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APPROXIMATE CONTROLLABILITY OF NONAUTONOMOUS PAST DELAY EQUATIONS

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Abstract

In this paper, we study the approximate controllability of partial differential equations with nonautonomous past delay in L^p -phase spaces. We illustrate our abstract results by the approximate controllability of a dynamical population equation.

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1 Introduction

In this paper, we study the approximate controllability of the controlled system with nonautonomous past

$$\begin{cases} x'(t) = Ax(t) + \Phi(\tilde{x}_t) + Bu(t), & t \geq 0, \\ x(0) = x, \quad x_0 = g. \end{cases} \tag{1.1}$$

Here the operator $(A, D(A))$ is a generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ in a Banach space X, the delay $r \leq +\infty$, and $\Phi : D(\Phi) \subset L^p([-r,0],X) \longrightarrow X,\ p \geq 1$, is an unbounded linear

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operator. Moreover, the modified history function $\tilde{x}_t$ is defined by

$$\tilde{x}_t(s) := \begin{cases} V(s,0)x(t+s) & \text{if } t+s > 0, \\ V(s,s+t)g(t+s) & \text{if } t+s \leq 0, \end{cases}$$

where $(V(t,s))_{-r\leq t\leq s\leq 0}$ is an exponentially bounded backward evolution family on X. The control operator B is defined from a Banach space U to X. The wellposedness and the asymptotic behavior of the non controlled equation (i.e., $B = 0$) are studied in several papers, see [3, 4, 10, 11, 19].

The approximate controllability has been studied in the case of ordinary delay, i.e., $V(\cdot,\cdot) = Id$, see e.g., [6, 14, 15, 16, 18, 20, 21]. Actually, S. Krause has studied [16] the approximate controllability of general boundary systems, see also [8, 17, 21] for recent results.

Here, we characterize the approximate controllability of systems with nonautonomous past (1.1), as in the ordinary delay case. We give also sufficient conditions to obtain this aim. We remark here that the modification of the delay can act positively or negatively on the approximate controllability of these systems.

We end this paper by the study of the approximate controllability of the population equation

$$\begin{cases} z'(t,x) &= \Delta_N z(t,x) - dz(t,x) + \int_0^r b(a)v(t,a,x)da - b_1 v(t,r,x) \\ &\qquad -b_2(x)u(t,x), \quad t \geq 0, x \in \Omega, \\ v'(t,a,x) &= -\frac{\partial}{\partial a}v(t,a,x) + \Delta_D v(t,a,x) - dv(t,a,x) - b(a)v(t,a,x),\ t \geq 0, \\ &\qquad x \in \Omega, 0 \leq a \leq r, \\ v(t,0,x) &= f(x)z(t,x), \quad t \geq 0,\ x \in \Omega, \\ v(t,a,x) &= \frac{\partial}{\partial n}z(t,x) = 0, \quad t \geq 0,\ 0 \leq a \leq r, x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $z(t,x)$ is the density of the population at time t and position $x \in \Omega$, and $v(t,a,x)$ is the density of the subpopulation of pregnant individuals with time of gestation a, that at time t is in position x. The control u is an external action on the total population z, for more details on this equation, see [3, 12]. In [3], we have shown how this population equation can be fitted in the abstract form (2.2). Here, we show that if b_2 is a bounded function on Ω, the population equation (3.5) is approximately controllable.

2 Preliminaries

Recall some definitions and basic results needed for the elaboration of this work, for more details, see [11, 13].

Definition 2.1. *A family $(V(t,s))_{t\leq s,t,s\in I}$ of bounded linear operators on a Banach space X is called an (exponentially bounded backward) evolution family if*

(i) $V(t,\tau)V(\tau,s) = V(t,s), V(t,t) = Id$ *for all* $t \leq \tau \leq s, t, \tau, s \in I$,

(ii) *the mapping* $(t,s) \longmapsto V(t,s)$ *is strongly continuous,*

(iii) $\|V(t,s)\| \leq Me^{w(s-t)}$ *for some* $M \geq 1, w \in \mathbb{R}$ *and all* $t \leq s, t, s \in I$.

Here $I = [-r,0]$ *or* $I = (-\infty, 0]$.

We will use evolution semigroup techniques for which we refer to [5]. To that purpose, we extend $(V(t,s))_{t\leq s,t,s\in I}$ to an evolution family $(\tilde{V}(t,s))_{t\leq s}$ on $\mathbb{R}$.

Definition 2.2. *(1) The evolution family* $(V(t,s))_{t\leq s,t,s\in I}$ *on* X *is extended to an evolution family* $(\tilde{V}(t,s))_{t\leq s}$ *by setting*

$$\tilde{V}(t,s) := \begin{cases} V(-r,-r), & t \leq s < -r, \ (\text{if } r < \infty) \\ V(-r,s), & t < -r, s \in I, \ (\text{if } r < \infty) \\ V(t,s), & t \leq s, t, s \in I, \\ V(t,0), & t \leq 0 \leq s, \\ V(0,0), & 0 \leq t \leq s. \end{cases}$$

(2) On the space $\tilde{E} := L^p(\mathbb{R}, X)$, *we define the corresponding evolution semigroup* $(\tilde{T}(t))_{t\geq 0}$ *by* $(\tilde{T}(t)\tilde{f})(s) := \tilde{V}(s,s+t)\tilde{f}(s+t), \ \tilde{f} \in \tilde{E}, s \in \mathbb{R}, t \geq 0$.

It is easy to prove that the semigroup $(\tilde{T}(t))_{t\geq 0}$ is strongly continuous on $\tilde{E}$. We denote its generator by $(\tilde{G}, D(\tilde{G}))$, where $D(\tilde{G})$ is a dense subset of $C_0(\mathbb{R}, X)$, the space of functions vanishing at infinity.

Since $(\tilde{G}, D(\tilde{G}))$ is a local operator, we can define its restriction the space $E := L^p(I, X)$ by

$$D(G) := \{\tilde{f}_{|I} : \tilde{f} \in D(\tilde{G})\}, \quad Gf := (\tilde{G}\tilde{f})_{|I}, \quad f = \tilde{f}_{|I} \in D(G).$$

This operator G is not a generator on E. However, if one identifies E with the subspace $\{f \in \tilde{E} : f(s) = 0, \ \forall s \notin I\}$, then E remains invariant under $(\tilde{T}(t))_{t\geq 0}$. As a consequence we obtain the following lemma.

Lemma 2.3. *The semigroup* $(T_0(t))_{t\geq 0}$ *induced by* $(\tilde{T}(t))_{t\geq 0}$ *on* E *is*

$$(T_0(t)f)(s) = \begin{cases} 0, & s+t > 0, \\ V(s,s+t)f(s+t), & s+t \leq 0, \end{cases} \tag{2.1}$$

and its generator is given by $G_0 = G$, $D(G_0) = \{f \in D(G) : f(0) = 0\}$.

It is shown in [2, 3, 11, 13] that the wellposedness of

$$\begin{cases} x'(t) = Ax(t) + \Phi(\tilde{x}_t), & t \geq 0, \\ x(0) = x, & x_0 = g \end{cases} \tag{2.2}$$

is equivalent to show that the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & G \end{pmatrix}$$

on the domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times D(G) : f(0) = x \right\}$$

generates a strongly continuous semigroup in the product Banach space $\mathcal{E} = X \times L^p([-r,0],X)$, $p \geq 1$. It has been shown in these references that this occurs for operators $\Phi : \mathcal{C}([-r,0],X) \cap L^p([-r,0],X) \longrightarrow X$ given by

$$\Phi(f) := \int_{-r}^{0} d\eta(\theta) f(\theta) \tag{2.3}$$

with $\eta : [-r,0] \longrightarrow \mathcal{L}(X)$ is of bounded variation such that $|\eta|([-r,0]) < \infty$, where $|\eta|$ is the positive Borel measure of $[-r,0]$ defined by the total variation on η.

To obtain the aim of this paper, we need the following results, see [11].

Lemma 2.4. *i) For each $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0(T_0)$, we define the bounded operator $\varepsilon_\lambda : X \longrightarrow E$ by*

$$(\varepsilon_\lambda x)(s) := e^{\lambda s} V(s,0)x, \qquad s \leq 0, x \in X.$$

Then $\varepsilon_\lambda x$ is an eigenvector of G with eigenvalue λ for every $x \in X$.
ii) For $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0(T_0)$, we have that

$$\lambda \in \rho(\mathcal{A}) \quad \textit{if and only if} \quad \lambda \in \rho(A + \Phi\varepsilon_\lambda).$$

Moreover, for these $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda,\mathcal{A})$ is given by

$$R(\lambda,\mathcal{A}) := \begin{pmatrix} r_\lambda & r_\lambda \Phi R(\lambda, G_0) \\ \varepsilon_\lambda r_\lambda & (\varepsilon_\lambda r_\lambda \Phi + Id) R(\lambda, G_0) \end{pmatrix} \tag{2.4}$$

with $r_\lambda := R(\lambda, A + \Phi\varepsilon_\lambda)$.

We consider a general controlled system

$$\begin{cases} X'(t) = \mathcal{A}X(t) + \mathcal{B}u(t), \quad t \geq 0, \\ X(0) = X_0, \end{cases} \tag{2.5}$$

where $\mathcal{A}$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Banach space X and $B : U \longrightarrow X$ a bounded linear control operator, with U is another Banach space. The characterization of the approximate controllability of (2.5) is given in the following lemma. For the proof, we refer to [9, Prop. 2.1].

Lemma 2.5. *The following assertions are equivalent:*
i) $(\mathcal{A},\mathcal{B})$ is approximately controllable.
ii) $\langle \mathcal{T}(t)\mathcal{B}u, x' \rangle = 0$ for all $t \geq 0$ and all $u \in U \Rightarrow x' = 0$.
iii) $\langle R(\lambda,\mathcal{A})\mathcal{B}u, x' \rangle = 0$ for all $\lambda \in \rho(\mathcal{A})$ and all $u \in U \Rightarrow x' = 0$.

3 Main Results

In this section we study the approximate controllability of the nonautonomous past delay controlled equation

$$\begin{cases} x'(t) = Ax(t) + \Phi(\tilde{x}_t) + Bu(t), \quad t \geq 0, \\ x(0) = x, \quad x_0 = g. \end{cases} \tag{3.1}$$

For this purpose, we consider the controlled Cauchy problem

$$\begin{cases} X'(t) = \mathcal{A}X(t) + \mathcal{B}u(t), \quad t \geq 0, \\ X(0) = X_0, \end{cases} \tag{3.2}$$

where $X(t) = \binom{x(t)}{\tilde{x}_t} \in \mathcal{E} = X \times L^p([-r,0],X)$, and $\mathcal{B}u(t) = \binom{Bu(t)}{0}$.

This Cauchy problem is wellposed and its mild solution is given by the variation of constants formula

$$X(t) = \mathcal{T}(t)X_0 + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} Bu(s) \\ 0 \end{pmatrix} ds, \quad t \geq 0.$$

For all $t \geq 0$, define the controllability operator

$$R_t : L^1([0,t],U) \longrightarrow E \times X, \quad R_t(u) := \int_0^t \mathcal{T}(t-s)\begin{pmatrix} Bu(s) \\ 0 \end{pmatrix} ds. \tag{3.3}$$

Definition 3.1. *By the approximate controllability (resp. X-approximate controllability, L_p-approximate controllability) of the system* (3.1)*, we design*

$$C\overline{\bigcup_{t\geq 0} rg(R_t)} = E \tag{3.4}$$

where $C = I$, $C = \Pi_1$ the projection on X and $C = \Pi_2$ the projection on L_p respectively.

We give now a characterization of the approximate controllability.

Proposition 3.2. *The equation* (3.1) *is approximately controllable if and only if*
$\langle R(\lambda, A+\Phi\varepsilon_\lambda)Bu, x'\rangle_{X,X^*} + \langle e^{\lambda\cdot}V(\cdot,0)R(\lambda, A+\Phi\varepsilon_\lambda)Bu, f'\rangle_{E,E^*} = 0$ *for all* $\lambda \in \rho(\mathcal{A})$ *and* $u \in U$ $\Rightarrow x' = f' = 0$.

Proof. As the approximate controllability of (3.1) is equivalent to one of (3.2), by Lemma 2.5 (iii), this is equivalent to

$$\langle R(\lambda,\mathcal{A})\begin{pmatrix} Bu \\ 0 \end{pmatrix}, \begin{pmatrix} x' \\ f' \end{pmatrix}\rangle = 0 \quad \text{for all} \quad \lambda \in \rho(\mathcal{A}) \quad \text{and} \quad u \in U \Rightarrow x' = f' = 0,$$

and by (2.4) this is equivalent to

$$\langle \begin{pmatrix} r_\lambda & r_\lambda \Phi R(\lambda, G_0) \\ \varepsilon_\lambda r_\lambda & (\varepsilon_\lambda r_\lambda \Phi + Id)R(\lambda, G_0) \end{pmatrix} \begin{pmatrix} Bu \\ 0 \end{pmatrix}, \begin{pmatrix} x' \\ f' \end{pmatrix}\rangle = 0 \ \text{ for all } \lambda \in \rho(\mathcal{A}) \text{ and } u \in U$$

$\Rightarrow x' = f' = 0$. Thus, we obtain the claim. □

The characterization of the L_p-approximate controllability is given in the following proposition.

Proposition 3.3. *The equation* (3.1) *is L_p-approximately controllable if and only if*

$$< e^{\lambda \cdot} V(\cdot,0) R(\lambda, A + \Phi\varepsilon_\lambda) Bu, f' >_{E,E^*} = 0 \text{ for all } \lambda \in \rho(\mathcal{A}) \text{ and } u \in U \Rightarrow f' = 0.$$

Proof. The condition (3.4) is equivalent to

$$\overline{\cup_{t\geq 0} \cup_{u\in U} \Pi_2 \mathcal{T}(t) \begin{pmatrix} Bu \\ 0 \end{pmatrix}} = E$$

which, by the Laplace transform and its uniqueness (see [1]), is equivalent to

$$\overline{\cup_{\lambda>\omega} \cup_{u\in U} \Pi_2 R(\lambda, \mathcal{A}) \begin{pmatrix} Bu \\ 0 \end{pmatrix}} = E.$$

This is equivalent to

$$\langle \Pi_2 R(\lambda, \mathcal{A}) \begin{pmatrix} Bu \\ 0 \end{pmatrix}, f' \rangle = 0 \quad \text{for all} \quad \lambda \in \rho(\mathcal{A}) \quad \text{and all} \quad u \in U \Rightarrow f' = 0,$$

and again by (2.4), we obtain our claim. □

By the same proof, we obtain the characterization of the X-approximate controllability (3.1).

Proposition 3.4. *The equation* (3.1) *is X-approximately controllable if and only if*

$$< R(\lambda, A + \Phi\varepsilon_\lambda) Bu, x' >_{X,X^*} = 0 \text{ for all } \lambda \in \rho(\mathcal{A}) \text{ and } u \in U \Rightarrow x' = 0.$$

We give sufficient conditions for the approximate controllability of the equation (3.1).

Proposition 3.5. *Assume that $rg(B)$ is dense in X. Then,*
(i) the equation (3.1) *is X-approximately controllable.*
(ii) span$\{e^{\lambda \cdot} V(\cdot,0)x, x \in X, \lambda \in \rho(\mathcal{A})\}$ is dense in E if and only if (3.1) *is L_p-approximately controllable if and only if*

Proof. (i) Let $< R(\lambda, A + \Phi\varepsilon_\lambda) Bu, x' >_{X,X^*} = 0$ for all $\lambda \in \rho(\mathcal{A})$ and $u \in U$. Since $rg(B)$ and $D(A)$ are dense and $R(\lambda, A + \Phi\varepsilon_\lambda)$ is bijective, then $x' = 0$.
(ii) Similarly, the assertion $< e^{\lambda \cdot} V(\cdot,0) R(\lambda, A + \Phi\varepsilon_\lambda) Bu, f' >_{E,E^*} = 0$ for all $\lambda \in \rho(\mathcal{A})$ and $u \in U$ becomes $< e^{\lambda \cdot} V(\cdot,0)x, f' >_{E,E^*} = 0$ for all
$\lambda \in \rho(\mathcal{A})$ and $x \in X$, and the additional assumption implies that $f' = 0$. The converse follows easily by Proposition 3.3 . □

We end this work by an example of a dynamical population system

$$\begin{cases} z'(t,x) & = \Delta_N z(t,x) - dz(t,x) + \int_0^r b(a) v(t,a,x) da - b_1 v(t,r,x) \\ & \quad -b_2(x) u(t,x), \quad t \geq 0, x \in \Omega, \\ v'(t,a,x) & = -\frac{\partial}{\partial a} v(t,a,x) + \Delta_D v(t,a,x) - dv(t,a,x) - b(a) v(t,a,x), t \geq 0, \\ & \quad x \in \Omega, 0 \leq a \leq r, \\ v(t,0,x) & = f(x) z(t,x), t \geq 0, x \in \Omega, \\ v(t,a,x) & = \frac{\partial}{\partial n} z(t,x) = 0, \quad t \geq 0, 0 \leq a \leq r, x \in \partial\Omega, \end{cases} \tag{3.5}$$

where $z(t,x)$ is the density of the population at time t and position $x \in \Omega$, and $v(t,a,x)$ is the density of the subpopulation of pregnant individuals with time of gestation a, that at time t is in position x, for more details on this equation, see [3, 12]. The control u is an external action on the total population z.

In [3] we have showed in details how to transform this population equation into the following abstract equation with nonautonomous past

$$\begin{cases} x'(t) = Ax(t) + \Phi(\tilde{x}_t) + Bu(t), \quad t \geq 0, \\ x(0) = x, \quad x_0 = g, \end{cases} \tag{3.6}$$

where the operator $A = \Delta_N - d$ generates an exponentially stable C_0-semigroup $(S(t))_{t\geq 0}$ on $X = L^2(\Omega)$.

The modified history function $\tilde{x}_t$ is

$$\tilde{x}_t(s,\cdot) := \begin{cases} V(s,0)x(t+s,\cdot) & \text{if } t+s > 0 \\ V(s,s+t)g(t+s,\cdot) & \text{if } t+s \leq 0, \end{cases}$$

where

$$V(t,s) = e^{-\int_{-s}^{-t} b(\sigma)d\sigma} e^{(s-t)(\Delta_D - d)}, \quad t \leq s \leq 0.$$

The delay operator Φ is given by

$$\Phi(\varphi) = \int_{-r}^{0} b(-s) f(\cdot)\varphi(s)ds - b_1 f(\cdot)\varphi(-r), \quad \varphi \in C([-r,0],X).$$

The control operator is $B : U = L^2(\Omega) \longrightarrow X, \quad Bu = b_2 u$, where b_2 and b_2^{-1} are bounded functions. Under these conditions, the operator is invertible. Hence, by Proposition 3.5, we have the following result.

Proposition 3.6. *The equation* (3.5) *is X and L_2-approximately controllable.*

Proof. By Proposition 3.5 (i), the equation (3.5) is X-approximately controllable.

To show the L_2-approximate controllability, we have to verify that $span\{e^{\lambda\cdot}V(\cdot,0)x, x \in X, \lambda \in \rho(\mathcal{A})\}$ is dense in $E = L^2([-r,0],L^2(\Omega))$. Let $\lambda_n, n \geq 1$, the sequence of eigenvalues of Dirichlet Laplacian $\triangle_D$ and $\{\phi_n, n \geq 1\}$ the basis of associated eigenvectors. For $f' \in E^* = L^2([-r,0],L^2(\Omega))$, assume that

$$< e^{\lambda\cdot}V(\cdot,0)\phi_n, f' >_E = \int_{-r}^{0} < e^{\lambda s}V(s,0)\phi_n, f'(s) >_X ds = 0$$

for all $n \geq 1$ and $\lambda \in \rho(\mathcal{A})$. Then

$$\int_{-r}^{0} e^{\lambda s} e^{-\int_0^{-s} b(\sigma)d\sigma} e^{-s(\Delta_D - d)} < \phi_n, f'(s) >_X ds = 0$$

for all $n \geq 1$ and $\lambda \in \rho(\mathcal{A})$, which gives

$$\int_{-r}^{0} e^{\lambda s} e^{-\int_0^{-s} b(\sigma)d\sigma} e^{-s(\lambda_n - d)} g_n'(s)ds = 0$$

for all $n \geq 1$ and $\lambda \in \rho(\mathcal{A})$, with $g'_n(s) :=< \phi_n, f'(s) >_X$ is a scalar function on $[-r, 0]$. Now, using Stone-Weierstrass we obtain that $g'_n = 0$ for all $n \geq 1$, and this yields that $f' = 0$. Thus, the population equation is L_2-approximately controllable. □

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