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POSITIVE SOLUTIONS OF THE SEMIPOSITONE STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

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Abstract

Let $\mathbb{T}$ be a time scale. We study the nonlinear Sturm-Liouville problem

$$-u^{\triangle\triangle}(t) = \lambda f(t, u(\sigma(t))), \quad t \in [0,1],$$
$$\alpha u(0) - \beta u^{\triangle}(0) = 0, \quad \gamma u(\sigma(1)) + \delta u^{\triangle}(\sigma(1)) = 0,$$

where $\alpha, \beta, \gamma, \delta \geq 0$, $\gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$. The inhomogeneous term $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$ is a sign-changing continuous function. Using Krasnosel'skiĭ's cone-theoretic theorem, we obtain existence theorems for at least one or two positive solutions.

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1 Introduction

We study

$$-u^{\triangle\triangle}(t) = \lambda f(t, u(\sigma(t))), \quad t \in [0,1], \tag{1.1}$$
$$\alpha u(0) - \beta u^{\triangle}(0) = 0, \quad \gamma u(\sigma(1)) + \delta u^{\triangle}(\sigma(1)) = 0, \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$, and $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$ is a sign-changing continuous function.

This manuscript is partially motivated by the results in the papers [1, 10, 12, 14]. In [1] Agarwal *et al.* showed the existence of positive solutions to semipositone (n, p) conjugate

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boundary value problems. In [10], Kosmatov showed the existence of at least one positive solution to the m-point eigenvalue problem $-[p(t)u'(t)]' = \lambda f(t,u(t)), 0 < t < 1, u'(0) = 0, \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1)$, where $\sum_{i=1}^{m-2} \alpha_i < 1$. Sun and Sun, [12], established the existence of at least one positive solution to the singular semipositone boundary value problem $-u'' + m^2 u = \lambda f(t,u) + g(t,u), 0 < t < 1, u'(0) = 0, u'(1) = 0$. Yu *et al.*, in [14], gave sufficient conditions for the existence of multiple positive solutions of the boundary value problem $x''' - \lambda f(t,x) = 0, 0 < t < 1, x(0) = x'(\eta) = x''(1) = 0$ where f is allowed to be singular at $t = 0$ or $t = 1$.

We present some basic definitions which can be found in Atici and Guseinov [2], Bohner and Peterson [3, 4], Hilger [7], and Kaymakcalan *et al.* [9].

A *time scale* $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the *forward jump operator*, σ, and the *backward jump operator*, ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be *right scattered*, and if $\sigma(t) = t$, t is said to be *right dense* (rd). If $\rho(t) < t$, t is said to be *left scattered*, and if $\rho(t) = t$, t is said to be *left dense* (ld). A function f is left-dense continuous, ld-continuous, if f is continuous at each left dense point in $\mathbb{T}$ and its right-sided limits exist at each right dense points in $\mathbb{T}$.

For $u\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume t is not left scattered if $t = \sup \mathbb{T}$), we define the *delta derivative* of $u(t)$, $u^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\left| u(\sigma(t)) - u(s) - u^{\triangle}(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$.

In [7], Hilger established the following result.

Theorem 1.1. *Assume that $f\colon \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$, (if $\mathbb{T}$ has a left-scattered maximum at m, let $t \in \mathbb{T} \setminus \{m\}$).*

(i) If f is differential at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differential at t with

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable at t and t is right-dense, then

$$f^{\triangle}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

The solution of the homogeneous dynamic equation (1.1) with the boundary conditions (1.2) is the Green's function

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\alpha t + \beta)(\gamma(\sigma(1) - \sigma(s)) + \delta), & t \leq s, \\ (\alpha \sigma(s) + \beta)(\gamma(\sigma(1) - t) + \delta), & \sigma(s) \leq t, \end{cases}$$

where $\rho = \gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$.

For the sake of completeness we present some properties of Green's function $G(t,s)$, which can be found in [4, 5, 6].

Lemma 1.2. *The Green's function satisfies*

(i) $G(t,s) \geq 0$, $(t,s) \in [0,\sigma(1)] \times [0,1]$;

(ii) $G(t,s) = G(\sigma(s),\rho(t))$, $t,s \in [\sigma(0),1]$;

(iii) $G(t,s) \leq G(\sigma(s),s)$, $(t,s) \in [0,\sigma(1)] \times [0,1]$;

(iv) $G(t,s) \geq \kappa_1 G(\sigma(s),s)$, $(t,s) \in \left[\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}\right] \times [0,1]$, *where*

$$\kappa_1 = \min\left\{\frac{\alpha\sigma(1)+4\beta}{4(\alpha\sigma(1)+\beta)}, \frac{\gamma\sigma(1)+4\delta}{4(\gamma\sigma(1)+\delta)}\right\};$$

(v) $G(t,s) \leq \Gamma$, $(t,s) \in [0,\sigma(1)] \times [0,1]$, *where*

$$\Gamma = \frac{1}{4}\begin{cases} \frac{\beta}{\alpha} + \frac{\delta}{\gamma} + \sigma(1), & \alpha\gamma \neq 0, \\ \frac{\delta}{\gamma} + \sigma(1), & \alpha = 0, \\ \frac{\beta}{\alpha} + \sigma(1), & \gamma = 0. \end{cases}$$

We assume that the set $[0,\sigma(1)]$ is such that

$$\zeta = \min\left\{\tau \in \mathbb{T}\colon \tau \geq \frac{\sigma(1)}{4}\right\} \quad \text{and} \quad \omega = \max\left\{\tau \in \mathbb{T}\colon \tau \leq \frac{3\sigma(1)}{4}\right\}$$

exist and satisfy

$$\frac{\sigma(1)}{4} \leq \zeta < \omega \leq \frac{3\sigma(1)}{4}.$$

We assume also that if $\sigma(\omega) = 1$, then $\sigma(\omega) < \sigma(1)$.

Define

$$\kappa = \min\left\{\kappa_1, \min_{s\in[\zeta,\omega]} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)}\right\}.$$

Consider the equation

$$-u^{\triangle\triangle}(t) = g(\sigma(t)), \quad t \in [0,1], \tag{1.3}$$

subject to the boundary conditions (1.2). The following result is obvious.

Lemma 1.3. *Let* $g \in C(\mathbb{T},\mathbb{R}^+)$. *Then*

$$u(t) = \int_0^{\sigma(1)} G(t,s)g(\sigma(s))\,\triangle s, \quad t \in [0,\sigma^2(1)],$$

is the unique nonnegative solution of (1.3), (1.2).

Let $g \equiv 1$ and define, for $t \in [0, \sigma^2(1)]$,

$$u_1(t) = \int_0^{\sigma(1)} G(t,s) \triangle s. \tag{1.4}$$

For the solution (1.4) we define the constants

$$A = \sup_{t \in [0,\sigma^2(1)]} u_1(t) = \sup_{t \in [0,\sigma^2(1)]} \int_0^{\sigma(1)} G(t,s) \triangle s$$

and

$$B = \sup_{t \in [0,\sigma^2(1)]} \int_\zeta^\omega G(t,s) \triangle s.$$

Definition 1.4. Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be closed and nonempty. Then $\mathcal{C}$ is said to be a cone if

1. $\alpha u + \beta v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$ and for all $\alpha, \beta \geq 0$, and
2. $u, -u \in \mathcal{C}$ implies $u \equiv 0$.

We introduce the Banach space $\mathcal{B} = C([0,\sigma^2(1)], \mathbb{R})$ with the norm $\|u\| = \sup_{t \in [0,\sigma^2(1)]} |u(t)|$. In the Banach space, as in [4], we define a cone $\mathcal{C} \subset \mathcal{B}$ by

$$\mathcal{C} = \{u \in \mathcal{B} \colon u(t) \geq 0 \text{ on } [0,\sigma^2(1)] \text{ and } \min_{t \in [\zeta,\sigma(\omega)]} |u(t)| \geq \kappa \|u\|\}.$$

Based on the construction of the cone $\mathcal{C}$, the following fixed point theorem due to Krasnosel'skiĭ [11] is central in obtaining our main results.

Theorem 1.5. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T \colon \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

At this point the standing assumptions of the inhomogeneous term of (1.1) are introduced. Let

(A_1) $f \in C(\mathbb{T} \times \mathbb{R}^+, \mathbb{R})$;

(A_2) there exists $M > 0$ such that $f(t,z) + M \geq 0$ on $\mathbb{T} \times [0,\infty)$;

Furthermore, using (A_2), we define the function $f_p\colon C([0,\sigma^2(1)]\times\mathbb{R},\mathbb{R}^+)$ by

$$f_p(t,z)=\begin{cases} f(t,z)+M, & (t,z)\in[0,\sigma(1)]\times[0,\infty),\\ f(t,0)+M, & (t,z)\in[0,\sigma(1)]\times(-\infty,0).\end{cases} \tag{1.5}$$

Consider the dynamic equation

$$-u^{\triangle\triangle}(t)=\lambda f_p(t,u(\sigma(t))-\lambda M u_1(\sigma(t))),\quad t\in[0,1], \tag{1.6}$$

subject to the boundary conditions (1.2).

Definition 1.6. By a positive solution of the boundary problem (1.1), (1.2) we understand a function $u\in C^2([0,\sigma^2(1)],\mathbb{R}^+)$, which satisfies (1.1) on the interval $[0,1]$ and the boundary conditions (1.2).

Lemma 1.7. *The function u is a positive solution of the boundary problem (1.1), (1.2) if and only if the function $v=u+\lambda M u_1$ is a solution of the boundary value problem (1.6), (1.2) satisfying*

$$v(\sigma(t))>\lambda M u_1(\sigma(t)), \tag{1.7}$$

for all $t\in(0,1)$.

Proof. Suppose function v a solution of the boundary problem (1.6), (1.2) with $v(\sigma(t))>\lambda M u_1(\sigma(t))$, $t\in(0,1)$. Note that function u_1 satisfies, for all $t\in[0,1]$,

$$-u^{\triangle\triangle}(t)=1.$$

Then, for all $t\in[0,1]$,

$$\begin{aligned}
-u^{\triangle\triangle}(t) &= -(v-\lambda M u_1)^{\triangle\triangle}(t)\\
&= -v^{\triangle\triangle}(t)-\lambda M\\
&= \lambda f_p(t,v(\sigma(t)))-\lambda M\\
&= \lambda(f(t,v(\sigma(t))-\lambda M u_1(\sigma(1)))+M)-\lambda M\\
&= \lambda f(t,u(\sigma(t))),
\end{aligned}$$

that is, u is a positive solution of (1.1).

Conversely, if u is a positive solution of (1.1), (1.2), then $v(\sigma(t))>\lambda M u_1(\sigma(t))$, $t\in(0,1)$. In addition,

$$\begin{aligned}
-v^{\triangle\triangle}(t) &= -(u+\lambda M u_1)^{\triangle\triangle}(t)\\
&= -u^{\triangle\triangle}(t)+\lambda M\\
&= \lambda f(t,u(\sigma(t)))+\lambda M\\
&= \lambda(f_p(t,v(\sigma(t))-\lambda u_1(\sigma(t)))-M)+\lambda M\\
&= \lambda(f_p(t,v(\sigma(t))-\lambda M u_1(\sigma(t))),
\end{aligned}$$

that is, v satisfies (1.6).

It is clear v satisfies (1.2) if and only if u satisfies (1.2). □

An integral operator $T:\mathcal{B}\to\mathcal{B}$ associated with the boundary value problem (1.6), (1.2) is defined by

$$T_\lambda u(t)=\lambda\int_0^{\sigma(1)}G(t,s)\,f_p(s,u(\sigma(s))-\lambda Mu_1(\sigma(s)))\,\triangle s,\quad t\in[0,\sigma^2(1)].$$

It is not difficult to establish the following two results.

Lemma 1.8. *The operator $T_\lambda\colon\mathcal{C}\to\mathcal{C}$ is completely continuous.*

Lemma 1.9. *A function $u\in\mathcal{B}$ is a positive solution of the boundary value problem (1.6), (1.2) if and only if $T_\lambda|_{\mathcal{C}}(u)=u$.*

The idea is obtain a positive (by Lemma 1.9) solution of the auxiliary boundary value problem (1.6), (1.2). By virtue of Lemma 1.7, it will follow that the boundary value problem (1.1), (1.2) has a positive solution.

2 Positive solutions

We present our main results.

Theorem 2.1. *Assume the hypotheses (A_1) and (A_2). Assume*

(H_1) $\lim_{z\to\infty}\frac{f(t,z)}{z}=\infty$ *holds uniformly on the interval* $[\zeta,\omega]$.

Then, for a sufficiently small $\lambda>0$, the boundary value problem (1.1), (1.2) has at least one positive solution.

Proof. Let $R_1>0$ be fixed. Define

$$\mathcal{B}_1=\{u\in\mathcal{B}\colon\ \|u\|<R_1\}.$$

Let

$$K=\max_{(t,z)\in[0,\sigma(1)]\times[0,R_1]}f_p(t,z).$$

Define $\Omega_1=\{u\in\mathcal{B}:\ \|u\|<R_1\}$, then, for $u\in\mathcal{C}\cap\partial\Omega_1$, we have

$$\begin{aligned}\|T_\lambda u\| &= \sup_{t\in[0,\sigma^2(1)]}\lambda\int_0^{\sigma(1)}G(t,s)\,f_p(s,u(\sigma(s))-\lambda Mu_1(\sigma(s)))\,\triangle s\\ &\le \sup_{t\in[0,\sigma^2(1)]}\lambda K\int_0^{\sigma(1)}G(t,s)\,\triangle s\\ &= \lambda KA\\ &\le R_1,\end{aligned}$$

provided $\lambda\in\left(0,\frac{R_1}{KA}\right]$. That is,

$$\|T_\lambda u\|\le\|u\|,\quad u\in\mathcal{C}\cap\partial\Omega_1, \tag{2.1}$$

for a sufficiently small $\lambda > 0$.

Fix λ from the interval above. It follows from (1.5) and (H_1) that

$$\lim_{z\to\infty} \frac{f_p(t,z)}{z} = \infty$$

uniformly on the interval $[\zeta,\omega]$. Hence there exists an $R^* > \max\left\{\frac{\lambda MA}{\kappa}, R_1\right\}$ such that

$$f_p(t,z) \geq \frac{2R^*}{\lambda\kappa B},$$

for all $t \in [\zeta,\omega]$ and all $z \geq R^*$.

Set $R_2 = \frac{2R^*}{\kappa}$ and define

$$\Omega_2 = \{u \in \mathcal{B} : \|u\| < R_2\}.$$

Then, for $u \in \mathcal{C} \cap \partial\Omega_2$, we have

$$u(\sigma(s)) - u_1(\sigma(s)) \geq \kappa\|u\| - \lambda M u_1(\sigma(s)) \geq 2R^* - \lambda MA > R^*$$

for all $s \in [\zeta,\omega]$. Hence

$$f_p(s, u(\sigma(s)) - \lambda M u_1(\sigma(s))) \geq \frac{2R^*}{\lambda\kappa B} = \frac{R_2}{\lambda B},$$

for all $s \in [\zeta,\omega]$.

Let $u \in \mathcal{C} \cap \Omega_2$. Then

$$\begin{aligned}
\|T_\lambda u\| &= \sup_{t\in[0,\sigma^2(1)]} \lambda \int_0^{\sigma(1)} G(t,s) f_p(s, u(\sigma(s)) - \lambda M u_1(\sigma(s))) \triangle s \\
&\geq \lambda \sup_{t\in[0,\sigma^2(1)]} \int_\zeta^\omega G(t,s) f_p(s, u(\sigma(s)) - \lambda M u_1(\sigma(s))) \triangle s \\
&\geq \lambda \sup_{t\in[0,\sigma^2(1)]} \int_\zeta^\omega G(t,s) \frac{R_2}{\lambda B} \triangle s \\
&= R_2,
\end{aligned}$$

that is,

$$\|T_\lambda u\| \geq \|u\|, \quad u \in \mathcal{C} \cap \partial\Omega_2. \tag{2.2}$$

By Lemma (1.8), the operator T_λ is completely continuous and preserves the cone $\mathcal{C}$. By Theorem 1.5, it follows from the inequalities (2.1) and (2.2) that T_λ has a fixed point $u_p \in \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. By Lemma 1.9, u_p is a positive solution of the boundary value problem (1.6), (1.2).

Suppose $u_p(0) > 0$ and $u_p(\sigma(1)) > 0$ and setting $m = \min\{u_p(\sigma(0)), u_p(\sigma(1))\}$. Then we obtain, for all $t \in [0,1]$,

$$u_p(\sigma(t)) - \lambda M u_1(\sigma(t)) \geq m - \lambda MA > 0$$

provided $\lambda < \frac{m}{MA}$. Thus the inequality (1.7) is fulfilled.

Suppose $u_p(0) = 0$ and $u_p(\sigma(1)) > 0$. Then, clearly, $u_p^{\triangle}(0) > 0$. If $t = 0$ is right-scattered, then $u_p(\sigma(0)) > \sigma(0) > 0$. With the choice of $\lambda > 0$ above, we can ensure (1.7) for all $t \in (0,1)$. If $t = 0$ is right-dense, then there exists constants $c_1, c_2 > 0$ and a neighborhood $(0,r]$ such that for all $t \in (0,r)$ we have, respectively, $u_p(t) > c_1 t$ and $u_1(\sigma(t)) < c_2 t$. Choosing $\lambda < \frac{c_1}{Mc_2}$, we obtain

$$u_p(\sigma(t)) - \lambda M u_1(\sigma(t)) > (c_1 - \lambda M c_2)\sigma(t) > 0$$

for $t \in (0, \sigma(r)]$. Setting, in addition, $\lambda < \left(\min\{u_p(\sigma(r)), u_p(\sigma(1))\}\right)/MA$ yields (1.7) for all $t \in [r,1)$, and thus (1.7) holds for all $t \in (0,1)$.

The cases of $u_p(0) > 0$ and $u_p(\sigma(1)) = 0$ and $u_p(0) = u_p(\sigma(1)) = 0$ are treated in a similar fashion.

Recall that we already have $\lambda > 0$ satisfy $\lambda \le \frac{R_1}{MA}$. In addition, from the considerations above we obtain, that in each case of boundary conditions (1.2), we can find a sufficiently small $\lambda > 0$ such that Lemma 1.7 applies to yield that $u = u_p - \lambda M u_1$ is a positive solution of the boundary value problem (1.1), (1.2). □

Example 2.2. Let $\mathbb{T} = (-\infty, 0] \cup \left\{\frac{1}{4}, \frac{1}{3}\right\} \cup \left\{1 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup [1,\infty)$ and consider the boundary value problem

$$-u^{\triangle\triangle}(t) = \lambda f(u(\sigma(t))), \quad t \in [0,1],$$
$$u(0) = 0, \quad u^{\triangle}(\sigma(1)) = 0,$$

where $f(z) = z^{3/2} - z^{1/2}$. The Green function $G(t,s)$ is

$$G(t,s) = \begin{cases} t, & t \le s \\ \sigma(s), & \sigma(s) \le t \end{cases}.$$

Note that $\sigma(1) = 1$ and $\kappa_1 = \frac{1}{4}, \zeta = \frac{1}{4}$, and $\omega = \frac{3}{4}$. Also, $\sigma(\omega) = \frac{4}{5} < 1$. Since, $G(\sigma(\omega), s) = \sigma(s)$, then $\kappa = \kappa_1 = \frac{1}{4}$. Finally, $A = \sup_{t\in[0,\sigma^2(1)]} \int_0^{\sigma(1)} G(t,s)\triangle s = \frac{24\pi^2 - 155}{6}$.

The function f satisfies $f(z) \ge -\frac{2\sqrt{3}}{9}$ for all $z \ge 0$ and it is easy to check that $\lim_{z\to\infty} \frac{f(z)}{z} \to \infty$. Hence f satisfies conditions (A1), (A2) and (H1). By Theorem 2.1 the boundary value problem has at least one positive solution for small values of λ.

Our next theorem is a multiplicity result.

Theorem 2.3. *Assume the hypotheses (A_1) and (A_2). Assume, in addition to (H_1),*

(H_2) $f(t,0) \not\equiv 0$ on $[0,\sigma(1)]$, and there exists $l > 0$ such that $f(t,z) \ge 0$ on $[0,\sigma(1)] \times [0,l]$.

Then, for a sufficiently small $\lambda > 0$, the boundary value problem (1.1), (1.2) has at least two positive solutions.

Proof. Let, without loss of generality, the solutions u_p and u in the proof of Theorem 2.1 satisfy, for all $t \in (0,1)$,

$$u(\sigma(t)) - \frac{1}{2}u_p(\sigma(t)) > 0 \tag{2.3}$$

and set $r = \|u_p\|$. Then $\|u\| \ge \frac{r}{2}$.

It follows from (H_2) that there exist constants $l, L > 0$ such that $0 \le f(t,z) \le L$, for all $(t,z) \in [0,\sigma(1)] \times [0,l]$. Define

$$g_p(t,z) = \begin{cases} f(t,z), & (t,z) \in [0,\sigma(1)] \times [0,l], \\ f(t,r), & (t,l) \in [0,\sigma(1)] \times (l,\infty), \end{cases} \tag{2.4}$$

which satisfies

$$0 \le g_p(t,z) \le L,$$

for all $(t,z) \in [0,\sigma(1)] \times [0,\infty)$.

We introduce the dynamic equation

$$-u^{\triangle\triangle}(t) = \lambda g_p(t,z), \quad t \in [0,1], \tag{2.5}$$

subject to the boundary condition (1.2). We use the same notation is in the proof of Theorem 2.1 to introduce an integral operator $T_\lambda \colon \mathcal{B} \to \mathcal{B}$ defined by

$$T_\lambda u(t) = \lambda \int_0^{\sigma(1)} G(t,s) g_p(s, u(\sigma(s))) \triangle s, \quad t \in [0, \sigma^2(1)].$$

The operator $T_\lambda \colon \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let $r_1 = \min\{\frac{r}{2}, l\}$ and introduce

$$K' = \max_{(t,z) \in [0,\sigma(1)] \times [0,r_1]} g_p(t,z).$$

As in the proof of Theorem 2.1, we define $\Omega_1' = \{u \in \mathcal{B} : \|u\| < R_1\}$, and, for $u \in \mathcal{C} \cap \partial\Omega_1'$, obtain

$$\begin{aligned} \|T_\lambda u\| &= \sup_{t \in [0,\sigma^2(1)]} \lambda \int_0^{\sigma(1)} G(t,s) g_p(s, u(\sigma(s))) \triangle s \\ &= \lambda K' A \\ &\le r_1, \end{aligned}$$

provided $\lambda \in (0, \frac{R_1}{K'A}]$. That is,

$$\|T_\lambda u\| \le \|u\|, \quad u \in \mathcal{C} \cap \partial\Omega_1', \tag{2.6}$$

if $\lambda > 0$ is sufficiently small.

It follows from (2.4) and (H_2) that

$$\lim_{z \to 0^+} \frac{g_p(t,z)}{z} = \infty$$

uniformly on the interval $[\zeta, \omega]$. Hence there exists an $0 < r_2 < r_1$ such that

$$g_p(t,z) \ge \frac{z}{\lambda \kappa B}, \tag{2.7}$$

for all $t \in [\zeta, \omega]$ and all $z \le r_2$.

Define

$$\Omega_2' = \{u \in \mathcal{B} : \|u\| < r_2\}.$$

Let $u \in \mathcal{C} \cap \Omega_2$. Then, $u(\sigma(s)) \geq \kappa r_2$, $s \in [\zeta, \omega]$, and, by (2.7),

$$\begin{aligned} \|T_\lambda u\| &= \sup_{t\in[0,\sigma^2(1)]} \lambda \int_0^{\sigma(1)} G(t,s) g_p(s, u(\sigma(s))) \triangle s \\ &\geq \lambda \sup_{t\in[0,\sigma^2(1)]} \int_\zeta^\omega G(t,s) g_p(s, u(\sigma(s))) \triangle s \\ &\geq \lambda \sup_{t\in[0,\sigma^2(1)]} \int_\zeta^\omega G(t,s) \frac{u(\sigma(s))}{\lambda\kappa B} \triangle s \\ &= r_2, \end{aligned}$$

that is,

$$\|T_\lambda u\| \geq \|u\|, \quad u \in \mathcal{C} \cap \partial\Omega_2'. \tag{2.8}$$

By Theorem 1.5, it follows from the inequalities (2.6) and (2.8) that T_λ has a fixed point $u' \in \mathcal{C} \cap (\overline{\Omega'}_1 \setminus \Omega_2')$. We conclude that u' is a positive solution of the boundary value problem (1.6), (1.2). Since $\|u'\| \leq r_1 \leq \min\{\frac{r}{2}, l\} \leq \|u\|$, then u' is also a (different from u) positive solution of (1.1), (1.2). In fact, by the first part of (H_2), $f(t,0) \not\equiv 0$ on the interval $[0, \sigma(1)]$, and since $G(t,s) > 0$, $(t,s) \in (0, \sigma(1)) \times (0,1)$, we see that

$$u(t) = \lambda \int_0^{\sigma(1)} G(t,s) f(s, u(\sigma(s))) \triangle s, \quad t \in [0, \sigma^2(1)],$$

is a positive solution of the boundary value problem (1.1), (1.2).

□

Example 2.4. Let $\mathbb{T} = (-\infty, 0] \cup \left\{\frac{1}{4}, \frac{1}{3}\right\} \cup \left\{1 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup [1, \infty)$ and consider the boundary value problem

$$\begin{gathered} -u^{\triangle\triangle}(t) = \lambda f(u(\sigma(t))), \quad t \in [0,1], \\ u(0) = 0, \quad u^\triangle(\sigma(1)) = 0, \end{gathered}$$

where $f(z) = z^{3/2} - 5z^{1/2} + 4$.

The function f satisfies $f(z) > 0$ for all $z \geq [0,1)$ and for all $z \geq 0$, we have $f(z) + M \geq 0$, where $M = \frac{10\sqrt{15}}{9}$. Finally, $\lim_{z\to\infty} \frac{f(z)}{z} \to \infty$. Hence f satisfies conditions (A1), (A2), (H1), and (H2). By Theorem 2.3 the boundary value problem has at least two positive solutions for small values of λ.

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