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HILLE TYPE OSCILLATION CRITERIA FOR A CLASS OF FIRST ORDER NEUTRAL PANTOGRAPH DIFFERENTIAL EQUATIONS OF EULER TYPE

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Abstract

In this paper, we investigate the unbounded delay neutral differential equation with positive and negative coefficients (and of Euler form)

$$\frac{d}{dt}\left[x(t)-R(t)x(\gamma t)\right]+\frac{P(t)}{t}x(\alpha t)-\frac{Q(t)}{t}x(\beta t)=0,\ t\geqslant t_0>0,$$

where $P,Q,R\in C([t_0,\infty),R^+),\alpha,\gamma\in(0,1)$, $\beta\in(0,1]$, and $\alpha\leqslant\beta$ are constants. Some Hille type oscillation criteria for the oscillation of all solutions are established.

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1 Introduction

The oscillation of neutral differential equation with positive and negative coefficients and constant delays has been investigated by many authors. For example, Chuanxi and Ladas [4], Farrel *et al.*[5], Ruan [12], Yu [17], Yu and Yan [18], and Shen and Debnath [13] have investigated the following neutral differential equation with positive and negative coefficients

$$\frac{d}{dt}\left[x(t)-R(t)x(t-r)\right]+P(t)x(t-\tau)-Q(t)x(t-\sigma)=0,\tag{1.1}$$

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where $P,Q,R \in C([t_0,\infty),R^+), r \in (0,\infty)$, and $\tau,\sigma \in [0,\infty)$. All kinds of sufficient conditions for the oscillation of all solutions have been obtained there.

On the other hand, it should be mentioned that there are many good results for the stability of delay differential equations with unbounded delays (see [16] for example). However, there are few results for the oscillation on such equations except [2], [8] and [14]. In particular, to the best of our knowledge, there is little in the way of results for the oscillation of neutral differential equations with unbounded delays and positive and negative coefficients.

In this paper, we consider the following unbounded delay neutral differential equation with positive and negative coefficients (and of Euler form)

$$\frac{d}{dt}\left[x(t)-R(t)x(\gamma t)\right]+\frac{P(t)}{t}x(\alpha t)-\frac{Q(t)}{t}x(\beta t)=0,\ t \geqslant t_0>0, \tag{1.2}$$

where

$$P,Q,R \in C([t_0,\infty),R^+), \alpha,\gamma \in (0,1), \beta \in (0,1], \text{ and } \alpha \leqslant \beta, \tag{1.3}$$

$$H(t)=P(t)-Q\left(\frac{\alpha}{\beta}t\right) \geqslant 0, \text{ and not identically zero.} \tag{1.4}$$

It should be noted that Eq. (1.2) is different from Eq. (1.1) which is concerned with constant delays. And, Eq. (1.2) includes a lot of differential equations. For example, the oscillation of solutions of the following unbounded delay equations of Euler form

$$x'(t)=\frac{p}{t}x(\alpha t) \text{ and } x'(t)=\frac{p}{t}x(\alpha t)-\frac{q}{t}x(\beta t),$$

where $p \in (0,\infty), q \in [0,\infty)$, and $\alpha,\beta \in (0,1)$, was investigated in [1] and [6], respectively. In [9] and [11], the authors studied the asymptotic behavior of solutions of the equation

$$x'(t)=ax(\alpha t)+bx(t),$$

where $a,b \in R$, $0<\alpha<1$, which arises in [10] as a mathematical model of the motion of a pantograph head on the electric locomotive.

The major object of this paper is to establish oscillation criteria for Eq. (1.2) under the following three cases:

$$R(t)+\int_{\frac{\alpha}{\beta}t}^{t}\frac{Q(s)}{s}ds \equiv 1, \tag{1.5}$$

$$R(t)+\int_{\frac{\alpha}{\beta}t}^{t}\frac{Q(s)}{s}ds \leqslant 1, \tag{1.6}$$

and

$$R(t)+\int_{\frac{\alpha}{\beta}t}^{t}\frac{Q(s)}{s}ds \geqslant 1. \tag{1.7}$$

By a solution of Eq. (1.2) we mean a function $x(t) \in C([\rho\bar{t},\infty),R)$ for some $\bar{t} \geqslant t_0$, such that $x(t)-R(t)x(\gamma t)$ is continuously differentiable, and $x(t)$ satisfies Eq. (1.2) for all $t \geqslant \bar{t}$, where $\rho=\min\{\alpha,\beta,\gamma\}$.

As is customary, a solution of Eq. (1.2) is said to oscillate if it has arbitrarily large zeroes; otherwise, the solution is called non-oscillatory.

2 Lemmas

In this section, we establish some lemmas which are also interesting in their own rights.

Lemma 2.1. *Assume that (1.3), (1.4) and (1.7) hold. Let $y(t)$ be an eventually positive solution of the inequality*

$$\frac{d}{dt}\left[x(t)-R(t)x(\gamma t)\right]+\frac{P(t)}{t}x(\alpha t)-\frac{Q(t)}{t}x(\beta t)\leqslant 0, \tag{2.1}$$

and set

$$z(t)=y(t)-R(t)y(\gamma t)-\int_{\frac{\alpha}{\beta}t}^{t}\frac{Q(s)}{s}y(\beta s)ds. \tag{2.2}$$

Then the oscillation of all solutions of the second ordinary differential equation

$$w''(t)+\left(\ln\frac{1}{\rho}\right)^{-1}\cdot\frac{H(t)}{t^2}w(t)=0,\ \ t\geqslant t_0>0, \tag{2.3}$$

implies that $z'(t)\leqslant 0$ and $z(t)<0$ eventually.

Proof. Let $t_1\geqslant t_0$ be such that

$$y(\gamma t)>0, y(\alpha t)>0, y(\beta t)>0, \quad \text{for } t\geqslant t_1.$$

Then, from (2.1) and (2.2) it follows that

$$z'(t)=-\frac{1}{t}\left(P(t)-Q\left(\frac{\alpha}{\beta}t\right)\right)y(\alpha t)=-\frac{H(t)}{t}y(\alpha t)\leqslant 0,\ t\geqslant t_1. \tag{2.4}$$

Therefore, if $z(t)<0$ does not hold eventually, then $z(t)>0$ eventually. Let $t_1\geqslant\frac{t_0}{\rho}$ be such that $y(\rho t)>0, z(t)>0$ for $t\geqslant t_1$. Set $M=2^{-1}\min\{y(t):\rho t_1\leqslant t\leqslant t_1\}$. Then $y(t)>M$ for $\rho t_1\leqslant t\leqslant t_1$. We claim that

$$y(t)>M,\ t\geqslant t_1. \tag{2.5}$$

If (2.5) does not hold, then there exists a $t^*>t_1$ such that $y(t)>M$ for $\rho t_1\leqslant t<t^*$ and $y(t^*)=M$. Using (1.7) and (2.2), we obtain

$$\begin{aligned} M=y(t^*) &= z(t^*)+R(t^*)y(\gamma t^*)+\int_{\frac{\alpha}{\beta}t^*}^{t^*}\frac{Q(s)}{s}y(\beta s)ds \\ &> \left(R(t^*)+\int_{\frac{\alpha}{\beta}t^*}^{t^*}\frac{Q(s)}{s}y(\beta s)ds\right)M\geqslant M. \end{aligned}$$

This is a contradiction and so (2.5) holds. Let $\lim_{t\to\infty}z(t)=l$. There exist two possible cases:

Case 1. $l=0$. There exists a $T_1>t_1$ such that $z(t)<M/2$ for $t\geqslant T_1$. Then for any $\bar{t}>T_1$, we have

$$\left(\ln\frac{1}{\rho}\right)^{-1}\int_{\bar{t}}^{\frac{t}{\rho}}\frac{z(s)}{s}ds\leqslant M<y(t),\ t\in[\bar{t},\bar{t}/\rho].$$

Case 2. $l > 0$. Then $z(t) \geqslant l$ for $t > t_1$. From (1.7), (2.2) and (2.5), it follows that

$$y(t) \geqslant l + R(t)y(\gamma t) + \int_{\frac{\alpha}{\beta}t}^{t} \frac{Q(s)}{s} y(\beta s)ds \geqslant l + M,\ t \geqslant t_1.$$

By induction, it is easy to see that $y(t) \geqslant kl + M$ for $t \geqslant \frac{t_1}{\rho^{k-1}}$ $(k = 1, 2, \ldots)$, and so $\lim_{t\to\infty} y(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{z(s)}{s} ds \leqslant 2z(T) < y(t),\ t \in [T, T/\rho].$$

Cases 1 and 2 show that

$$y(t) > \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{z(s)}{s} ds,\ t \in [T, T/\rho].$$

Now we prove that

$$y(t) > \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{z(s)}{s} ds,\ t \geqslant T/\rho. \tag{2.6}$$

Otherwise, there exists a $t^* > T/\rho$ such that

$$y(t^*) = \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t^*}{\rho}} \frac{z(s)}{s} ds,\ \ y(t) > \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{z(s)}{s} ds,\ \ t \in (T/\rho, t^*).$$

This implies by (1.7) and (2.2) that

$$\begin{aligned}
&\left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t^*}{\rho}} \frac{z(s)}{s} ds \\
&= z(t^*) + R(t^*)y(\gamma t^*) + \int_{\frac{\alpha}{\beta}t^*}^{t^*} \frac{Q(s)}{s} y(\beta s)ds \\
&> \left(\ln \frac{1}{\rho}\right)^{-1} \int_{t^*}^{\frac{t^*}{\rho}} \frac{z(s)}{s} ds + \left(R(t^*) + \int_{\frac{\alpha}{\beta}t^*}^{t^*} \frac{Q(s)}{s} ds\right) \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{t^*} \frac{z(s)}{s} ds \\
&\geqslant \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{\frac{t^*}{\rho}} \frac{z(s)}{s} ds.
\end{aligned}$$

This is a contradiction and so (2.6) holds. Thus, for $t > T/\rho$, we have

$$y(\alpha t) \geqslant \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{t} \frac{z(s)}{s} ds. \tag{2.7}$$

Substituting (2.7) into (2.4) leads to

$$z'(t) + \frac{H(t)}{t} \left(\ln \frac{1}{\rho}\right)^{-1} \int_{T}^{t} \frac{z(s)}{s} ds \leqslant 0,\ t > T/\rho.$$

Hence

$$z'(t)+\frac{H(t)}{t^2}\left(\ln\frac{1}{\rho}\right)^{-1}\int_T^t z(s)ds\leqslant 0,\ t>T/\rho.$$

Put

$$w(t)=\int_T^t z(s)ds,\ t>T/\rho.$$

One can easily find that $w'(t)=z(t), w''(t)=z'(t)$ and

$$w''(t)+\left(\ln\frac{1}{\rho}\right)^{-1}\cdot\frac{H(t)}{t^2}w(t)\leqslant 0,\ t>T/\rho.$$

By Lemma 2.4 in [15], Eq. (2.3) has an eventually positive solution. This is a contradiction and the proof is complete. □

Lemma 2.2. *Assume that (1.3), (1.4) and (1.6) hold. Let $y(t)$ be an eventually positive solution of the inequality (2.1) and $z(t)$ be defined by (2.2). Then*

$$z'(t)\leqslant 0,\quad z(t)>0\quad \textit{and}\quad z'(t)+\frac{H(t)}{t}z(\alpha t)\leqslant 0. \tag{2.8}$$

Proof. (2.4) implies that $z(t)$ is eventually decreasing. In view of $y(t)\geqslant z(t)$, (2.4) yields

$$z'(t)+\frac{H(t)}{t}z(\alpha t)\leqslant 0.$$

Now we prove $z(t)>0$. For otherwise, then eventually $z(t)<0$, and there exist $t_2\geqslant t_1$ and $c>0$ such that $z(t)\leqslant -c$ for $t\geqslant t_2$, that is,

$$y(t)\leqslant -c+R(t)y(\gamma t)+\int_{\frac{\alpha}{\beta}t}^t\frac{Q(s)}{s}y(\beta s)ds, t\geqslant t_2. \tag{2.9}$$

We consider the following two possible cases:

Case 1. $y(t)$ is unbounded, that is, $\limsup_{t\to\infty}y(t)=\infty$. Thus, there exists a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $s_n\geqslant \frac{t_2}{\rho}, n=1,2,\ldots, s_n\to\infty, y(s_n)\to\infty$ as $n\to\infty$, and $y(s_n)=\max\{y(t): t_2\leqslant t\leqslant s_n\}, n=1,2,\ldots$. From (1.6) and (2.9), it follows that

$$\begin{aligned} y(s_n) &\leqslant -c+R(s_n)y(\gamma s_n)+\int_{\frac{\alpha}{\beta}s_n}^{s_n}\frac{Q(s)}{s}y(\beta s)ds\\ &< -c+y(s_n)\left[R(s_n)+\int_{\frac{\alpha}{\beta}s_n}^{s_n}\frac{Q(s)}{s}ds\right]\\ &\leqslant -c+y(s_n). \end{aligned}$$

This is a contradiction.

Case 2. $y(t)$ is bounded, i.e., $\limsup_{t\to\infty}y(t)=l<\infty$. Choose a sequence of points $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n\to\infty$ and $y(\sigma_n)\to l$ as $n\to\infty$. Let $y(\sigma_n^*)=\max\{y(s):\rho\sigma_n\leqslant s\leqslant$

$\rho\sigma_n\}, \sigma_n^* \in [\rho\sigma_n, \rho\sigma_n], n = 1, 2, ...,$ where $\rho = \max\{\alpha, \beta, \gamma\}$. Then $\sigma_n^* \to \infty$ as $n \to \infty$ and $\limsup_{t\to\infty} y(\sigma_n^*) \leqslant l$. Thus, by (1.6) and (2.9), we get

$$\begin{aligned} y(\sigma_n) &\leqslant -c + R(\sigma_n)y(\gamma\sigma_n) + \int_{\frac{\alpha}{\beta}\sigma_n}^{\sigma_n} \frac{Q(s)}{s} y(\beta s)ds \\ &< -c + y(\sigma_n^*)\left[R(\sigma_n) + \int_{\frac{\alpha}{\beta}\sigma_n}^{\sigma_n} \frac{Q(s)}{s} ds\right] \\ &\leqslant -c + y(\sigma_n^*). \end{aligned}$$

Taking the superior limit as $n \to \infty$ yields

$$l \leqslant -c + \limsup_{t\to\infty} y(\sigma_n^*) \leqslant -c + l,$$

which is also a contradiction. The proof of Lemma 2.2 is complete. □

3 Main results

In this section, we will establish some Hille type oscillation criteria for Eq. (1.2) by using the lemmas in Section 2 and the following Lemma which is due to E. Hille [7]. Some examples are also given to illustrate the applications of our results.

Lemma 3.1. [7] *Consider the ordinary differential equation*

$$y''(t) + p(t)y(t) = 0, \ t \geqslant t_0, \tag{3.1}$$

where $p(t) \in C([t_0, \infty), R^+)$. Then all solutions of Eq. (3.1) oscillate if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4}.$$

Theorem 3.2. *Assume that (1.3), (1.4) and (1.5) hold and that*

$$\liminf_{t\to\infty} t \int_t^\infty \frac{H(s)}{s^2} ds > \frac{1}{4} \ln\frac{1}{\rho}. \tag{3.2}$$

Then all solutions of Eq. (1.2) oscillate.

Proof. Suppose that Eq. (1.2) has an eventually positive solution $y(t)$. Let $z(t)$ be defined by (2.2). Then by Lemma 2.2, $z(t) > 0$ eventually. On the other hand, By Lemma 3.1, (3.2) implies that all solutions of Eq. (2.3) oscillate. By Lemma 2.1, it follows that $z(t) < 0$. This contradiction completes the proof. □

Example 3.3. Consider the differential equation

$$\frac{d}{dt}[x(t) - 0.97x(e^{-\pi}t)] + \frac{1.97 + \frac{3}{200\pi}}{t} x(e^{-\frac{5\pi}{2}}t) - \frac{3}{200\pi t} x(e^{-\frac{\pi}{2}}t) = 0, \ t \geqslant 2. \tag{3.3}$$

Simple calculation shows that

$$R(t)+\int_{\frac{\alpha}{\beta}t}^{t}\frac{Q(s)}{s}ds=1,\quad \liminf_{t\to\infty} t\int_{t}^{\infty}\frac{H(s)}{s^2}ds=1.97>\frac{1}{4}\ln\frac{1}{\rho}=\frac{5\pi}{8}.$$

One can easily see that Eq. (3.3) satisfies all conditions of Theorem 3.2, and so every solution of Eq. (3.3) oscillates. Indeed, $x(t)=\sin(\ln t)$ is such a solution.

Theorem 3.4. *Assume that (1.3), (1.4), (1.6), and (3.2) hold, and that*

$$R(\alpha t)H(t)\geqslant H(\gamma t). \tag{3.4}$$

Then every solution of Eq. (1.2) oscillates.

Proof. Assume, for the sake of contradiction, that Eq. (1.2) has a non-oscillatory solution $y(t)$, we shall assume that $y(t)$ is eventually positive. The case where $y(t)$ is eventually negative is similar and will be omitted. Let $z(t)$ be defined by (2.2). Then, by Lemma 2.2, there exists $t_1\geqslant t_0$ such that

$$z(t)>0,\quad \text{for } t\geqslant t_1. \tag{3.5}$$

Thus, in view of (3.4), we have

$$z'(t)=-\frac{H(t)}{t}y(\alpha t)\leqslant-\frac{H(t)}{t}z(\alpha t)-\frac{H(\gamma t)}{t}y(\alpha\gamma t)=\frac{H(t)}{t}z(\alpha t)-\frac{d}{dt}z(\gamma t),$$

which implies that $z(t)$ is a positive solution of the inequality

$$\frac{d}{dt}[w(t)-w(\gamma t)]+\frac{H(t)}{t}w(\alpha t)\leqslant 0. \tag{3.6}$$

This shows that all conditions of Lemma 2.2 are satisfied, hence, $u(t)=z(t)-z(\gamma\, t)$ is eventually positive. On the other hand, by Lemma 3.1 and noting that (3.6) all conditions of Lemma 2.2 are also satisfied , there follows that $u(t)=z(t)-z(\gamma t)<0$ eventually, which is a contradiction and the proof is complete. □

Example 3.5. Consider the differential equation

$$\frac{d}{dt}\left[x(t)-\frac{1}{2}x\left(e^{-2}t\right)\right]+\frac{c+\sqrt{t}}{t}x\left(e^{-2}t\right)-\frac{c}{t}x\left(e^{-1}t\right)=0,\ t\geqslant 1, \tag{3.7}$$

where $0<c\leqslant 1/2$.

Standard calculation give us $\liminf_{t\to\infty} t\int_t^\infty \frac{H(s)}{s^2}ds=+\infty$ and $\frac{1}{4}\ln\frac{1}{\rho}=\frac{1}{2}$. And one can without difficulty find that Eq. (3.7) also satisfies the another conditions of Theorem 3.4 and so every solution of Eq. (3.7) is oscillatory.

Theorem 3.6. *Assume that (1.3), (1.4), (1.7), and (3.2) hold, and that* $\frac{Q(\alpha t)}{P(\beta\, t)-Q(\alpha t)}$ *is non-decreasing. Also suppose that there exist nonnegative constants* h_1 *and* h_2 *such that*

$$H(t)R(\alpha t)\leqslant h_1H(\gamma t), \tag{3.8}$$

$$H(t)Q(\alpha t) \leqslant h_2[P(\beta t) - Q(\alpha t)], \tag{3.9}$$

and

$$h_1 + h_2 \ln \frac{\beta}{\alpha} = 1. \tag{3.10}$$

Then every solution of Eq. (1.2) oscillates.

Proof. Otherwise, Eq. (1.2) has an eventually positive solution $y(t)$. Let $z(t)$ be defined by (2.2). From Lemma 2.1, it follows that $z(t) < 0$. And, by (3.9) and (3.10), we have

$$\begin{aligned} z'(t) &= -\frac{1}{t}[P(t) - Q(\frac{\alpha}{\beta}t)]y(\alpha t) = -\frac{H(t)}{t}y(\alpha t) \\ &= -\frac{H(t)}{t}[z(\alpha t) + R(\alpha t)y(\alpha\gamma t) - \int_{\frac{\alpha}{\beta}t}^{t} \frac{Q(\alpha s)}{s} y(\alpha\beta s)ds] \\ &\geqslant -\frac{H(t)}{t}z(\alpha t) - \frac{h_1}{t}H(t)y(\alpha\gamma t) - \frac{H(t)}{t}\int_{\frac{\alpha}{\beta}t}^{t} \frac{Q(\alpha s)}{P(\beta s) - Q(\alpha s)} \frac{d}{ds}[-z(\beta s)]\,ds \\ &\geqslant -\frac{H(t)}{t}z(\alpha t) + h_1\frac{d}{dt}[z(\gamma t)] - \frac{h_2}{t}[z(\alpha t) - z(\beta t)]. \end{aligned}$$

That is

$$\frac{d}{dt}[z(t) - h_1 z(\gamma t)] + \frac{H(t)+h_2}{t}z(\alpha t) - \frac{h_2}{t}z(\beta t) \geqslant 0.$$

This implies that $-z(t)$ is a positive solution of the inequality

$$\frac{d}{dt}[w(t) - h_1 w(\gamma t)] + \frac{H(t)+h_2}{t}w(\alpha t) - \frac{h_2}{t}w(\beta t) \leqslant 0,$$

which will yield a contradiction by Lemma 2.1 and Lemma 2.2. Therefore, the proof of Theorem 3.6 is complete. □

Example 3.7. If we take $h_1 = h_2 = 1/2$, then the equation

$$\frac{d}{dt}\left[x(t) - \frac{et+2}{2(et+3)}x(e^{-1}t)\right] + \frac{1}{t}x(e^{-1}t) - \frac{1}{2t}x(t) = 0,\ t \geqslant e \tag{3.11}$$

satisfies the conditions of Theorem 3.6. Hence, all solutions of the equation oscillate.

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