d$. Let $\{m_i\} \subset Aut(D)$ be a sequence such that $\{m_i(x)\} \to p$ for some $x \in D$. Then for any compact subset $K \subset D$ and any boundary neighborhood $\hat{D}$ of p, $m_i(K) \subset \hat{D}, i >> 0$. In particular, $\{m_i(y)\} \to p$ for any $y \in D$.

Proof: By normal family argument, through a subsequence, $\{m_i\}$ will converge on compacta to a holomorphic mapping $m : D \to X$ such that $m(D) \subset \partial D$ and $m(x) = p$. By remark 2.4 , $m(K) \subset \hat{D}$ for $i >> 0$.

Remark 3.6. When Aut(D) is co-compact we can choose $m_i \in Aut(D)$ and $x \in K \subset D$ such that $m_i(x) \to p$.

<u>*Proof of Theorem 1*</u>

Since D_1 and D_2 are locally biholomorphic at $p_1 \in \partial D_1$ and $p_2 \in \partial D_2$, there is a biholomorphism f between two boundary neighborhood $\hat{D_1}$ and $\hat{D_2}$ of p_1 and p_2 respectively. Choose two sequences of relatively compact open subsets $\{X_i\}$ and $\{Y_i\}$ in $\hat{D}_1$ and $\hat{D}_2$ respectively so that

$(i) X_i \subset\subset X_{i+1},\ \ Y_i \subset\subset Y_{i+1}$

$(ii) \cup_{i=1}^{\infty} X_i = \hat{D_1},\ \ \cup_{i=1}^{\infty} Y_i = \hat{D_2}$

(iii) X_i is biholomorphic to Y_i under f (i.e. $f(X_i) = Y_i$).

Clearly the sequence of relatively compact open subsets $D_1^i = g_i^{-1}(X_i)$, $D_2^i = h_i^{-1}(Y_i)$ will satisfy the following three properties, where $\{g_i\} \subset Aut(D_1)$ and $\{h_i\} \subset Aut(D_2)$ are the corresponding sequences obtained in lemmas 3.4, 3.5. (with respect to the sequences $\{X_i\}$ and $\{f(X_i)\}$ in our definition of local biholomorphisms at two boundary points p_1 and p_2)

$(i) D_1^i \subset\subset D_1^{i+1},\ \ D_2^i \subset\subset D_2^{i+1}$

$(ii) \cup_1^\infty D_1^i = D_1, \ \cup_1^\infty D_2^i = D_2$
(iii) the composition of mappings $F_i = h_i^{-1} \circ f \circ g_i$ is a biholomorphism between D_1^i and D_2^i.

Let K and L be the compact subsets (i.e. fundamental domains) in D_1 and D_2 respectively which are obtained in Remark 3.1. It is easy to show that $\{F_i, D_1^i, K, L\}$ satisfies the non-divergent condition in lemma 3.2. Thus $\{F_i\}$, through a subsequence, will converge to a holomorphic map $F : D_1 \to D_2$. On the other hand, one can repeat the same argument to $\{F_i^{-1}, D_2^i, L, K\}$. In this way, one can then prove F_i^{-1} converge to a holomorphic map $G : D_2 \to D_1$. Let a be a fixed point in K. By taking subsequences and readjustments of indices, one can assume $g_i(a) \in X_i$, $f \circ g_i(a) \in Y_i$ for all i. It is clear from the proofs of lemmas 3.4 and 3.5; one can choose the above sequence $\{h_i\} \subset Aut(D_2)$ satisfying further property, namely, $(h_i^{-1} \circ f \circ g_i)(a) \in L$. Let's say $\{(h_i^{-1} \circ f \circ g_i)(a)\}$ converge to $b \in L$. Apparently we obtain the following two conclusions:

$(i) G \circ F(a) = b$

$(ii) \mid det(G \circ F)(a) \mid = 1$

To conclude the proof of biholomorphism, we apply the result due to Dektyarev-Graham-Wu [2, 5] which is a generalization of a theorem of H. Cartan [8].

Remark 3.7. It is possible to weaken the condition of local biholomorphism which would give a more general result.

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