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SOLUTIONS OF A PAIR OF DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Abstract

In this paper, we consider the common solutions of a pair of differential equations and give some of their applications in the uniqueness problem of entire functions.

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1 Introduction and main results

In the study of the solutions of complex differential equations, the growth of a solution is a very important property. For linear differential equations of the form

$$f^{(n)}+a_{n-1}(z)f^{(n-1)}+\cdots+a_0(z)f=a(z), \tag{1.1}$$

where $a(z)$, $a_0(z),\cdots$, $a_{n-1}(z)$ are polynomials, it is known that any entire solution of (1.1) must be of finite order, and if some of the coefficients $a_j(z)(0\leq j\leq n-1)$ are replaced by transcendental entire functions, then the equation (1.1) has at least one solution of infinite order. This can be proved by mainly using the Wiman-Valiron theory (see[5], [6]).

It is assumed that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory (see [8]). We say that two entire functions f and g share the polynomial Q CM (counting multiplicities), if $f-Q$ and $g-Q$ have the same zeros with the same multiplicities. If $f(z_0)=z_0$, then z_0 is called a fixed point of f. We say that f and g have the same fixed points, if f and g share z CM.

In this paper, by using the Nevanlinna Theory (see [8]), we consider the common solutions of a pair of differential equations

$$f^{(n)}+a_{n-1}(z)f^{(n-1)}+\cdots+a_0(z)f=a(z),$$

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$$f^{(n+1)}+b_n(z)f^{(n)}+\cdots+b_0(z)f=b(z),$$

with some special entire coefficients $\{a_i\}$ and $\{b_j\}$, $i=0,\cdots,n-1,\ j=0,\cdots,n$, and give some of their applications in the uniqueness theorem of meromorphic functions. The results improve the theorems given by Yang (see [9]).

Theorem 1.1. *Let α be an entire function and β be a nonconstant entire function, Q be a polynomial of degree $q<n$, $n(\geq 2)$ be a positive integer. Then the following pair of differential equations*

$$f^{(n)}-e^{\alpha}f=Q, \quad f'-e^{\beta}f=Q-Q', \tag{1.2}$$

has no common solutions.

Corollary 1.2. *If a pair of differential equations*

$$f^{(n)}-e^{\alpha}f=z, \quad f'-e^{\beta}f=z-1, \tag{1.3}$$

has common solutions, where α and β are entire functions and $n(\geq 2)$ is a integer, then α and β must be constants and f assumes the form $f=Ce^z-z$, C is a nonzero constant.

Proof. By Theorem 1.1, β must be a constant, and so f is of finite order. From the first equation of (1.3), we know that α is also a constant. Solving (1.3), we get $f=Ce^z-z$, C is a nonzero constant.

Theorem 1.3. *Let α and β be two nonconstant entire functions, $e^{\alpha-\beta}\not\equiv 1$. Then the following pair of differential equations*

$$f^{(n)}-e^{\alpha}f=z, \quad f^{(n+1)}-e^{\beta}f=z, \tag{1.4}$$

has no common solutions.

Similarly as the proof of Corollary 1.2, we have the following result.

Corollary 1.4. *If a pair of differential equations*

$$f^{(n)}-e^{\alpha}f=z, \quad f^{(n+1)}-e^{\beta}f=z,$$

has common solutions, where α and β are entire functions, then f assumes the form $f=Ce^z-z$.

2 Some Lemmas

Lemma 2.1. *[7] Let f be a nonconstant entire function, Q be a polynomial of degree q, and $n>q$. If f, f', $f^{(n)}$ share Q CM, then*

$$m(r,\frac{1}{f-Q})=m(r,\frac{1}{f-Q'})+S(r,f)=S(r,f).$$

Lemma 2.2. *[3] Let f be a transcendental meromorphic function, k be a positive number, P be a nonzero polynomial. For any $\varepsilon>0$, then*

$$T(r,f)\leq(1+\frac{1}{k}+\varepsilon)\{N(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}-P})\}+S(r,f).$$

Lemma 2.3. *Let f be a common entire solution of a pair of differential equations*

$$\frac{f^{(n)}-Q}{f-Q}=e^{\alpha},\quad \frac{f'-Q}{f-Q}=e^{\beta},$$

where $n(\geq 2)$ be a positive integer and Q be a nonconstant polynomial of degree q, $n>q$, α and $\beta(\not\equiv\alpha)$ are nonconstant entire functions. Then

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,f).$$

Proof. Obviously f must be a transcendental entire function. From the conditions of Lemma 2.3, we know that f, f', $f^{(n)}$ share Q CM. Set

$$A(z)=\frac{f^{(n)}-f'}{f-Q},\quad B=\frac{f^{(n)}-Q}{f'-Q}.$$

Then from the lemma of logarithmic derivatives , Lemma 2.1, and $n>q$, we have

$$\begin{aligned} m(r,A)=m(r,\frac{f^{(n)}-f'-Q'+Q'}{f-Q}) &= m(r,\frac{f^{(n)}-Q^{(n)}+Q^{(n)}-Q'}{f-Q})+S(r,f)\\ &\leq m(r,\frac{Q'}{f-Q})+S(r,f). \end{aligned}$$

Hence we get

$$T(r,A)=S(r,f).$$

By the second fundamental theorem of Nevanlinna theory, we have

$$\begin{aligned} T(r,B) &\leq N(r,B)+N(r,\frac{1}{B})+N(r,\frac{1}{B-1})+S(r,B)\\ &\leq N(r,\frac{1}{A})+S(r,B)\\ &\leq T(r,A)=S(r,f). \end{aligned}$$

Noticing that $e^{\beta}=\frac{A}{B-1}$ and $e^{\alpha}=Be^{\beta}$, we get $T(r,e^{\beta})=S(r,f)$ and $T(r,e^{\alpha})=S(r,f)$, Lemma 2.3 is thus proved.

Lemma 2.4. *Let f be a common entire solution of a pair of differential equations*

$$\frac{f^{(n)}-z}{f-z}=e^{\alpha},\quad \frac{f^{(n+1)}-z}{f-z}=e^{\beta},$$

where α and $\beta\not\equiv\alpha$ are nonconstant entire functions. Then

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,f).$$

Proof. If $n=1$, we easily get the result from Lemma 2.3. If $n\geq 2$, set

$$A(z)=\frac{f^{(n+1)}-f^{(n)}}{f-z},\quad B=\frac{f^{(n+1)}-z}{f^{(n)}-z}.$$

By the same way as the proof of Lemma 2.3, we get $T(r,A)=S(r,f)$ and $T(r,B)=S(r,f)$.

Noticing that $e^{\beta}=\frac{A}{B-1}$ and $e^{\alpha}=Be^{\beta}$, we get $T(r,e^{\beta})=S(r,f)$ and $T(r,e^{\alpha})=S(r,f)$, Lemma 2.4 is thus proved.

3 Proof of Theorems

Proof of Theorem 1.1. Suppose that the pair of (1.2) has a solution $f(z)$, then $f(z)$ must be a nonconstant entire function which satisfies

$$f^{(n)}-e^{\alpha}f=Q, \quad f'-e^{\beta}f=Q-Q'. \tag{3.1}$$

Set $f=F-Q$, then F satisfies

$$\frac{F^{(n)}-Q}{F-Q}=e^{\alpha}, \quad \frac{F'-Q}{F-Q}=e^{\beta}. \tag{3.2}$$

From Lemma 2.3, we have

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,F).$$

Taking the derivative on both sides of the second equation of (3.1) gives

$$f''=e^{2\beta}f+\beta'e^{\beta}f+e^{\beta}(Q-Q')+(Q'-Q'').$$

In the same manner, we have

$$f'''=(e^{3\beta}+3\beta'e^{2\beta}+\beta''e^{\beta}+\beta'^2e^{\beta})f+(e^{2\beta}+2\beta'e^{\beta})(Q-Q')+e^{\beta}(Q'-Q'')+(Q''-Q'''). \tag{3.3}$$

By inducting in number n and noticing $n>q$, we obtain that

$$f^{(n)}=(e^{n\beta}+p_1e^{(n-1)\beta}+\cdots+p_{n-1}e^{\beta})f+q_1e^{(n-1)\beta}+q_2e^{(n-2)\beta}+\cdots+q_{n-1}e^{\beta}+q_n, \tag{3.4}$$

where $p_i(i=1,2,\cdots n-1)$ and $q_j(j=1,2,\cdots n)$ are the differential polynomials in β and Q. Set

$$e^{n\beta}+p_1e^{(n-1)\beta}+p_2e^{(n-2)\beta}+\cdots+p_{n-1}e^{\beta}=M, \tag{3.5}$$

$$q_1e^{(n-1)\beta}+q_2e^{(n-2)\beta}+\cdots+q_{n-1}e^{\beta}+q_n=N, \tag{3.6}$$

then $f^{(n)}=Mf+N$. From (3.2), we know that f and $f^{(n)}-Q$ have the same zeros with same multiplicities, by Lemma 2.2, for any $\varepsilon>0$, then

$$\begin{aligned} T(r,f) &\leq (1+\frac{1}{k}+\varepsilon)\{N(r,\frac{1}{f})+N(r,\frac{1}{f^{(n)}-Q})\}+S(r,f)\\ &\leq 2(1+\frac{1}{k}+\varepsilon)N(r,\frac{1}{f})+S(r,f). \end{aligned}$$

If $N\equiv Q$, then from (3.6), we get

$$(n-1)T(r,e^{\beta})=S(r,e^{\beta}).$$

This contradicts that β is a nonconstant entire function. If $N\not\equiv Q$, then

$$\begin{aligned} T(r,f) &\leq 2(1+\frac{1}{k}+\varepsilon)N(r,\frac{1}{f})+S(r,f)\\ &\leq 2(1+\frac{1}{k}+\varepsilon)N(r,\frac{1}{N-Q})+S(r,f). \end{aligned}$$

From Lemma 2.3, we get

$$T(r,f)=S(r,f).$$

This is a contradiction. Theorem 1.1 is thus proved.

Proof of Theorem 1.3

Suppose that the pair of equations (1.4) has a solution $F(z)$, then $F(z)$ must be a non-constant entire function which satisfies

$$F^{(n)}-e^{\alpha}F=z,\quad F^{(n+1)}-e^{\beta}F=z. \tag{3.7}$$

By differentiating the first equation and combining with the second equation of (3.7), we obtain

$$F'e^{\alpha}+Fe^{\alpha}\alpha'=e^{\beta}F+z-1.$$

Let $p=e^{-\alpha}$, $G=e^{\beta-\alpha}-\alpha'$. Then $F'=GF+(z-1)p$ and

$$F''=(G^2+G')F+p[(z-1)G+1]+(z-1)p'.$$

By mathematical induction, for any positive integer n,

$$F^{(n)}=F(G^n+H_{n-1})+pH_{n-1}+p'H_{n-2}+\cdots+(z-1)p^{(n-1)}. \tag{3.8}$$

(Here we denote by H_j a differential polynomial with degree j of G, which may not be the same each time it occurs). From (3.7) and (3.8), we have

$$e^{\alpha}F+z=F(G^n+H_{n-1})+pH_{n-1}+p'H_{n-2}+\cdots+(z-1)p^{(n-1)}, \tag{3.9}$$

$$e^{\alpha}=G^n+H_{n-1}+\frac{1}{F}\{pH_{n-1}+p'H_{n-2}+\cdots+(z-1)p^{(n-1)}-z\}. \tag{3.10}$$

If $pH_{n-1}+p'H_{n-2}+\cdots+(z-1)p^{(n-1)}\not\equiv z$, set $F=f-z$. Then (3.7) becomes

$$\frac{f^{(n)}-z}{f-z}=e^{\alpha},\quad \frac{f^{(n+1)}-z}{f-z}=e^{\beta}.$$

From Lemma 2.4, we have

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,f).$$

Together with (3.10), we have

$$\begin{aligned} T(r,F) &\leq nT(r,G)+\sum_{i=1}^{n-1}T(r,H_i)+\sum_{i=0}^{n-1}T(r,p^{(i)})+O(1)\\ &\leq S(r,F). \end{aligned}$$

This is a contradiction. Hence we get

$$pH_{n-1}+p'H_{n-2}+\cdots+(z-1)p^{(n-1)}\equiv z. \tag{3.11}$$

From (3.10), we get

$$e^{\alpha} \equiv (e^{\alpha-\beta})^n + l_1(e^{\alpha-\beta})^{n-1} + \cdots + l_{n-1}e^{\alpha-\beta} + l_n, \tag{3.12}$$

where $l_j(j = 1, \cdots n)$ are polynomials in z, α, β and their derivatives. If there exists an set I with $measI = +\infty$ such that

$$T(r, e^{\alpha-\beta}) = o\{T(r, e^{\alpha})\}, \quad r \in I, \tag{3.13}$$

then from (3.12) and (3.13), we have $T(r, e^{\alpha}) = o(T(r, e^{\alpha}))$: this is impossible. So there exists a set E with finite linear measure such that

$$T(r, e^{\alpha-\beta}) = O\{T(r, e^{\alpha})\}, \quad r \notin E, \tag{3.14}$$

and

$$\sum_{i=1}^{n} T(r, l_i) = O\{T(r, e^{\alpha-\beta})\}, \quad r \notin E.$$

From (3.12), we obtain

$$T(r, e^{\alpha}) = nT(r, e^{\alpha-\beta}) + o\{T(r, e^{\alpha-\beta})\}, \quad r \notin E. \tag{3.15}$$

On the other hand, from (3.11) and $p = e^{-\alpha}$, we have

$$ze^{\alpha} = H_{n-1} + \frac{p'}{p}H_{n-2} + \cdots + \frac{p^{(n-2)}}{p}H_1 + (z-1)\frac{p^{(n-1)}}{p}, \tag{3.16}$$

and

$$T(r, e^{\alpha}) = (n-1)T(r, e^{\alpha-\beta}) + o\{T(r, e^{\alpha-\beta})\}, \quad r \notin E. \tag{3.17}$$

From (3.15) and (3.17), we have

$$T(r, e^{\alpha}) = o\{T(r, e^{\alpha-\beta})\}, \quad r \notin E.$$

This is a contradiction. Thus we complete the proof of Theorem 1.3.

4 Applications

In 1986, Jank *et al.* (see [4]) proved the next result.

Theorem A[4]. Let f be a nonconstant meromorphic function, let $a \neq 0$ be a finite constant. If f, f', f'' share the value a CM, then $f = f'$.

In 1998, Gundersen and Yang(see [2]) proved that every solution of the differential equation

$$F^{(n)} - e^{\alpha(z)}F = 1$$

is infinite order, where $\alpha(z)$ is a nonconstant entire function. And they proved the following theorem.

Theorem B[2]. Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If the value a is shared by f, $f^{(n)}$, $f^{(n+1)}$, then $f = f'$.

Theorem A, B suggest the following question of Yi-Yang.

Question 1. Let f be a non-constant meromorphic function, let $a \neq 0$ be a finite constant, and let n and m be positive integers satisfying $n < m$. If f, $f^{(n)}$, $f^{(m)}$ share the value a CM, where n and m are not both even or both odd, must $f \equiv f^{(n)}$?

The following example shows that the answer to Question 1 is, in general, negative. Let n and m be positive integers satisfying $m > n+1$, and let b be a constant which satisfies $b^n = b^m \neq 1$. Set $a = b^n$ and $f = e^{bz} + a - 1$. Then f, $f^{(n)}$, $f^{(m)}$ share the value a CM, but $f \not\equiv f^{(n)}$.

Regarding to Theorem A, a natural question is:

Question 2. what can be said when the value a is replaced by a fixed point ?

Regarding to the question 2, there have been some results.

Theorem C[7]. Let f be a nonconstant entire function, $n \geq 2$ be a positive integer. If f, f', $f^{(n)}$ have the same fixed points, then $f = f'$.

Theorem D[1]. Let f be a nonconstant entire function, n be a positive integer. If f, $f^{(n)}$, $f^{(n+1)}$ have the same fixed points, then $f = f'$.

Using the theorems in our paper, we give Theorem C, D in a short proof.

Proof. From the conditions of Theorem C, we get

$$\frac{f^{(n)} - z}{f - z} = e^{\alpha}, \quad \frac{f' - z}{f - z} = e^{\beta}.$$

Set $F = f - z$, then we get the following pair of differential equations

$$F^{(n)} - e^{\alpha} F = z, \quad F' - e^{\beta} F = z - 1. \tag{4.1}$$

From Corollary 1.2, we get $F = Ce^z - z$, which implies that $f = Ce^z$ and $f = f'$, C is a nonzero constant. Similarly we get the proof of Theorem D.

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