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STEPANOV-LIKE PSEUDO ALMOST PERIODIC FUNCTIONS AND THEIR APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract

This paper introduces and examines a new class of functions called Stepanov-like pseudo almost periodic functions (or S^p-pseudo almost periodic functions), which generalizes in a natural fashion the classical notion of pseudo almost periodicity. We then make extensive use of these new functions to study the existence and uniqueness of a pseudo almost periodic solution to the semilinear equation

$$u'(t) = Au(t) + F(t,u(t)),$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is the infinitesimal generator of an exponentially stable C_0-semigroup on a Banach space $\mathbb{X}$ and $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is S^p-pseudo almost periodic for $p > 1$ and jointly continuous.

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1 Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $p \geq 1$. In a recent paper by N'Guérékata and Pankov [11], the concept of Stepanov-like almost automorphy (or S^p-almost automorphy) was introduced. Such a notion generalizes in a natural fashion the classical almost automorphy in the sense of Bochner. Moreover, the concept of S^p-almost automorphy was, subsequently, utilized to study the existence of weak S^p-almost automorphic solutions to some parabolic evolution equations.

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Recently, Diagana and N'Guérékata [4] made use of S^p-almost automorphy to study the existence and uniqueness of an almost automorphic solution to the semilinear equation

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a densely closed linear operator, which is also the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$ and $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is S^p-almost automorphic for $p > 1$ and jointly continuous. One should point out that such a result generalizes the existence results obtained in N'Guérékata [10], as the space of S^p-almost automorphic functions contains the space $AA(\mathbb{X})$ of almost automoprhic functions.

The present paper is definitely inspired by the above-mentioned papers and consists of introducing a similar notion within the framework of pseudo almost periodicity, that is, the class of functions called Stepanov-like pseudo almost periodic functions (or S^p-pseudo almost periodic functions), which, obviously, generalizes in a natural fashion the concept of pseudo almost periodicity.

Properties of these S^p-pseudo almost periodic functions are investigated. In particular, we prove a useful composition result (Theorem 2.14). Next, we make extensive use of the previous results to study the existence and uniqueness of a pseudo almost periodic solution to Eq. (1.1) when the forcing term $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is S^p-pseudo almost periodic for $p > 1$ and jointly continuous.

The existence of almost automorphic, almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions to differential equations is one of the most attractive topics in the qualitative theory of differential equations due to their significance and applications in physical sciences. The concept of pseudo almost periodicity, which is the central issue in this paper was first introduced in the literature by Zhang [13, 14, 15] in the earlier nineties – and is a natural generalization of the concept of almost periodicity.

Some contributions upon pseudo almost periodic solutions to differential equations have recently been made in [1, 2, 5, 6, 7, 8]. However, the existence of pseudo almost periodic solutions to semilinear equations of the form Eq. (1.1) in the case when the forcing term F belongs to the class of S^p-pseudo almost periodic functions is quite new and original and is the main motivation of the present paper. The paper is organized as follows. In Section 2, we recall some results on the notion of pseudo almost periodicity, and introduce and examine properties of the concept of S^p-pseudo almost periodicity. Applications to abstract differential equations are presented in Section 3.

2 Pseudo Almost Periodicity

Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$$

is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

Definition 2.1. A function $f \in C(\mathbb{R},\mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon \text{ for each } t \in \mathbb{R}.$$

The number τ above is called an ε-*translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 2.2. A function $F \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t+\tau,y) - F(t,y)\| < \varepsilon \text{ for each } t \in \mathbb{R},\ y \in K.$$

The collection of those functions is denoted by $AP(\mathbb{R}\times\mathbb{Y})$.

Define the classes of functions $PAP_0(\mathbb{X})$ and $PAP_0(\mathbb{R}\times\mathbb{X})$ respectively as follows:

$$PAP_0(\mathbb{X}) := \left\{u \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|u(s)\|ds = 0\right\},$$

and $PAP_0(\mathbb{R}\times\mathbb{X})$ as the collection of functions $F \in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|F(t,u)\|dt = 0$$

uniformly in $u \in \mathbb{Y}$.

Definition 2.3. A function $f \in BC(\mathbb{R},\mathbb{X})$ is called pseudo almost periodic if it can be expressed as $f = h + \varphi$, where $h \in AP(\mathbb{X})$ and $\varphi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAP(\mathbb{X})$.

Remark 2.4. The functions h and φ in Definition 2.3 are respectively called the *almost periodic* and the *ergodic perturbation* components of f. Moreover, the decomposition given in Definition 2.3 is unique.

Lemma 2.5. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset PAP(\mathbb{X})$ *be a sequence of functions. If* f_n *converges uniformly to some* f*, then* $f \in PAP(\mathbb{X})$.

Proof. First of all, note that f is necessarily a bounded continuous function from $\mathbb{R}$ into $\mathbb{X}$. Now for each $n \in \mathbb{N}$, write $f_n = h_n + \varphi_n$ where $\{h_n\}_{n\in\mathbb{N}} \subset AP(\mathbb{X})$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subset PAP_0(\mathbb{X})$. From [13, Lemma 1.3] we have that

$$\|h_n\| \le \|f_n\|,$$

and hence there exists $h \in AP(\mathbb{X})$ such that $\|h_n - h\|_\infty \to 0$ as $n \to \infty$.

Using the previous fact, it easily follows that there exists a function $\varphi \in BC(\mathbb{R},\mathbb{X})$ such that $\|\varphi_n - \varphi\|_\infty \to 0$ as $n \to \infty$.

Now

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\|\varphi(t)\|dt &\leq \frac{1}{2T}\int_{-T}^{T}\|\varphi_n(t)-\varphi(t)\|dt+\frac{1}{2T}\int_{-T}^{T}\|\varphi_n(t)\|dt\\ &\leq \|\varphi_n-\varphi\|_\infty+\frac{1}{2T}\int_{-T}^{T}\|\varphi_n(t)\|dt,\end{aligned}$$

and hence $\varphi\in PAP_0(\mathbb{X})$. Consequently, $f=h+\varphi\in PAP(\mathbb{X})$. □

Definition 2.6. A function $F\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to pseudo almost periodic in $t\in\mathbb{R}$ uniformly in $y\in\mathbb{Y}$ if it can be expressed as $F=G+\Phi$, where $G\in AP(\mathbb{R}\times\mathbb{Y})$ and $\phi\in PAP_0(\mathbb{R}\times\mathbb{Y})$. The collection of such functions will be denoted by $PAP(\mathbb{R}\times\mathbb{Y})$.

Definition 2.7. ([12]) The Bochner transform $f^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in $\mathbb{X}$, is defined by

$$f^b(t,s):=f(t+s).$$

Remark 2.8. ([12]) A function $\varphi(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t,s)=f^b(t,s)\,,$$

if and only if

$$\varphi(t+\tau,s-\tau)=\varphi(s,t)$$

for all $t\in\mathbb{R}$, $s\in[0,1]$ and $\tau\in[s-1,s]$.

Obviously, if $f=h+\varphi$, then $f^b=h^b+\varphi^b$. Moreover, $(\lambda f)^b=\lambda f^b$ for each scalar λ.

Definition 2.9. The Bochner transform $F^b(t,s,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $u\in\mathbb{X}$ of a function $F(t,u)$ on $\mathbb{R}\times\mathbb{X}$, with values in $\mathbb{X}$, is defined by

$$F^b(t,s,u):=F(t+s,u)$$

for each $u\in\mathbb{X}$.

Definition 2.10. Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b\in L^\infty\big(\mathbb{R},L^p((0,1),\mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(\tau)\|^p\,d\tau\right)^{1/p}.$$

Definition 2.11. A function $f\in BS^p(\mathbb{X})$ is called S^p-pseudo almost periodic (or Stepanov-like pseudo almost periodic) if it can be expressed as $f=h+\varphi$, where $h^b\in AP\big(L^p((0,1),\mathbb{X})\big)$ and $\varphi^b\in PAP_0\big(L^p((0,1),\mathbb{X})\big)$. The collection of such functions will be denoted by $PAPS^p(\mathbb{X})$.

In other words, a function $f\in L^p(\mathbb{R},\mathbb{X})$ is said to be S^p-pseudo almost periodic if its Bochner transform $f^b:\mathbb{R}\to L^p((0,1),\mathbb{X})$ is pseudo almost periodic in the sense that there exist two functions $h,\varphi:\mathbb{R}\mapsto\mathbb{X}$ such that $f=h+\varphi$, where $h^b\in AP(L^p((0,1),\mathbb{X}))$ and $\varphi^b\in PAP_0(L^p((0,1),\mathbb{X}))$, that is,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\phi(\sigma)\|^p d\sigma\right)^{1/p}dt=0.$$

Remark 2.12. It is clear that if $1 \leq p < q < \infty$ and $f \in L^q(\mathbb{R}, \mathbb{X})$ is S^q-pseudo almost periodic, then f is S^p-pseudo almost periodic. Also if $f \in PAP(\mathbb{X})$, then f is S^p-pseudo almost periodic for any $1 \leq p < \infty$.

Similarly one gets the following definition.

Definition 2.13. A function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}, (t,u) \mapsto F(t,u)$ with $F(\cdot,u) \in L^p(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$, is said to be S^p-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$ if $t \mapsto F(t,u)$ is S^p-pseudo almost periodic for each $u \in \mathbb{X}$.

This means, there exist two functions $H, \Phi : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ such that $F = H + \Phi$, where $H^b \in AP(\mathbb{R} \times L^p((0,1), \mathbb{X}))$ and $\Phi^b \in PAP_0(\mathbb{R} \times L^p((0,1), \mathbb{X}))$, that is,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|\Phi(\sigma,u)\|^p d\sigma \right)^{1/p} dt = 0$$

uniformly in $u \in \mathbb{X}$

The collection of those S^p-pseudo almost periodic functions $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ will be denoted by $PAPS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

We have the following useful composition theorem.

Theorem 2.14. *Let $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be a S^p-pseudo almost periodic. Suppose that $F(t,u)$ is Lipschitzian in $u \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$, that is there exists $L > 0$ such*

$$\|F(t,u) - F(t,v)\| \leq L \,.\, \|u - v\| \tag{2.1}$$

for all $t \in \mathbb{R}, (u,v) \in \mathbb{X} \times \mathbb{X}$.

If $\phi \in PAPS^p(\mathbb{X})$, then $\Gamma : \mathbb{R} \to \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, \phi(\cdot))$ belongs to $PAPS^p(\mathbb{X})$.

Proof. Write $F^b = H^b + \Phi^b$, where $H^b \in AP(\mathbb{R} \times L^p((0,1), \mathbb{X}))$ and $\Phi^b \in PAP_0(\mathbb{R} \times L^p((0,1), \mathbb{X}))$, that is,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|\Phi(\sigma,u)\|^p d\sigma \right)^{1/p} dt = 0$$

uniformly in $u \in \mathbb{X}$. Similarly, write $\phi^b = \phi_1^b + \phi_2^b$, where $\phi_1^b \in AP(L^p((0,1), \mathbb{X}))$ and $\phi_2^b \in PAP_0(L^p((0,1), \mathbb{X}))$, that is,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} \|\varphi_2^b(\sigma)\|^p d\sigma \right)^{1/p} dt = 0. \tag{2.2}$$

It is obvious to see that $F^b(\cdot, \phi(\cdot)) : \mathbb{R} \mapsto L^p((0,1), \mathbb{X})$. Now decompose F^b as follows

$$\begin{aligned} F^b(\cdot, \phi^b(\cdot)) &= H(\cdot, \phi_1^b(\cdot)) + F^b(\cdot, \phi^b(\cdot)) - H(\cdot, \phi_1^b(\cdot)) \\ &= H(\cdot, \phi_1^b(\cdot)) + F^b(\cdot, \phi^b(\cdot)) - F^b(\cdot, \phi_1^b(\cdot)) + \Phi^b(\cdot, \phi_1^b(\cdot)). \end{aligned}$$

Using the theorem of composition of almost periodic functions, it is easy to see that $H^b(\cdot, \phi_1^b(\cdot)) \in AP(L^p((0,1), \mathbb{X}))$. Now, set

$$G^b(\cdot) := F^b(\cdot, \phi^b(\cdot)) - F^b(\cdot, \phi_1^b(\cdot)).$$

Clearly, $G^b(\cdot) \in PAP_0(L^p((0,1),\mathbb{X}))$. Indeed, for $T > 0$,

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|G^b(\sigma)\|^p d\sigma\right)^{1/p} dt$$

$$\begin{aligned}
&= \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|F^b(\sigma,\phi^b(\sigma)) - F^b(\sigma,\phi_1^b(\sigma))\|^p d\sigma\right)^{1/p} dt \\
&\leq \frac{L}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\phi^b(\sigma) - \phi_1^b(\sigma)\|^p d\sigma\right)^{1/p} dt \\
&\leq \frac{L}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\phi_2^b(\sigma)\|^p d\sigma\right)^{1/p} dt,
\end{aligned}$$

and hence using Eq. (2.2), it follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|G^b(\sigma)\|^p d\sigma\right)^{1/p} dt = 0.$$

Using the theorem of composition of functions of $PAP_0(L^p((0,1),\mathbb{X}))$ (see [9]) it is easy to see that $\Phi^b(\cdot,\phi_1^b(\cdot)) \in PAP_0(L^p((0,1),\mathbb{X}))$, that is,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\Phi^b(\sigma,\phi_1^b(\sigma))\|^p d\sigma\right)^{1/p} dt = 0.$$

□

3 Applications to Differential Equations

In this section we make use of the previous properties of S^p-pseudo almost periodic functions to study the existence of pseudo almost periodic solutions to the abstract differential equation

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely closed linear operator and $f \in PAPS^p(\mathbb{X}) \cap C(\mathbb{R},\mathbb{X})$ for $p > 1$.

Throughout the rest of the paper, we set $q = 1 - \dfrac{1}{p}$. Note that $q \neq 0$, as $p \neq 1$.

Theorem 3.1. *Under previous assumptions, if A (possibly unbounded) is the generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$, then Eq. (3.1) has a unique mild solution $u \in PAP(\mathbb{X})$.*

Proof. By assumption there exist $K > 0$ and $\omega > 0$ such that $\|T(t)\|_{L(\mathbb{X})} \leq Ke^{-\omega t}$ for all $t \geq 0$. Let us first prove uniqueness. Assume that $u : \mathbb{R} \to \mathbb{X}$ is bounded and satisfies the homogeneous equation

$$u'(t) = Au(t), \quad t \in \mathbb{R}. \tag{3.2}$$

Then $u(t) = T(t-s)u(s)$, for any $t \geq s$. Thus $\|u(t)\| \leq MKe^{-\omega(t-s)}$, where $\|u(s)\| \leq M$. Take a sequence of real numbers (s_n) such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $(s_{n_k}) \subset (s_n)$ such that $s_{n_k} < t$ for all $k = 1, 2, \ldots$. By letting $k \to \infty$, we get $u(t) = 0$.

Now, if f u_1, u_2 are bounded solutions to Eq. (3.1), then $v = u_1 - u_2$ is a bounded solution to Eq. (3.2). In view of the above, $v = u_1 - u_2 = 0$, that is, $u_1 = u_2$.

Let $f = h + \varphi$, where $h^b \in AP\big(L^p((0,1),\mathbb{X})\big)$ and $\varphi^b \in PAP_0\big(L^p((0,1),\mathbb{X})\big)$. Now let us investigate the existence. Consider for each $n = 1, 2, \ldots$, the integral

$$v_n(t) = \int_{n-1}^{n} T(\xi) f(t-\xi)d\xi = \int_{n-1}^{n} T(\xi)h(t-\xi)d\xi + \int_{n-1}^{n} T(\xi)\varphi(t-\xi)d\xi.$$

Set

$$Y_n(t) = \int_{n-1}^{n} T(\xi)h(t-\xi)d\xi \ \text{ and } \ X_n(t) = \int_{n-1}^{n} T(\xi)\varphi(t-\xi)d\xi.$$

It is clear that $Y_n \in AP(\mathbb{X})$. Now letting $t - \xi = r$, we obtain

$$X_n(t) = \int_{t-n}^{t-n+1} T(t-r)\varphi(r)dr.$$

Therefore

$$\|X_n\| \leq K \int_{t-n}^{t-n+1} e^{-\omega(t-r)} \|\varphi(r)\| dr.$$

We now use the Hölder's inequality to obtain

$$\begin{aligned}
\|X_n(t)\| &\leq K \left(\int_{t-n}^{t-n+1} e^{-q\omega(t-r)}dr\right)^{\frac{1}{q}} \left(\int_{t+n-1}^{t+n} \|\varphi(r)\|^p dr\right)^{\frac{1}{p}} \\
&\leq \frac{K}{\sqrt[q]{q\omega}} \left(e^{-q\omega(n-1)} - e^{-q\omega n}\right)^{\frac{1}{q}} \|\varphi\|_{S^p} \\
&\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}} (e^{q\omega} - 1)^{\frac{1}{q}} \|\varphi\|_{S^p} \\
&\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}} (e^{q\omega} + 1)^{\frac{1}{q}} \|\varphi\|_{S^p}.
\end{aligned}$$

Since the series

$$\frac{K}{\sqrt[q]{q\omega}} (e^{q\omega} + 1)^{\frac{1}{q}} \times \sum_{n=1}^{\infty} e^{-\omega n}$$

is convergent, we deduce from the Weierstrass test that the sequence of functions $\sum_{n=1}^{N} X_n(t)$ is uniformly convergent on $\mathbb{R}$.

Now let

$$N(t) := \sum_{n=1}^{\infty} X_n(t), \quad t \in \mathbb{R}.$$

Observe that

$$N(t) = \int_{-\infty}^{t} T(t-r)\varphi(r)dr, \quad t \in \mathbb{R}$$

and clearly $N(t) \in C(\mathbb{R},\mathbb{X})$. Moreover, for any $t \in \mathbb{R}$, we have

$$\|N(t)\| \leq \sum_{n=1}^{\infty} \|X_n(t)\| \leq C_q(K,\omega)\,.\,\|\varphi\|_{S^p}$$

where $C_q(K,\omega)$ depends only on the fixed constants q,K and ω, i.e. the parameters of the problem.

Let us show that each $X_n \in PAP_0(\mathbb{X})$. For that, note that

$$\begin{aligned} \|X_n(t)\| &\leq \left(\int_{t-n}^{t-n+1} e^{-q\omega(t-r)}dr\right)^{\frac{1}{q}} \left(\int_{t+n-1}^{t+n} \|\varphi(r)\|^p dr\right)^{\frac{1}{p}} \\ &\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\,.\left(\int_{t+n-1}^{t+n} \|\varphi(r)\|^p dr\right)^{\frac{1}{p}} \\ &= C_q(K,\omega)\left(\int_{t+n-1}^{t+n} \|\varphi(r)\|^p dr\right)^{\frac{1}{p}}, \end{aligned}$$

and hence $X_n \in PAP_0(\mathbb{X})$, as $\varphi^b \in PAP_0(L^p((0,1),\mathbb{X}))$.

Thus we conclude that each $v_n(t) \in PAP(\mathbb{X})$ and hence $\sum_{n=1}^{N} v_n(t) \in PAP(\mathbb{X})$. Consequently its uniform limit $u(t) = \sum_{n=1}^{\infty} v_n(t) \in PAP(\mathbb{X})$, by Lemma 2.5.

□

Using Theorem 3.1 one easily proves the following theorem.

Theorem 3.2. *Under previous assumptions, if A (possibly unbounded) is the generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ and if $F \in PAPS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ satisfies the Lipschitz condition Eq. (2.1), then Eq. (1.1) has a unique mild solution $\tilde{u} \in PAP(\mathbb{X})$ whenever* $\dfrac{KL}{\omega} < 1$.

Proof. Consider the nonlinear operator Γ defined by

$$(\Gamma u)(t) = \int_{-\infty}^{t} T(t-r)F(r,u(r))dr$$

for each $t \in \mathbb{R}$.

Using the proof of Theorem 3.1, one can easily see that $\Lambda u \in PAP(\mathbb{X})$ whenever $u \in PAP(\mathbb{X}) \subset PAPS^p(\mathbb{X})$. Thus Λ maps $PAP(\mathbb{X})$ into itself. To complete the proof, one has to prove that Λ has a unique fixed-point.

Now

$$\|\Lambda u - \Lambda v\|_\infty \leq \frac{KL}{\omega}\|u-v\|_\infty,$$

and hence Λ has unique fixed-point $\tilde{u}$ whenever $\frac{KL}{\omega} < 1$. Of course, $\tilde{u}$ is the only pseudo almost periodic solution to Eq. (1.1).

□

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