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ON ABSOLUTE CESÀRO SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract

In this paper, a general theorem concerning the $\varphi - |C,1|_k$ summability factors of infinite series has been proved.

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1 Introduction

Let (φ_n) be a sequence of positive real numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By (t_n), we denote the n-th $(C,1)$ means of the sequence (na_n).The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty, \tag{1.1}$$

and it is said to be summable $\varphi - |C,1|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty. \tag{1.2}$$

If we take $\varphi = n$, then $\varphi - |C,1|_k$ summability reduces to $|C,1|_k$ summability.

2 The Known Result.

Concerning the $|C,1|_k$ summability factors, Mazhar [2] has proved the following theorem.

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Theorem 2.1. *If*

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} nlogn \mid \Delta^2 \lambda_n \mid = O(1), \tag{2.2}$$

$$\sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v} = O(logm) \quad as \quad m \to \infty, \tag{2.3}$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid C, 1 \mid_k$, $k \geq 1$.

3 The Main Result.

The aim of this paper is to generalize Theorem 2.1 for the $\varphi - \mid C, 1 \mid_k$ summability. Now we shall prove the following theorem.

Theorem 3.1. *Let* (φ_n) *be a sequence of positive real numbers and the conditions (2.1)-(2.2) of Theorem 2.1 are satisfied. If*

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} \mid t_v \mid^k = O(logm) \quad as \quad m \to \infty \tag{3.1}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1}} = O(\frac{\varphi_v^{k-1}}{v^k}), \tag{3.2}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - \mid C, 1 \mid_k$, $k \geq 1$.

It should be noted that if we take $\varphi_n = n$ in Theorem 3.1, then we get Theorem 2.1. Because in this case condition (3.1) reduces to condition (2.3) and condition (3.2) reduces to

$$\sum_{n=v}^{m} \frac{1}{n^2} = O(\frac{1}{v}), \tag{3.3}$$

but this always holds.

4 Proof of the Theorem 3.1.

Let T_n be the n-th $(C, 1)$ means of the sequence $(na_n\lambda_n)$. Applying Abel's transformation, we get that

$$\begin{aligned} T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v &= \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{r=0}^{v} r a_r + \frac{\lambda_n}{n+1} \sum_{r=0}^{n} r a_r \\ &= \frac{1}{n+1} \sum_{v=0}^{n-1} (v+1) \Delta\lambda_v t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}. \end{aligned}$$

Since $\mid T_{n,1}+T_{n,2}\mid^{k}< 2^{k}(\mid T_{n,1}\mid^{k}+\mid T_{n,2}\mid^{k})$, in order to complete the proof of the Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\mid T_{n,r}\mid^{k}<\infty, \quad for \quad r=1,2. \tag{4.1}$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\mid T_{n,1}\mid^{k} &= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\mid \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta\lambda_v(v+1)t_v\mid^{k}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\{\sum_{v=1}^{n-1}v\mid\Delta\lambda_v\mid\mid t_v\mid\}^{k}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\sum_{v=1}^{n-1}v\mid\Delta\lambda_v\mid\mid t_v\mid^{k}\times\{\sum_{v=1}^{n-1}v\mid\Delta\lambda_v\mid\}^{k-1}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\{\sum_{v=1}^{n}v\mid\Delta\lambda_v\mid\mid t_v\mid^{k}\}\\
&= O(1)\sum_{v=1}^{m}v\mid\Delta\lambda_v\mid\mid t_v\mid^{k}\{\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\}\\
&= O(1)\sum_{v=1}^{m}v\mid\Delta\lambda_v\mid\frac{\varphi_v^{k-1}}{v^k}\mid t_v\mid^{k}\\
&= O(1)\sum_{v=1}^{m-1}\mid\Delta(v\mid\Delta\lambda_v\mid)\mid\sum_{r=1}^{v}\frac{\varphi_r^{k-1}}{r^k}\mid t_r\mid^{k}+m\mid\Delta\lambda_m\mid\sum_{r=1}^{m}\frac{\varphi_r^{k-1}}{r^k}\mid t_r\mid^{k}\\
&= O(1)\sum_{v=1}^{m-1}\mid\Delta\lambda_v\mid logv+\sum_{v=1}^{m-1}v\mid\Delta^2\lambda_v\mid logv+m\mid\Delta\lambda_m\mid logm\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem 3.1. Finally,

$$\begin{aligned}
\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\mid T_{n,2}\mid^{k} &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\mid\lambda_n t_n\mid^{k}\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\mid t_n\mid^{k}\mid\sum_{v=n}^{\infty}\Delta\lambda_v\mid\\
&= O(1)\sum_{v=1}^{\infty}\mid\Delta\lambda_v\mid\sum_{n=1}^{v}\frac{\varphi_n^{k-1}}{n^k}\mid t_n\mid^{k}\\
&= O(1)\sum_{v=1}^{\infty}\mid\Delta\lambda_v\mid logv\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem 3.1. Therefore we get that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\mid T_{n,r}\mid^{k}= O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2.$$

This completes the proof of Theorem 3.1.

References

[1] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. Lond. Math. Soc.* **7** (1957), pp. 113-141.

[2] S. M. Mazhar, On $|C,1|_k$ summability factors of infinite series. *Indian J. Math.* **14** (1972), pp. 45-48.

[3] H. Seyhan, *The absolute summability methods* . Ph.D. Thesis, Kayseri (1995), pp. 1-57.