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INTERVALS OF SABIDUSSI GRAPHS

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Dedicated to Gert Sabidussi on the occasion of his retirement from the University of Montreal.

Abstract

A Sabidussi graph is defined from a total order T and a graph G as follows. Choose a vertex of G and denote it by 0. Denote by ${}^{V(T)}V(G)$ the family of the functions $f : V(T) \longrightarrow V(G)$ such that $\{q \in V(T) : f(q) \neq 0\}$ is finite. The Sabidussi graph ${}^{T}G$ is defined on ${}^{V(T)}V(G)$ by: given $f \neq g \in ({}^{V(T)}V(G))$, $\{f,g\} \in E({}^{T}G)$ if $\{f(m), g(m)\} \in E(G)$, where m is the smallest element of $\{q \in V(T) : f(q) \neq g(q)\}$ in the total order T.

Given a graph Γ, a subset X of $V(\Gamma)$ is an interval of Γ if for $a, b \in X$ and $x \in V(\Gamma) \setminus X$, $\{a,x\} \in E(\Gamma)$ if and only if $\{b,x\} \in E(\Gamma)$. Moreover, a subset X of $V(\Gamma)$ is a strong interval of Γ provided that X is an interval of Γ and for every interval Y of Γ, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$.

The intervals and the strong intervals of the Sabidussi graphs ${}^{\mathbb{Q}}G$ are characterized, where $\mathbb{Q}$ is the set of rational numbers with the usual total order.

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1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple graph. We denote the characteristic function of $E(\Gamma)$ in $\binom{V(\Gamma)}{2}$ by $[x,y]_\Gamma$ so that $[x,y]_\Gamma = 1$ if and only if $\{x,y\} \in E(\Gamma)$. We extend this to subsets X of $V(\Gamma)$ by defining $[x,X]_\Gamma = 1$ exactly when $[x,y]_\Gamma = 1$ for each $y \in X$ and to pairs of disjoint subsets X,Y of $V(\Gamma)$ by setting $[X,Y]_\Gamma = 1$ if and only if $[x,y]_\Gamma = 1$ for all $x \in X$, $y \in Y$.

Given a graph Γ, associate with each subset X of $V(\Gamma)$ the *subgraph* $\Gamma(X)$ of Γ induced by X defined on $V(\Gamma(X)) = X$ by $[x,y]_{\Gamma(X)} = [x,y]_\Gamma$ for $x \neq y \in X$. The *complement* of a graph Γ is the graph $\overline{\Gamma}$ defined on $V(\Gamma)$ by $[x,y]_{\overline{\Gamma}} = 1 - [x,y]_\Gamma$ for $x \neq y \in V(\Gamma)$.

1.1 The intervals

We use the following notation. Given sets X and Y, $X \subseteq Y$ means that X is a subset of Y whereas $X \subset Y$ means that X is a proper subset of Y.

Given a graph Γ, a subset X of $V(\Gamma)$ is an *interval* ([2, Subsection 9.8] and [8]) or an *autonomous* subset [10] or a *homogeneous* subset [3, 11] or a *clan* [1, Subsection 3.2] of Γ if for each $x \in V(\Gamma) \setminus X$, there is $\alpha \in \{0,1\}$ such that $[x,X]_\Gamma = \alpha$. The following properties of the intervals of a graph are well known (see, for example, [1, Subsection 3.3]).

Proposition 1.1. *Given a graph Γ, the assertions below hold:*

A1 *$\emptyset$, $V(\Gamma)$ and $\{x\}$, where $x \in V(\Gamma)$, are intervals of Γ;*

A2 *(i) given a subset W of $V(\Gamma)$, if X is an interval of Γ, then $X \cap W$ is an interval of $\Gamma(W)$;*

(ii) given an interval X of Γ, we have for every $Y \subseteq X$: Y is an interval of $\Gamma(X)$ if and only if Y is an interval of Γ;

A3 *(i) for every family $\mathcal{F}$ of intervals of Γ, the intersection $\cap\mathcal{F}$ of all the elements of $\mathcal{F}$ is an interval of Γ;*

(ii) given intervals X and Y of Γ, if $X \cap Y \neq \emptyset$, then $X \cup Y$ is an interval of Γ;

(iii) for every family $\mathcal{F}$ of intervals of Γ, the union $\cup\mathcal{F}$ of all the elements of $\mathcal{F}$ is an interval of Γ provided that for any $X,Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ such that $X \cup Y \subseteq Z$;

(iv) given intervals X and Y of Γ, if $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is an interval of Γ;

A4 *for any intervals X and Y of Γ, if $X \cap Y = \emptyset$, then there is $\alpha \in \{0,1\}$ such that $[X,Y]_\Gamma = \alpha$.*

Following Assertion A1, $\emptyset$, $V(\Gamma)$ and $\{x\}$, where $x \in V(\Gamma)$, are called *trivial*. A graph all of whose intervals are trivial is *indecomposable* [8] or *prime* [10] or *primitive* [1]. Otherwise, it is *decomposable*.

Given a graph Γ, a partition P of $V(\Gamma)$ is an *interval partition* of Γ when all the elements of P are intervals of Γ. Using Assertion A4, for each interval partition P of G, we can define the *quotient* Γ/P of Γ by P on $V(\Gamma/P) = P$ as follows. For any $X \neq Y \in P$, $[X,Y]_{\Gamma/P} = [x,y]_\Gamma$, where $x \in X$ and $y \in Y$.

The following strengthening of the notion of interval is due to Gallai [3, 11]. It is used to decompose finite graphs in an intrinsic and unique way. Given a graph Γ, an interval X if Γ is *strong* if for every interval Y of Γ not disjoint from X, we have $X \subseteq Y$ or $Y \subseteq X$. Properties analogous to those stated in Proposition 1.1 hold for strong intervals.

Proposition 1.2. *Given a graph Γ, the assertions below hold:*

B1 *$\emptyset$, $V(\Gamma)$ and $\{x\}$, where $x \in V(\Gamma)$, are strong intervals of Γ;*

B2 *(i) given an interval X of Γ, we have for every $Y \subset X$: Y is a strong interval of $\Gamma(X)$ if and only if Y is a strong interval of Γ;*

(ii) given a strong interval X of Γ, we have for every $Y \subseteq X$: Y is a strong interval of $\Gamma(X)$ if and only if Y is a strong interval of Γ;

B3 *(i) for every family $\mathcal{F}$ of strong intervals of Γ, the intersection $\cap\mathcal{F}$ of all the elements of $\mathcal{F}$ is a strong interval of Γ;*

(ii) for every family $\mathcal{F}$ of strong intervals of Γ, the union $\cup\mathcal{F}$ of all the elements of $\mathcal{F}$ is a strong interval of Γ provided that for any $X, Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ such that $X \cup Y \subseteq Z$;

For a proof of Assertion B2.(i), we refer to [1, Lemma 3.11]. For convenience, we denote the family of the nonempty strong intervals of a graph Γ by $\mathcal{S}(\Gamma)$ and the family of the maximal elements of $\mathcal{S}(\Gamma) \setminus \{V(\Gamma)\}$ under inclusion by $P(\Gamma)$. In the finite case, $P(\Gamma)$ yields the following decomposition theorem.

Theorem 1.3 (Gallai [3, 11])**.** *Given a finite graph Γ, with $|V(\Gamma)| \geq 2$, the family $P(\Gamma)$ realizes an interval partition of $V(\Gamma)$. Furthermore, the corresponding quotient $\Gamma/P(\Gamma)$ is either indecomposable, with $|P(T)| \geq 3$, or there exists $\alpha \in \{0,1\}$ such that for any $X \neq Y \in P(\Gamma)$, $[X,Y]_{\Gamma/P(\Gamma)} = \alpha$.*

In the infinite case, we have (see, for example, [7, Theorem 4.2]):

Lemma 1.4. *Given an infinite graph Γ, if $P(\Gamma) \neq \emptyset$, then $P(\Gamma)$ is an interval partition of Γ.*

Theorem 1.3 is still true for an infinite graph Γ when $P(\Gamma) \neq \emptyset$. Indeed, $P(\Gamma)$ is an interval partition of Γ by Lemma 1.4. Since the elements of $P(\Gamma)$ are the maximal elements of $\mathcal{S}(\Gamma) \setminus \{V(\Gamma)\}$ under inclusion, all the strong intervals of $\Gamma/P(\Gamma)$ are trivial. Then, it suffices to apply [7, Theorem 4.1] to $\Gamma/P(\Gamma)$. Given an infinite graph Γ, a strong interval X of Γ is *limit* [10] if $P(\Gamma(X)) = \emptyset$. Given $x \in V(\Gamma)$, note that $\{x\}$ is a limit interval because $\mathcal{S}(\Gamma(\{x\})) = \{V(\Gamma(\{x\}))\} = \{\{x\}\}$. We denote by $\mathcal{L}(\Gamma)$ the family of limit strong intervals of Γ. The *decomposition tree* of Γ is the following family ordered by inclusion:

$$\mathcal{D}(\Gamma) = \bigcup_{X \in \mathcal{S}(\Gamma) \setminus \mathcal{L}(\Gamma)} \{X\} \cup P(\Gamma(X)).$$

Recall that the total order $1 + \mathbb{Z}$ defined on $\{-\infty\} \cup \mathbb{Z}$ is the extension of the usual total order $\mathbb{Z}$ on the set of integers by adding one element denoted by $-\infty$ which is smaller than all the integers. Consider the graph Γ defined on $V(\Gamma) = \{-\infty\} \cup \mathbb{Z}$ by: given $x \neq y \in V(\Gamma)$, $\{x,y\} \in E(\Gamma)$ if $\max(x,y)$ is even. It is easy to verify that the subsets $\{-\infty\} \cup \{\ldots, n-1, n\}$

of $\{-\infty\} \cup \mathbb{Z}$, where $n \in \mathbb{Z}$, are the only non trivial intervals of Γ. Therefore, they are the only non trivial strong intervals of Γ as well. As previously noted, $\{x\} \in \mathcal{L}(\Gamma)$ for $x \in V(\Gamma)$. Moreover, $V(\Gamma) \in \mathcal{L}(\Gamma)$ because $\bigcup_{n\in\mathbb{Z}}(\{-\infty\} \cup \{\dots,n-1,n\}) = \{-\infty\} \cup \mathbb{Z}$. Lastly, for each $n \in \mathbb{Z}$, we have $P(\Gamma(\{-\infty\} \cup \{\dots,n-1,n\})) = \{\{n\},\{-\infty\} \cup \{\dots,n-2,n-1\}\}$. Consequently, $\mathcal{D}(\Gamma) = \{\{-\infty\} \cup \{\dots,n-1,n\}; n \in \mathbb{Z}\} \cup \{\{n\}; n \in \mathbb{Z}\}$. Clearly, $\{-\infty\} \notin \mathcal{D}(\Gamma)$. Sometimes we add the singletons to the decompostion tree depending on its use.

1.2 The Sabidussi graphs

Sabidussi graphs are defined as follows. Consider a total order T defined on a set S and a graph $G = (V,E)$, with $|V| \geq 2$. Choose a vertex of G and denote it by 0. Denote by ${}^{S}V$ the family of the functions $f : S \longrightarrow V$ such that $\{q \in S : f(q) \neq 0\}$ is finite. In particular, the function $\overline{0} : S \longrightarrow V$, defined by $\overline{0}(q) = 0$ for every $q \in S$, belongs to ${}^{S}V$. The graph ${}^{T}G$ is defined on ${}^{S}V$ as follows: given $f \neq g \in ({}^{S}V)$, $[f,g]_{{}^{T}G} = [f(\delta(f,g)),g(\delta(f,g))]_{G}$, where $\delta(f,g)$ denotes the smallest element of $\{q \in S : f(q) \neq g(q)\}$ in the total order T. The graph ${}^{T}G$ is called *Sabidussi graph*. By replacing G by $\overline{G}$ in what precedes, we obtain $\overline{{}^{T}G}$ instead of ${}^{T}G$.

Sabidussi [13] introduced this construction to obtain graphs idempotent under the lexicographic product. Given graphs Γ and Γ', recall that the *lexicographic product* $\Gamma[\Gamma']$ of Γ' by Γ is defined on $V(\Gamma[\Gamma']) = V(\Gamma) \times V(\Gamma')$ as follows. Given $(x,x'),(y,y') \in V(\Gamma[\Gamma'])$, $\{(x,x'),(y,y')\} \in E(\Gamma[\Gamma'])$ if either $x \neq y$ and $\{x,y\} \in E(\Gamma)$ or $x = y$ and $\{x',y'\} \in E(\Gamma')$. An infinite graph Γ is *idempotent* under the lexicographic product if $\Gamma[\Gamma]$ and Γ are isomorphic. The lexicographic product of directed graphs is defined similarly. For a total order T and a graph G, we obtain that $({}^{T}G)[{}^{T}G]$ is isomorphic to ${}^{2[T]}G$, where 2 denotes the usual total order on $\{0,1\}$. Consequently, the Sabidussi graph ${}^{T}G$ is idempotent under the lexicographic product if $2[T]$ is isomorphic to T. For instance, consider the usual total order on the set of rational numbers, which is denoted by $\mathbb{Q}$ as well. We have $2[\mathbb{Q}]$ is isomorphic to $\mathbb{Q}$. In the sequel, we consider the Sabidussi graph ${}^{\mathbb{Q}}G$ for some graph $G = (V,E)$, with $|V| \geq 2$. For convenience, ${}^{\mathbb{Q}}G$ is denoted by Π. We propose to characterize the intervals and the strong intervals of Π. This leads us to some of their remarkable properties which suitably illustrate the idempotency of Π. In fact, Sabidussi graphs are the only known graphs idempotent under the lexicographic product. We hope that our structural study will provide a more general construction of such graphs in terms of decomposition tree which will permit a complete characterization. In 1961, Sabidussi conjectured the following algebraic property of graphs idempotent under the lexicographic product. Let Γ and Γ' be permutation groups acting on sets X and X', respectively. The *wreath product* (also called the *composition* or the *corona*) of Γ *around* Γ' is the group $\Gamma \wr \Gamma'$ whose elements are the pairs $(\phi,\{\psi_x \,;\, x \in X\})$, where $\phi \in \Gamma$ and $\psi_x \in \Gamma'$, and which acts on $X \times X'$ by $(\phi,\{\psi_x : x \in X\})(x,x') = (\phi(x),\psi_x(x'))$. Given graphs Γ and Γ', the family $\{\{x\} \times V(\Gamma') \,;\, x \in V(\Gamma)\}$ clearly constitutes an interval partition of the lexicographic product $\Gamma[\Gamma']$. It follows from Proposition 1.1 (A4) that the wreath product $\mathrm{Aut}(\Gamma) \wr \mathrm{Aut}(\Gamma')$ is a subgroup of $\mathrm{Aut}(\Gamma[\Gamma'])$. In particular, $\mathrm{Aut}(\Gamma) \wr \mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(\Gamma[\Gamma])$ for every graph Γ. The relationship between the wreath product of the automorphism groups of two graphs and the automorphism group of the lexicographic product of the two graphs was studied by Sabidussi [13, 12] who corrected the first attempt at characterizing the graphs for which the two resulting groups coincide. The characterization was generalized by the first author to hypergraphs [6] and to directed graphs [4] and to directed hypergraphs [5]. The latter work also mentions the next conjecture.

Conjecture 1.5 (Sabidussi, 1961 (unpublished))**.** *If* Γ *is a graph idempotent under the lexicographic product, then* $\mathrm{Aut}(\Gamma) \wr \mathrm{Aut}(\Gamma)$ *is a proper subgroup of* $\mathrm{Aut}(\Gamma[\Gamma])$.

The second author proved this conjecture in 2003 [9] after studying the relationship between the structures of the decomposition tree of $\Gamma[\Gamma]$ and of Γ. This explains our present approach.

1.3 Notation

Given $\mathcal{X} \subseteq {}^{\mathbb{Q}}V$, we denote $\mathcal{X} \setminus \{\overline{0}\}$ by $\mathcal{X}^\star$. Let $f \in {}^{\mathbb{Q}}V$. We denote the family $\{q \in \mathbb{Q} : f(q) \neq 0\}$ by $\sigma(f)$ and $|\sigma(f)|$ by $n(f)$. We use the following notation when $f \neq \overline{0}$.

- $s(f) = \min(\sigma(f))$ and $S(f) = \max(\sigma(f))$.
- Set $\sigma(f) = \{q_1^f, \ldots, q_{n(f)}^f\}$, where $s(f) = q_1^f < \cdots < q_{n(f)}^f = S(f)$.

We consider s as a function $({}^{\mathbb{Q}}V)^\star \longrightarrow \mathbb{Q}$ and hence for $\mathcal{X} \subseteq ({}^{\mathbb{Q}}V)^\star$, we can extend it by setting $s(\mathcal{X}) = \{s(f) : f \in \mathcal{X}\}$. Given a nonempty subset $\mathcal{X}$ of ${}^{\mathbb{Q}}V$ such that $s(\mathcal{X}^\star)$ admits a smallest element q, we denote $\{f(q)\ ;\ f \in \mathcal{X} \cap s^{-1}(\{q\})\}$ by $\mathcal{X}\downarrow$. Given $q \in \mathbb{Q}$ and $x \in V \setminus \{0\}$, ${}^q x$ is the element of ${}^{\mathbb{Q}}V$ defined by $\sigma({}^q x) = \{q\}$ and $({}^q x)(q) = x$. More generally, given $\emptyset \neq X \subseteq V \setminus \{0\}$, ${}^q X$ denotes the set $\{{}^q x\ ;\ x \in X\}$.

Remark 1.6. If Π is connected, then there is $f \in ({}^{\mathbb{Q}}V)^\star$ such that $[\overline{0}, f]_\Pi = 1$. Consequently, $[0, f(s(f))]_G = 1$. Conversely, assume that there is $x \in V \setminus \{0\}$ such that $[0,x]_G = 1$. Firstly, consider $f \in ({}^{\mathbb{Q}}V)^\star$. For $q < s(f)$, we have $[\overline{0}, {}^q x]_\Pi = [f, {}^q x]_\Pi = 1$. Secondly, consider $f \neq g \in ({}^{\mathbb{Q}}V)^\star$. For $q < \min(s(f), s(g))$, we have $[f, {}^q x]_\Pi = [g, {}^q x]_\Pi = 1$. Consequently, Π is connected if and only if there is $x \in V \setminus \{0\}$ such that $[0,x]_G = 1$. By considering $\overline{G}$ instead of G, we obtain that $\overline{\Pi}$ is connected if and only if there is $y \in V \setminus \{0\}$ such that $[0,y]_{\overline{G}} = 1$ or, equivalently, $[0,y]_G = 0$.

Assumption 1.7. In the sequel, we assume that there are $x, y \in V \setminus \{0\}$ such that $[0,x]_G \neq [0,y]_G$. It follows from Remark 1.6 that Π and $\overline{\Pi}$ are connected.

To continue, we define a poset $<$ on ${}^{\mathbb{Q}}V$ as follows. First, for every $f \in ({}^{\mathbb{Q}}V)^\star$, we have $\overline{0} < f$. Second, given $f, g \in ({}^{\mathbb{Q}}V)^\star$, $f \leq g$ if $f(q) = g(q)$ for every $q \leq S(f)$. Consequently, if $f \leq g$, then $s(f) = s(g)$ and $\sigma(f) \subseteq \sigma(g)$. Furthermore, if $f < g$, then $[f, g]_\Pi = [0, g(q_{n(f)+1}^g)]_G$.

Given $\emptyset \neq \mathcal{X} \subseteq {}^{\mathbb{Q}}V$, denote by $\mathcal{X}^-$ the set of $f \in {}^{\mathbb{Q}}V$ such that $f \leq g$ for every $g \in \mathcal{X}$. We have $\mathcal{X}^- \neq \emptyset$ because $\overline{0} \in \mathcal{X}^-$. Assume that $\mathcal{X} \neq \{\overline{0}\}$ and consider $g \in \mathcal{X}^\star$. We have $\mathcal{X}^- \subseteq \{g\}^-$. For $1 \leq i \leq n(g)$, let g_i be the element of ${}^{\mathbb{Q}}V$ defined by $\sigma(g_i) = \{q_1^g, \ldots, q_i^g\}$ and $g_i(q) = g(q)$ for every $q \leq q_i^g$. Since $(\{g\}^-, <)$ is the total order $\overline{0} < g_1 < g_2 < \cdots < g_{n(g)} = g$, $\mathcal{X}^-$ admits a largest element denoted by $\wedge\mathcal{X}$. For convenience, given $f_1, \ldots, f_p \in {}^{\mathbb{Q}}V$, we denote $\wedge\{f_1, \ldots, f_p\}$ by $f_1 \wedge \ldots \wedge f_p$.

Consider $f \in {}^{\mathbb{Q}}V$ and $q \in \mathbb{Q}$ such that $q > S(f)$ if $f \neq \overline{0}$. Denote by $\lfloor f \rfloor$ the family of $g \in {}^{\mathbb{Q}}V$ such that $f \leq g$. For example, given $x \in V \setminus \{0\}$, ${}^q f_x$ is the element of $\lfloor f \rfloor$ defined by $\sigma({}^q f_x) = \sigma(f) \cup \{q\}$ and $({}^q f_x)(q) = x$. More generally, given $\emptyset \neq \mathcal{X} \subseteq {}^{\mathbb{Q}}V$, $\lfloor \mathcal{X} \rfloor$ denotes the union of $\lfloor h \rfloor$, where $h \in \mathcal{X}$. We use the following subsets of $\lfloor f \rfloor$.

- $\lfloor f\rfloor^{>q} = \{f\}\cup\{g\in\lfloor f\rfloor\setminus\{f\} : q^g_{n(f)+1} > q\}$; for instance, $\lfloor\bar{0}\rfloor^{>q} = \{\bar{0}\}\cup s^{-1}((q,+\infty))$.
- $\lfloor f\rfloor^{\geq q} = \{f\}\cup\{g\in\lfloor f\rfloor\setminus\{f\} : q^g_{n(f)+1} \geq q\}$; for instance, $\lfloor\bar{0}\rfloor^{\geq q} = \{\bar{0}\}\cup s^{-1}([q,+\infty))$.
- Given $\emptyset \neq X \subseteq V\setminus\{0\}$, $\lfloor f\rfloor^{=q}_X$ is the family of $g\in\lfloor f\rfloor\setminus\{f\}$ such that $q^g_{n(f)+1} = q$ and $g(q)\in X$. For instance, $\lfloor\bar{0}\rfloor^{=q}_X = \lfloor {}^qX\rfloor$.
- Given $\emptyset \neq X \subseteq V\setminus\{0\}$, $\lfloor f\rfloor^{\geq q}_X = \lfloor f\rfloor^{>q}\cup\lfloor f\rfloor^{=q}_X$. For instance, $\lfloor\bar{0}\rfloor^{\geq q}_X = \{\bar{0}\}\cup s^{-1}((q,+\infty))\cup\lfloor {}^qX\rfloor$.

2 The intervals of Π

2.1 Preliminary properties

Lemma 2.1. *Let I be an interval of Π, with $|I|>1$. Consider $f\in I^\star$ such that there exists $f'\in I$ satisfying $f'(s(f))\neq f(s(f))$ and $f'(r)=0$ for every $r<s(f)$. Then, ${}^qx\in I$, where $q=s(f)$ and $x=f(q)$.*

Proof : For a contradiction, suppose that ${}^qx\notin I$. We have $[{}^qx, f']_\Pi = [x, f'(q)]_G$ and hence $[{}^qx, I]_\Pi = [x, f'(q)]_G$. For every $g\in I\cap\lfloor {}^qx\rfloor$, ${}^qx<g$ because ${}^qx\notin I$, and thus $n(g)\geq 2$. Set $W=\{g(q^g_2)\ ;\ g\in I\cap\lfloor {}^qx\rfloor\}$. As $f\in I\cap\lfloor {}^qx\rfloor$, we have $W\neq\emptyset$. For each $y\in W$, consider $g\in I\cap\lfloor {}^qx\rfloor$ such that $g(q^g_2)=y$. As $\delta({}^qx,g)=q^g_2$, we have $[{}^qx,g]_\Pi=[0,y]_G$ so that $[0,y]_G = [x,f'(q)]_G$. Consequently, we have $[0,W]_G=[x,f'(q)]_G$. For each $y\in(V\setminus\{0\})\setminus W$, consider $r\in(q^f_1=q,q^f_2)$ and the element h of ${}^{\mathbb{Q}}V$ defined by $\sigma(h)=\{q,r\}$, $h(q)=x$ and $h(r)=y$. Since $y\notin W$, we have $h\notin I$. Therefore, $[h,I]_\Pi=[h,f']_\Pi=[x,f'(q)]_G$. In particular, we obtain that $[h,f]_\Pi=[x,f'(q)]_G$. As $\delta(f,h)=r$, $[h,f]_\Pi=[0,y]_G$. Consequently, $[0,(V\setminus\{0\})\setminus W]_G=[x,f'(q)]_G$ and hence $[0,V\setminus\{0\}]_G=[x,f'(q)]_G$, which contradicts Assumption 1.7.

Lemma 2.2. *Let I be an interval of Π, with $|I|>1$. Given $f\in I^\star$, consider $r\in\mathbb{Q}$ such that $r>S(f)$ (resp. there is $i\in\{1,\dots,n(f)-1\}$ such that $q^f_i<r<q^f_{i+1}$). If there exists $f'\in I$ such that $f'(s(f))\neq f(s(f))$ and $f'(r')=0$ for every $r'<s(f)$, then there exists $g\in I^\star$ satisfying:*

(1) $f<g$, $n(g)=n(f)+1$ and $S(g)=r$

(resp. (2) $\sigma(g)=\{q^f_1,\dots,q^f_i,r\}$ and $g(g^f_j)=f(q^f_j)$ for every $j\in\{1,\dots,i\}$).

Proof : By Assumption 1.7, there exists $x\in V\setminus\{0\}$ such that $[0,x]_G\neq[f,f']_\Pi$. Denote by g the element of ${}^{\mathbb{Q}}V$ satisfying (1) (resp. (2)) and such that $g(r)=x$. As $\delta(f,g)=r$, we have $[f,g]_\Pi=[0,x]_G$. Furthermore, since $\delta(f,f')=\delta(g,f')=s(f)$ and since $f(s(f))=g(s(f))$, we obtain that $[f,f']_\Pi=[g,f']_\Pi$. It follows that $[g,f]_\Pi\neq[g,f']_\Pi$ and hence $g\in I$.

Proposition 2.3. *Let I be an interval of Π, with $|I|>1$. Given $f\in I^\star$, if there exists $f'\in I$ such that $f'(s(f))\neq f(s(f))$ and $f'(r)=0$ for every $r<s(f)$, then $\lfloor f\rfloor\subseteq I$.*

Proof : For a contradiction, suppose that there exists $g\in(\lfloor f\rfloor\setminus\{f\})\setminus I$. Consider the set $\mathcal{J}$ of $h\in(\lfloor f\rfloor\setminus\{f\})\cap I$ such that $n(h)=n(f)+1$ and $S(h)=q^h_{n(f)+1}<q^g_{n(f)+1}$. By the

preceding lemma, $\mathcal{J} \neq \emptyset$. Set $W = \{h(S(h))\ ;\ h \in \mathcal{J}\}$. Given $x \in W$, consider $h \in \mathcal{J}$ such that $h(S(h)) = x$. As $\delta(f,g) = q^g_{n(f)+1}$ and $\delta(g,h) = S(h)$, we have $[f,g]_\Pi = [0, g(q^g_{n(f)+1})]_G$ and $[g,h]_\Pi = [0,x]_G$. For convenience, denote $[0, g(q^g_{n(f)+1})]_G$ by α. Since I is an interval of Π, we obtain that $[f,g]_\Pi = [g,h]_\Pi$, that is, $[0,x]_G = \alpha$. Consequently, $[0,W]_G = \alpha$. Now, let x be an element of $(V \setminus \{0\}) \setminus W$. Since $\mathcal{J} \neq \emptyset$ by Lemma 2.2, consider $h \in \mathcal{J}$. Given $r \in (S(h), q^g_{n(h)})$, we have ${}^r f_x \notin I$ because $x \notin W$. Therefore, $[h, ({}^r f_x)]_\Pi = [f, ({}^r f_x)]_\Pi$. Since $\delta(h, ({}^r f_x)) = S(h)$, we have $[h, ({}^r f_x)]_\Pi = [0, h(S(h))]_G$. We obtain $[h, ({}^r f_x)]_\Pi = \alpha$ because $h(S(h)) \in W$ and $[0,W]_G = \alpha$. As $\delta(f, ({}^r f_x)) = r$, we have $[f, ({}^r f_x)]_\Pi = [0,x]_G$ and hence $[0,x]_G = \alpha$. It follows that $[0, (V \setminus \{0\}) \setminus W]_G = \alpha$ so that $[0, V \setminus \{0\}]_G = \alpha$, which contradicts Assumption 1.7.

2.2 The intervals I of Π such that $|s(I^\star)| > 1$

Lemma 2.4. *If I is an interval of Π such that $|s(I^\star)| > 1$, then $s(I^\star)$ is an interval of $\mathbb{Q}$ and $\overline{0} \in I$.*

Proof : Consider $f,g \in I$ such that $s(f) < s(g)$ and consider $q \in (s(f), s(g))$. By Assumption 1.7, there exists $x \in V \setminus \{0\}$ such that $[0,x]_G \neq [0, f(s(f))]_G$. As $\delta({}^q x, f) = s(f)$ and $\delta({}^q x, g) = q$, we have $[{}^q x, f]_\Pi = [0, f(s(f))]_G$ and $[{}^q x, g]_\Pi = [0,x]_G$. Consequently, $[{}^q x, f]_\Pi \neq [{}^q x, g]_\Pi$ and hence ${}^q x \in I$. Firstly, we conclude that $s({}^q x) = q \in s(I^\star)$ and thus $s(I^\star)$ is an interval of $\mathbb{Q}$. Secondly, as $\delta(\overline{0}, f) = s(f)$ and $\delta(\overline{0}, {}^q x) = q$, we have $[\overline{0}, f]_\Pi = [0, f(s(f))]_G$ and $[\overline{0}, {}^q x]_\Pi = [0,x]_G$. Therefore, $[\overline{0}, f]_\Pi \neq [\overline{0}, {}^q x]_\Pi$ and hence $\overline{0} \in I$.

The three corollaries below are immediate consequences of Lemmas 2.1, 2.2 and 2.4, and of Proposition 2.3.

Corollary 2.5. *Let I be an interval of Π such that $|s(I^\star)| > 1$. For every $f \in I^\star$, ${}^q x \in I$, where $q = s(f)$ and $x = f(q)$.*

Proof : By Lemma 2.4, we have $\overline{0} \in I$. It is then sufficient to apply Lemma 2.1 by considering $\overline{0}$ for f'.

Corollary 2.6. *Let I be an interval of Π such that $|s(I^\star)| > 1$. Given $f \in I^\star$, consider $r \in \mathbb{Q}$ such that $r > S(f)$ (resp. there is $i \in \{1, \dots, n(f)-1\}$ such that $q^f_i < r < q^f_{i+1}$). There exists $g \in I^\star$ satisfying $n(g) = n(f)+1$, $f < g$ and $S(g) = r$ (resp. $\sigma(g) = \{q^f_1, \dots, q^f_i, r\}$ and $g(g^f_i) = f(q^f_i)$ for every $j \in \{1, \dots, i\}$).*

Proof : By Lemma 2.4, we have $\overline{0} \in I$. It is then sufficient to apply Lemma 2.2 by considering $\overline{0}$ for f'.

Corollary 2.7. *Let I be an interval of Π such that $|s(I^\star)| > 1$. For every $f \in I^\star$, $\lfloor f \rfloor \subseteq I$.*

Proof : By Lemma 2.4, we have $\overline{0} \in I$. It is then sufficient to apply Proposition 2.3 by considering $\overline{0}$ for f'.

The next result follows from Lemma 2.4 as well.

Proposition 2.8. *Let I be an interval of Π such that $|s(I^\star)| > 1$. For any $f,g \in I^\star$, if $s(f) < s(g)$, then $s^{-1}((s(g), +\infty)) \subseteq I$.*

Proof : Set $\mathcal{J} = \{h \in I^\star : s(f) < s(h) < s(g)\}$ and $W = \{h(s(h))\ ;\ h \in \mathcal{J}\}$. By Lemma 2.4, $\mathcal{J} \neq \emptyset$ and hence $\emptyset \neq W \subseteq V \setminus \{0\}$. For a contradiction, suppose that $W \subset V \setminus \{0\}$. Let $y \in W$ and $x \in (V \setminus \{0\}) \setminus W$. There is $h \in \mathcal{J}$ such that $h(s(h)) = y$ and we consider ${}^r x$, where $r \in (s(h), s(g))$. As $x \notin W$, we have ${}^r x \notin I$. Therefore, $[h, ({}^r x)]_\Pi = [g, ({}^r x)]_\Pi$. Since $\delta(h, ({}^r x)) = s(h)$ and $\delta(g, ({}^r x)) = r$, we have $[h, ({}^r x)]_\Pi = [0, y]_G$ and $[g, ({}^r x)]_\Pi = [0, x]_G$. It would follow that $[0, W]_G = [0, (V \setminus \{0\}) \setminus W]_G$, which contradicts Assumption 1.7. Consequently, $W = V \setminus \{0\}$. By Assumption 1.7, there are $z, z' \in W$ such that $[0, z]_G \neq [0, z']_G$. It follows that there are $h, h' \in \mathcal{J}$ such that $h(s(h)) = z$ and $h'(s(h')) = z'$. Now, consider any $g' \in ({}^{\mathbb{Q}}V)^\star$ such that $s(g') > s(g)$. As $\delta(h, g') = s(h)$ and $\delta(h', g') = s(h')$, we have $[h, g']_\Pi = [0, z]_G$ and $[h', g']_\Pi = [0, z']_G$. Therefore, $[h, g']_\Pi \neq [h', g']_\Pi$ and hence $g' \in I$. It results that $s^{-1}((s(g), +\infty)) \subseteq I$.

The following characterization completes the subsection.

Theorem 2.9. *Given $I \subseteq ({}^{\mathbb{Q}}V)$ such that $|s(I^\star)| > 1$, I is an interval of Π in precisely one of the three cases below.*

1. *$I = \lfloor \overline{0} \rfloor$, that is, $I = {}^{\mathbb{Q}}V$.*
2. *$I = \lfloor \overline{0} \rfloor^{>q}$, where $q \in \mathbb{Q}$.*
3. *$I = \lfloor \overline{0} \rfloor_X^{\geq q}$, where $q \in \mathbb{Q}$ and X is a nonempty subset of $V \setminus \{0\}$ such that $\{0\} \cup X$ is an interval of G.*

Proof : To commence, assume that I is an interval of Π. By Lemma 2.4, $s(I^\star)$ is an interval of $\mathbb{Q}$ and $\overline{0} \in I$. It follows from Proposition 2.8 that either $s(I^\star) = \mathbb{Q}$ or there is $q \in \mathbb{Q}$ such that $s(I^\star) = (q, +\infty)$ or $[q, +\infty)$. Consider $f \in I$ and assume further that $s(f) > q$ in the case where $s(I^\star) = [q, +\infty)$. There is $g \in I$ such that $s(g) < s(f)$. It follows from Proposition 2.8 that $s^{-1}((s(f), +\infty)) \subseteq I$. Consequently, if $s(I^\star) = \mathbb{Q}$, then $I = {}^{\mathbb{Q}}V$. Similarly, if $s(I^\star) = (q, +\infty)$, then $I = \{\overline{0}\} \cup s^{-1}((q, +\infty))$, that is, $I = \lfloor \overline{0} \rfloor^{>q}$. Assume that $s(I^\star) = [q, +\infty)$. Let $x \in I \downarrow$. Consider $f \in I^\star$ such that $s(f) = q$ and $f(q) = x$. By Corollary 2.5, we have ${}^q x \in I$ and, by Corollary 2.7, $\lfloor {}^q x \rfloor \subseteq I$. Therefore, $I = \{\overline{0}\} \cup s^{-1}((q, +\infty)) \cup \lfloor {}^q(I\downarrow) \rfloor$, that is, $I = \lfloor \overline{0} \rfloor_{I\downarrow}^{\geq q}$. Lastly, we have to verify that $\{0\} \cup (I\downarrow)$ is an interval of G. For every $y \in V \setminus (\{0\} \cup (I\downarrow))$, we have ${}^q y \notin I$. Consequently, there is $\alpha = 0$ or 1 such that $[{}^q y, \{\overline{0}\} \cup \lfloor {}^q(I\downarrow) \rfloor]_\Pi = \alpha$. It results that $[y, \{0\} \cup (I\downarrow)]_G = \alpha$.

Conversely, consider $q \in \mathbb{Q}$. For any $f \in \{\overline{0}\} \cup s^{-1}((q, +\infty))$ and $g \in ({}^{\mathbb{Q}}V) \setminus (\{\overline{0}\} \cup s^{-1}((q, +\infty))$, we have $\delta(f, g) = s(g)$ and hence $[f, g]_\Pi = [0, g(s(g))]_G$. Therefore, $[g, \{\overline{0}\} \cup s^{-1}((q, +\infty))]_\Pi = [0, g(s(g))]_G$. It follows that $\{\overline{0}\} \cup s^{-1}((q, +\infty))$ is an interval of Π. Finally, consider also a nonempty subset X of $V \setminus \{0\}$ such that $\{0\} \cup X$ is an interval of G. Let $g \in ({}^{\mathbb{Q}}V) \setminus (\{\overline{0}\} \cup s^{-1}((q, +\infty)) \cup \lfloor {}^q X \rfloor)$. As previously, if $s(g) < q$, then $[g, \{\overline{0}\} \cup s^{-1}((q, +\infty)) \cup \lfloor {}^q X \rfloor]_\Pi = [0, g(s(g))]_G$. Thus, assume that $s(g) = q$. Similarly, we have $[g, \{\overline{0}\} \cup s^{-1}((q, +\infty))]_\Pi = [0, g(q)]_G$. For every $f \in \lfloor {}^q X \rfloor$, we have $\delta(f, g) = q$ and hence $[f, g]_\Pi = [f(q), g(q)]_G$. Since $f(q) \in X$ and $g(q) \notin X$, and since $\{0\} \cup X$ is an interval of G, we obtain that $[f(q), g(q)]_G = [0, g(q)]_G$. Consequently, $[g, \lfloor {}^q X \rfloor]_\Pi = [0, g(q)]_G$. It follows that $\{\overline{0}\} \cup s^{-1}((q, +\infty)) \cup \lfloor {}^q X \rfloor$ is an interval of Π.

When G is indecomposable, we obtain the following:

Corollary 2.10. *Assume that G is indecomposable. Given $I \subseteq {}^{\mathbb{Q}}V$ such that $|s(I^\star)| > 1$, I is an interval of Π if and only if $I = \lfloor \overline{0} \rfloor$, $\lfloor \overline{0} \rfloor^{>q}$ or $\lfloor \overline{0} \rfloor^{\geq q}$, where $q \in \mathbb{Q}$.*

Proof : Given a nonempty subset X of $V \setminus \{0\}$, if $\{0\} \cup X$ is an interval of G, then $\{0\} \cup X = V$ and $X = V \setminus \{0\}$. But, for $q \in \mathbb{Q}$, we have $\lfloor \overline{0} \rfloor^{\geq q}_{V\setminus\{0\}} = \lfloor \overline{0} \rfloor^{\geq q}$ because $\lfloor {}^{q}(V \setminus \{0\}) \rfloor = s^{-1}(\{q\})$.

2.3 The intervals I of Π such that $|s(I^{\star})| = 1$ and $|I\downarrow| > 1$

Lemma 2.11. *If I is an interval of Π such that $|s(I^{\star})| = 1$ and $|I\downarrow| > 1$, then $\overline{0} \notin I$ and $I\downarrow$ is an interval of G.*

Proof : Denote the unique element of $s(I^{\star})$ by q. Given $x \in I\downarrow$, consider $f \in I^{\star}$ such that $f(q) = x$. By Assumption 1.7, there is $y \in V \setminus \{0\}$ such that $[0,y]_G \neq [0,x]_G$. Given $r > q$, we have ${}^{r}y \notin I$. Since $\delta({}^{r}y, \overline{0}) = r$ and $\delta({}^{r}y, f) = q$, we obtain that $[{}^{r}y, \overline{0}]_{\Pi} = [0,y]_G$ and $[{}^{r}y, f]_{\Pi} = [0,x]_G$. Consequently, $[{}^{r}y, \overline{0}]_{\Pi} \neq [{}^{r}y, f]_{\Pi}$. Necessarily, $\overline{0} \notin I$ because ${}^{r}y \notin I$ and $f \in I$.

To show that $I\downarrow$ is an interval of G, consider $x, x' \in I\downarrow$ and $y \notin I\downarrow$. There are $f, f' \in I$ such that $f(q) = x$ and $f'(q) = x'$. Firstly, assume that $y \neq 0$. Clearly, ${}^{q}y \notin I$ because $y \notin I\downarrow$. Therefore, $[{}^{q}y, f]_{\Pi} = [{}^{q}y, f']_{\Pi}$. As $\delta({}^{q}y, f) = \delta({}^{q}y, f') = q$, we have $[{}^{q}y, f]_{\Pi} = [y,x]_G$ and $[{}^{q}y, f']_{\Pi} = [y,x']_G$. Consequently, $[y,x]_G = [y,x']_G$. Lastly, when $y = 0$, we proceed as previously by considering $\overline{0}$ instead of ${}^{q}y$.

In the preceding statement, we obtain that $I\downarrow$ is a non trivial interval of G. An immediate consequence follows.

Corollary 2.12. *If there is an interval I of Π such that $|s(I^{\star})| = 1$ and $|I\downarrow| > 1$, then G is decomposable.*

When G is decomposable, we obtain the next characterization.

Theorem 2.13. *Given a subset I of ${}^{\mathbb{Q}}V$ such that $|s(I^{\star})| = 1$ and $|I\downarrow| > 1$, I is an interval of Π if and only if $I = \lfloor {}^{q}X \rfloor$, where $q \in \mathbb{Q}$ and X is an interval of G such that $|X| > 1$ and $X \subseteq V \setminus \{0\}$.*

Proof : Denote the unique element of $s(I^{\star})$ by q. To begin, assume that I is an interval of Π. By the previous lemma, $\overline{0} \notin I$ and $I\downarrow$ is an interval of G. As $\overline{0} \notin I$, we have $I \subseteq \lfloor {}^{q}(I\downarrow) \rfloor$. Given $x \in I\downarrow$, let $f \in I$ such that $f(q) = x$. Since $|I\downarrow| > 1$, there exists $f' \in I$ such that $f'(q) \neq x$. By applying Lemma 2.1 to f and f', we have ${}^{q}x \in I$. Then, by applying Proposition 2.3 to ${}^{q}x$ and f', we obtain that $\lfloor {}^{q}x \rfloor \subseteq I$. Consequently, $I = \lfloor {}^{q}(I\downarrow) \rfloor$.

Conversely, consider $q \in \mathbb{Q}$ and X an interval of G such that $0 \notin X$ and $|X| > 1$. For every $y \in V \setminus X$, we have $[y, X]_G = \alpha_y$, where $\alpha_y = 0$ or 1. Let $g \in ({}^{\mathbb{Q}}V) \setminus \lfloor {}^{q}X \rfloor$. Firstly, if $g \neq \overline{0}$ and $s(g) < q$, then $[g, \lfloor {}^{q}X \rfloor]_{\Pi} = [0, g(s(g))]_G$. Secondly, if $g = \overline{0}$ or $g \neq \overline{0}$ and $s(g) > q$, then $g(q) = 0 \notin X$. For every $f \in \lfloor {}^{q}X \rfloor$, we have $\delta(f, g) = q$ and hence $[g, f]_{\Pi} = [0, f(q)]_G$. Since $f(q) \in X$, $[0, f(q)]_G = \alpha_0$. Therefore, $[g, \lfloor {}^{q}X \rfloor]_{\Pi} = \alpha_0$. Lastly, if $g \neq \overline{0}$ and $s(g) = q$, then $g(q) \notin X$ because $g \notin \lfloor {}^{q}X \rfloor$. For every $f \in \lfloor {}^{q}X \rfloor$, we have $\delta(f, g) = q$ and thus $[g, f]_{\Pi} = [g(q), f(q)]_G$. As $f(q) \in X$, $[g(q), f(q)]_G = \alpha_{g(q)}$. Consequently, $[g, \lfloor {}^{q}X \rfloor]_{\Pi} = \alpha_{g(q)}$.

2.4 The intervals I of Π such that $|s(I^{\star})| = 1$ and $|I\downarrow| = 1$

Given $f \in ({}^{\mathbb{Q}}V)^{\star}$, we transform naturally an isomorphism from $\mathbb{Q}$ onto $\mathbb{Q}((S(f), +\infty))$ into an isomorphism from $\Pi(\lfloor f \rfloor)$ onto Π. We use the following notation.

- θ_f denotes an isomorphism from $\mathbb{Q}$ onto $\mathbb{Q}((S(f),+\infty))$.
- $\Theta_f : \lfloor f \rfloor \longrightarrow {}^{\mathbb{Q}}V$ is defined by $\theta_f(g) = (g_{/(S(f),+\infty)}) \circ \theta_f$ for $g \in \lfloor f \rfloor$.
- Given a function $g : (S(f),+\infty) \longrightarrow V$ such that $\{q > S(f) : g(q) \neq 0\}$ is finite, $f+g$ is the element of ${}^{\mathbb{Q}}V$ defined by $(f+g)_{/(-\infty,S(f)]} = f_{/(-\infty,S(f)]}$ and $(f+g)_{/(S(f),+\infty)} = g$. Clearly, $f+g \in \lfloor f \rfloor$ and $\sigma(f+g) = \sigma(f) \cup \{q > S(f) : g(q) \neq 0\}$. Now, $\Omega_f : {}^{\mathbb{Q}}V \longrightarrow \lfloor f \rfloor$ is defined by $\Omega_f(g) = f + (g \circ (\theta_f)^{-1})$ for $g \in {}^{\mathbb{Q}}V$.

We will use the following properties of Θ_f and of Ω_f.

Lemma 2.14.

1. $\Theta_f(f) = \overline{0}$.
2. *For every* $g \in \lfloor f \rfloor \setminus \{f\}$, $n(\Theta_f(g)) = n(g) - n(f)$ *and for* $i \in \{1,\dots,n(g)-n(f)\}$, *we have* $q_i^{\Theta_f(g)} = (\theta_f)^{-1}(q^g_{n(f)+i})$ *and* $\Theta_f(g)(q_i^{\Theta_f(g)}) = g(q^g_{n(f)+i})$.
3. *For any* $g \neq h \in \lfloor f \rfloor$, $\delta(\Theta_f(g), \Theta_f(h)) = (\theta_f)^{-1}(\delta(g,h))$.
4. $\Omega_f(\overline{0}) = f$.
5. *For every* $g \in ({}^{\mathbb{Q}}V)^{\star}$, $n(\Omega_f(g)) = n(f) + n(g)$. *If* $i \in \{1,\dots,n(f)\}$, *then* $q_i^{\Omega_f(g)} = q_i^f$ *and* $\Omega_f(g)(q_i^{\Omega_f(g)}) = f(q_i^f)$. *If* $i \in \{n(f)+1,\dots,n(f)+n(g)\}$, *then* $q_i^{\Omega_f(g)} = \theta_f(q^g_{i-n(f)})$ *and* $\Omega_f(g)(q_i^{\Omega_f(g)}) = g(q^g_{i-n(f)})$.

Proof : The first and fourth points are clear by the definition of Θ_f and of Ω_f. For the second, consider $g \in \lfloor f \rfloor \setminus \{f\}$ and $q \in \mathbb{Q}$. The following assertions are equivalent:

- $\Theta_f(g)(q) \neq 0$;
- $g_{/(S(f),+\infty)}(\theta_f(q)) \neq 0$;
- $\theta_f(q) \in \sigma(g) \cap (S(f),+\infty)$;
- there is $j \in \{n(f)+1,\dots,n(g)\}$ such that $\theta_f(q) = q^g_j$;
- there is $i \in \{1,\dots,n(g)-n(f)\}$ such that $q = (\theta_f)^{-1}(q^g_{n(f)+i})$.

For the third point, consider $g \neq h \in \lfloor f \rfloor$. For each $q \in \mathbb{Q}$, the following assertions are equivalent:

- $\Theta_f(g)(q) \neq \Theta_f(h)(q)$;
- $g_{/(S(f),+\infty)}(\theta_f(q)) \neq h_{/(S(f),+\infty)}(\theta_f(q))$;
- $\theta_f(q) \in \{r > S(f) \ : \ g(r) \neq h(r)\}$;
- $q \in (\theta_f)^{-1}(\{r > S(f) \ : \ g(r) \neq h(r)\})$.

It follows that $\min(\{q \in \mathbb{Q} \ : \ \Theta_f(g)(q) \neq \Theta_f(h)(q)\}) = \min((\theta_f)^{-1}(\{r > S(f) \ : \ g(r) \neq h(r)\}))$. Since θ_f is an isomorphism from $\mathbb{Q}$ onto $\mathbb{Q}((S(f),+\infty))$, we obtain that $\min((\theta_f)^{-1}(\{r > S(f) \ : \ g(r) \neq h(r)\})) = (\theta_f)^{-1}(\min(\{r > S(f) \ : \ g(r) \neq h(r)\}))$. As $g \neq h \in \lfloor f \rfloor$, we have $\{r > S(f) \ ; \ g(r) \neq h(r)\} = \{r \in \mathbb{Q} \ ; \ g(r) \neq h(r)\}$. Consequently, $\delta(\Theta_f(g), \Theta_f(h)) = (\theta_f)^{-1}(\delta(g,h))$.

For the last point, consider $q \in \mathbb{Q}$. The following assertions are equivalent:

- $\Omega_f(g)(q) \neq 0$;
- $(f + (g \circ (\theta_f)^{-1}))(q) \neq 0$;
- either $q \leq S(f)$ and $q \in \sigma(f)$ or $q > S(f)$ and $g((\theta_f)^{-1}(q)) \neq 0$;

- either $q \in \sigma(f)$ or $q > S(f)$ and $(\theta_f)^{-1}(q) \in \sigma(g)$;
- $q \in \sigma(f) \cup \theta_f(\sigma(g))$.

Therefore, $\sigma(\Omega_f(g)) = \sigma(f) \cup \theta_f(\sigma(g))$ and thus $n(\Omega_f(g)) = n(f) + n(g)$. More precisely, for each $i \in \{1,\dots,n(f)+n(g)\}$, we obtain that either $i \le n(f)$ and $q_i^{\Omega_f(g)} = q_i^f$ or $i > n(f)$ and $q_i^{\Omega_f(g)} = \theta_f(q^g_{i-n(f)})$. Finally, it follows from the definition of $\Omega_f(g)$ that for $1 \le i \le n(f)$, $\Omega_f(g)(q_i^{\Omega_f(g)}) = f(q_i^f)$ and for $n(f)+1 \le i \le n(f)+n(g)$, $\Omega_f(g)(q_i^{\Omega_f(g)}) = g(q^g_{i-n(f)})$.

The next result is an easy consequence.

Proposition 2.15. *For each $f \in ({}^{\mathbb{Q}}V)^\star$, the function Θ_f realizes an isomorphism from $\Pi(\lfloor f \rfloor)$ onto Π and $(\Theta_f)^{-1} = \Omega_f$. Moreover, for any $g,h \in \lfloor f \rfloor$, we have: $g < h$ if and only if $\Theta_f(g) < \Theta_f(h)$.*

Proof : Given $g \in \lfloor f \rfloor$, we have:

$$(\Omega_f \circ \Theta_f)(g) = \Omega_f(g_{/(S(f),+\infty)} \circ \theta_f) = f + g_{/(S(f),+\infty)} = g.$$

Conversely, given $g \in {}^{\mathbb{Q}}V$, we have:

$$(\Theta_f \circ \Omega_f)(g) = \Theta_f(f + (g \circ (\theta_f)^{-1})) = (f + (g \circ (\theta_f)^{-1}))_{/(S(f),+\infty)} \circ \theta_f$$

and

$$(f + (g \circ (\theta_f)^{-1}))_{/(S(f),+\infty)} \circ \theta_f = (g \circ (\theta_f)^{-1}) \circ \theta_f = g.$$

Consequently, Θ_f is bijective and $(\Theta_f)^{-1} = \Omega_f$.

Now, consider $g \neq h \in \lfloor f \rfloor$. We have $[g,h]_\Pi = [g(\delta(g,h)), h(\delta(g,h))]_G$ and $[\Theta_f(g), \Theta_f(h)]_\Pi = [\Theta_f(g)(\delta(\Theta_f(g),\Theta_f(h))), \Theta_f(h)(\delta(\Theta_f(g),\Theta_f(h)))]_G$. It follows from the third assertion of Lemma 2.14 that $\delta(\Theta_f(g),\Theta_f(h)) = (\theta_f)^{-1}(\delta(g,h))$. Furthermore, $\Theta_f(g)((\theta_f)^{-1}(\delta(g,h))) = g(\delta(g,h))$ and $\Theta_f(h)((\theta_f)^{-1}(\delta(g,h))) = h(\delta(g,h))$. Therefore, $[g,h]_\Pi = [\Theta_f(g),\Theta_f(h)]_\Pi$.

Lastly, consider $g,h \in \lfloor f \rfloor$ such that $g < h$. We have $n(g) < n(h)$ and for $i \in \{1,\dots,n(g)\}$, $q_i^g = q_i^h$ and $g(q_i^g) = h(q_i^h)$. Obviously, if $g = f$, then $\Theta_f(g) = \overline{0} \neq \Theta_f(h)$ because Θ_f is injective. Therefore, $\Theta_f(g) = \overline{0} < \Theta_f(h)$. Assume that $f < g$. It follows from the second assertion of Lemma 2.14 that $n(\Theta_f(g)) = n(g) - n(f) < n(h) - n(f) = n(\Theta_f(h))$ and for $i \in \{1,\dots,n(g)-n(f)\}$, $q_i^{\Theta_f(g)} = (\theta_f)^{-1}(q^g_{n(f)+i}) = (\theta_f)^{-1}(q^h_{n(f)+i}) = q_i^{\Theta_f(h)}$ and $\Theta_f(g)(q_i^{\Theta_f(g)}) = g(q^g_{n(f)+i}) = h(q^h_{n(f)+i}) = \Theta_f(h)(q_i^{\Theta_f(h)})$. Consequently, $\Theta_f(g) < \Theta_f(h)$.

Conversely, consider $g,h \in {}^{\mathbb{Q}}V$ such that $g < h$. Firstly, assume that $g = \overline{0}$ so that $\Omega_f(g) = f$. As $g < h$, we have $h \neq \overline{0}$ and hence $\Omega_f(h) \neq f$ because Ω_f is injective. Therefore, $\Omega_f(h) \in \lfloor f \rfloor \setminus \{f\}$, that is, $f = \Omega_f(g) < \Omega_f(h)$. Secondly, assume that $g \neq \overline{0}$. Since $\Omega_f(g), \Omega_f(h) \in \lfloor f \rfloor$, we have $q_i^{\Omega_f(g)} = q_i^f = q_i^{\Omega_f(h)}$ and $\Omega_f(g)(q_i^{\Omega_f(g)}) = f(q_i^f) = \Omega_f(h)(q_i^{\Omega_f(h)})$ for $i \in \{1,\dots,n(f)\}$. Furthermore, as $g < h$, we have $n(g) < n(h)$ and for $i \in \{1,\dots,n(g)\}$, we have $q_i^g = q_i^h$ and $g(q_i^g) = h(q_i^h)$. Then, it follows from the last assertion of Lemma 2.14 that $n(\Omega_f(g)) = n(f) + n(g) < n(f) + n(h) = n(\Omega_f(h))$ and for $i \in \{n(f)+$

$1,\ldots,n(f)+n(g)\}$, $q_i^{\Omega_f(g)} = \theta_f(q^g_{i-n(f)}) = \theta_f(q^h_{i-n(f)}) = q_i^{\Omega_f(h)}$ and $\Omega_f(g)(q_i^{\Omega_f(g)}) = g(q^g_{i-n(f)}) = h(q^h_{i-n(f)}) = \Omega_f(h)(q_i^{\Omega_f(h)})$. Consequently, $\Omega_f(g) < \Omega_f(h)$.

To conclude the subsection, we obtain the following characterization.

Theorem 2.16. *Given $I \subseteq {}^{\mathbb{Q}}V$ such that $|I| > 1$, $|s(I^\star)| = 1$ and $|I\downarrow| = 1$, I is an interval of Π in precisely one of the four cases below.*

1. *$I = \lfloor f \rfloor$, where $f \in ({}^{\mathbb{Q}}V)^\star$.*
2. *$I = \lfloor f \rfloor^{>q}$, where $f \in ({}^{\mathbb{Q}}V)^\star$ and $q > S(f)$.*
3. *$I = \lfloor f \rfloor_X^{\geq q}$, where $f \in ({}^{\mathbb{Q}}V)^\star$, $q > S(f)$ and X is a nonempty subset of $V \setminus \{0\}$ such that $\{0\} \cup X$ is an interval of G.*
4. *$I = \lfloor f \rfloor_X^{=q}$, where $f \in ({}^{\mathbb{Q}}V)^\star$, $q > S(f)$ and X is an interval of G such that $0 \notin X$ and $|X| > 1$.*

Proof : To commence, we verify that $\lfloor f \rfloor$ is an interval of Π for every $f \in {}^{\mathbb{Q}}V$. We proceed by induction on $n(f)$. If $n(f) = 0$, then $f = \overline{0}$ and $\lfloor \overline{0} \rfloor = {}^{\mathbb{Q}}V$ is an interval of Π. If $n(f) = 1$, then $f =^q x$, where $q = s(f)$ and $x = f(s(f))$. For each $g \in ({}^{\mathbb{Q}}V)^\star \setminus \lfloor {}^q x \rfloor$, we distinguish the following cases:

- if $g = \overline{0}$, then $[\overline{0}, \lfloor {}^q x \rfloor]_\Pi = [0,x]_G$;
- if $g \neq \overline{0}$ and $s(g) < q$, then $[g, \lfloor {}^q x \rfloor]_\Pi = [g(s(g)),0]_G$;
- if $g \neq \overline{0}$ and $q < s(g)$, then $[g, \lfloor {}^q x \rfloor]_\Pi = [0,x]_G$;
- if $g \neq \overline{0}$ and $s(g) = q$, then $g(q) \neq x$ and $[g, \lfloor {}^q x \rfloor]_\Pi = [g(q),x]_G$.

Consequently, $\lfloor {}^q x \rfloor$ is an interval of Π. Now, consider $f \in ({}^{\mathbb{Q}}V)^\star$ such that $n(f) \geq 2$. We proved that $\lfloor {}^q x \rfloor$ is an interval of Π, where $q = s(f)$ and $x = f(s(f))$. It follows from Lemma 2.14 that that $n(\Theta_{({}^q x)}(f)) = n(f) - 1$. By the induction hypothesis, $\lfloor \Theta_{({}^q x)}(f) \rfloor$ is an interval of Π. It follows from Proposition 2.15 applied to ${}^q x$ that $\lfloor \Theta_{({}^q x)}(f) \rfloor = \Theta_{({}^q x)}(\lfloor f \rfloor)$ and hence that $\lfloor f \rfloor$ is an interval of $\Pi(\lfloor {}^q x \rfloor)$. As $\lfloor {}^q x \rfloor$ is an interval of Π, $\lfloor f \rfloor$ is as well by Proposition 1.1.

To continue, consider $I \subseteq {}^{\mathbb{Q}}V$, with $|I| > 1$, satisfying: there is $q \in \mathbb{Q}$ such that $s(I^\star) = \{q\}$ and there is $x \in V \setminus \{0\}$ such that $I\downarrow = \{x\}$. Denote $\wedge I$ by f. We have ${}^q x \leq f$ and $I \subseteq \lfloor f \rfloor$. As $\lfloor f \rfloor$ is an interval of Π, we have: I is an interval of Π if and only if I is an interval of $\Pi(\lfloor f \rfloor)$. Moreover, it follows from Proposition 2.15 that I is an interval of $\Pi(\lfloor f \rfloor)$ if and only if $\Theta_f(I)$ is an interval of Π. For a contradiction, suppose that there is $p \in \mathbb{Q}$ such that $s((\Theta_f(I))^\star) = \{p\}$ and there is $y \in V \setminus \{0\}$ such that $(\Theta_f(I))\downarrow = \{y\}$. It follows that $\overline{0} < ({}^p y) \leq \wedge(\Theta_f(I))$. By Proposition 2.15, we have $\wedge(\Theta_f(I)) = \Theta_f(\wedge I)$. By applying Ω_f, we would obtain that $f < \Omega_f({}^p y) \leq \wedge I$. Consequently, either $|s((\Theta_f(I))^\star)| > 1$ or $|s((\Theta_f(I))^\star)| = 1$ and $|(\Theta_f(I))\downarrow| > 1$. To conclude, we distinguish the two cases below for application of Theorem 2.9 or Theorem 2.13 to $\Theta_f(I)$.

1. Assume that $|s((\Theta_f(I))^\star)| > 1$. By Theorem 2.9, $\Theta_f(I)$ is an interval of Π in one of the three cases below.

(a) $\Theta_f(I) = {}^{\mathbb{Q}}V$, that is, $I = \lfloor f \rfloor$.

(b) There is $p \in \mathbb{Q}$ such that $\Theta_f(I) = \{\overline{0}\} \cup s^{-1}((p,+\infty))$ or, equivalently, $I = \lfloor f \rfloor^{>\theta_f(p)}$.

(c) There is $p \in \mathbb{Q}$ such that $\Theta_f(I) = \{\overline{0}\} \cup s^{-1}((p,+\infty)) \cup \lfloor^p X \rfloor$, where X is a nonempty subset of $V \setminus \{0\}$ such that $\{0\} \cup X$ is an interval of G. We obtain that $I = \lfloor f \rfloor_X^{\geq \theta_f(p)}$.

2. Assume that there is $p \in \mathbb{Q}$ such that $s((\Theta_f(I))^\star) = \{p\}$. Denote $(\Theta_f(I)) \downarrow$ by X. Clearly, $X \subseteq V \setminus \{0\}$ and, as previously observed, we have $|X| > 1$. By Theorem 2.13, $\Theta_f(I)$ is an interval of Π if and only if X is an interval of G and $\Theta_f(I) = \lfloor^p X \rfloor$, that is, $I = \lfloor f \rfloor_X^{=\theta_f(p)}$.

When G is indecomposable, the preceding theorem is stated as follows.

Corollary 2.17. *Assume that G is indecomposable. Given $I \subseteq {}^{\mathbb{Q}}V$ such that $|I| > 1$, $|s(I^\star)| = 1$ and $|I\downarrow| = 1$, I is an interval of Π if and only if there exists $f \in ({}^{\mathbb{Q}}V)^\star$ and there is $q > S(f)$ such that $I = \lfloor f \rfloor, \lfloor f \rfloor^{>q}$ or $\lfloor f \rfloor^{\geq q}$.*

We summarize Theorems 2.9, 2.13, 2.16 and Corollaries 4, 5, 6 as below in Theorem 2.18 and Corollary 2.19. To simplify their statement, we extend the total order $\mathbb{Q}$ to $\{-\infty\} \cup \mathbb{Q}$ by considering $-\infty$ smaller than all the rational numbers. We also extend the function S to ${}^{\mathbb{Q}}V$ by $S(\overline{0}) = -\infty$. In particular, we obtain that ${}^q\overline{0}_x = {}^q x$ for $q \in \mathbb{Q}$ and $x \in V \setminus \{0\}$.

Theorem 2.18. *Given $I \subseteq {}^{\mathbb{Q}}V$ such that $|I| > 1$, I is an interval of Π in precisely one of the four cases below.*

1. *$I = \lfloor f \rfloor$, where $f \in {}^{\mathbb{Q}}V$.*

2. *$I = \lfloor f \rfloor^{>q}$, where $f \in {}^{\mathbb{Q}}V$ and $q > S(f)$.*

3. *$I = \lfloor f \rfloor_X^{\geq q}$, where $f \in {}^{\mathbb{Q}}V$, $q > S(f)$ and X is a nonempty subset of $V \setminus \{0\}$ such that $\{0\} \cup X$ is an interval of G.*

4. *$I = \lfloor f \rfloor_X^{=q}$, where $f \in {}^{\mathbb{Q}}V$, $q > S(f)$ and X is an interval of G such that $0 \notin X$ and $|X| > 1$.*

Corollary 2.19. *Assume that G is indecomposable. Given $I \subseteq {}^{\mathbb{Q}}V$ such that $|I| > 1$, I is an interval of Π if and only if there exists $f \in {}^{\mathbb{Q}}V$ and there is $q > S(f)$ such that $I = \lfloor f \rfloor, \lfloor f \rfloor^{>q}$ or $\lfloor f \rfloor^{\geq q}$.*

3 The strong intervals of Π

We examine specific strong intervals of Π in the four lemmas below.

Lemma 3.1. *For every $f \in {}^{\mathbb{Q}}V$, $\lfloor f \rfloor$ is a strong interval of Π.*

Proof : We proceed by induction on $n(f)$ as for the beginning of the proof of Theorem 2.16. If $n(f)=0$, then $f=\overline{0}$ and $\lfloor f\rfloor = {}^{\mathbb{Q}}V$ is a strong interval of Π. If $n(f)=1$, then $f={}^{q}x$, where $q=s(f)$ and $x=f(s(f))$. Consider an interval I of Π such that $|I|>1$ and $I\cap\lfloor {}^{q}x\rfloor\neq\emptyset$. Let $g\in I\cap\lfloor {}^{q}x\rfloor$. We distinguish the three cases below.

- Assume that $|s(I^{\star})|>1$. By Corollary 2.5 applied to g, ${}^{q}x\in I$ and $\lfloor {}^{q}x\rfloor\subseteq I$ by Corollary 2.7.
- Assume that $|s(I^{\star})|=1$ and $|I\downarrow|>1$. By Theorem 2.13, we have $I=\lfloor {}^{r}X\rfloor$, where $X\subseteq V\setminus\{0\}$. As $g\in I\cap\lfloor {}^{q}x\rfloor$, we obtain that $r=q$ and $x\in X$ so that $\lfloor {}^{q}x\rfloor\subseteq\lfloor {}^{q}X\rfloor$.
- Assume that $|s(I^{\star})|=1$ and $|I\downarrow|=1$. It follows from the proof of Theorem 2.16 that $\overline{0}<\wedge I$ and $I\subseteq\lfloor\wedge I\rfloor$. Consequently, ${}^{q}x\leq g$ and $\wedge I\leq g$. It results that either ${}^{q}x\leq\wedge I$ or $\wedge I<{}^{q}x$. As $\wedge I\neq\overline{0}$, we obtain that ${}^{q}x\leq\wedge I$ and thus $I\subseteq\lfloor\wedge I\rfloor\subseteq\lfloor {}^{q}x\rfloor$.

Now, consider $f\in{}^{\mathbb{Q}}V$ such that $n(f)=2$. We showed that $\lfloor {}^{q}x\rfloor$ is a strong interval of Π, where $q=s(f)$ and $x=f(s(f))$. By Lemma 2.14, we have $n(\Theta_{({}^{q}x)}(f))=n(f)-1$. By the induction hypothesis, we obtain that $\lfloor\Theta_{({}^{q}x)}(f)\rfloor$ is a strong interval of Π. It follows from Proposition 2.15 applied to ${}^{q}x$ that $\lfloor\Theta_{({}^{q}x)}(f)\rfloor=\Theta_{({}^{q}x)}(\lfloor f\rfloor)$ and hence that $\lfloor f\rfloor$ is a strong interval of $\Pi(\lfloor {}^{q}x\rfloor)$. As $\lfloor {}^{q}x\rfloor$ is a strong interval of Π, $\lfloor f\rfloor$ is also by Proposition 1.2 (B2.(ii)).

Lemma 3.2. *For every $q\in\mathbb{Q}$, $\lfloor\overline{0}\rfloor^{>q}$ is a strong interval of Π.*

Proof : Consider an interval I of Π such that $I\setminus\lfloor\overline{0}\rfloor^{>q}\neq\emptyset$ and $I\cap\lfloor\overline{0}\rfloor^{>q}\neq\emptyset$. We have to show that $\lfloor\overline{0}\rfloor^{>q}\subseteq I$. So, assume that $I\neq{}^{\mathbb{Q}}V$. Let $f\in I\setminus\lfloor\overline{0}\rfloor^{>q}$ and $g\in I\cap\lfloor\overline{0}\rfloor^{>q}$. We obtain that $f\neq\overline{0}$ and $s(f)\leq q$. Moreover, either $g=\overline{0}$ or $g\neq\overline{0}$ and $s(g)>q$. In the first instance, $\overline{0}\in I$ and, since $|I|>1$, it follows from Theorems 2.13 and 2.16 that $|s(I^{\star})|>1$. Therefore, $|s(I^{\star})|>1$ in both instances. By Theorem 2.9, one of the following cases occurs.

- There is $r\in\mathbb{Q}$ such that $I=\lfloor\overline{0}\rfloor^{>r}$. We obtain that $r<s(f)\leq q$ and thus $\lfloor\overline{0}\rfloor^{>q}\subset\lfloor\overline{0}\rfloor^{>r}$.
- There exist $r\in\mathbb{Q}$ and $\emptyset\neq X\subseteq V\setminus\{0\}$ such that $I=\lfloor\overline{0}\rfloor_{X}^{\geq r}$, where $X\subseteq V\setminus\{0\}$. We obtain that $r\leq s(f)\leq q$. For every $h\in\lfloor\overline{0}\rfloor^{>q}\setminus\{\overline{0}\}$, we have $s(h)>q\geq r$ and hence $h\in\lfloor\overline{0}\rfloor_{X}^{\geq r}$. Consequently, $\lfloor\overline{0}\rfloor^{>q}\subset\lfloor\overline{0}\rfloor_{X}^{\geq r}$ because $\overline{0}\in\lfloor\overline{0}\rfloor_{X}^{\geq r}$.

Lemma 3.3. *Consider $q\in\mathbb{Q}$ and a nonempty subset X of $V\setminus\{0\}$ such that $\{0\}\cup X$ is an interval of G. We have $\lfloor\overline{0}\rfloor_{X}^{\geq q}$ is a strong interval of Π if and only if $\{0\}\cup X$ is a strong interval of G.*

Proof : By Theorem 2.9, $\lfloor\overline{0}\rfloor_{X}^{\geq q}$ is an interval of Π. To begin, suppose that $\{0\}\cup X$ is not a strong interval of G. Since $\{0\}\cup X$ is an interval of G, there exists an interval Y of G such that $Y\cap(\{0\}\cup X)$, $Y\setminus(\{0\}\cup X)$ and $(\{0\}\cup X)\setminus Y$ are all nonempty. Firstly, assume that $0\notin Y$. By Theorem 2.13, $\lfloor\overline{0}\rfloor_{Y}^{=q}$ is an interval of Π. Let $y\in Y\cap(\{0\}\cup X)$ and $z\in Y\setminus(\{0\}\cup X)$. As $y,z\in Y$, we have $y,z\in V\setminus\{0\}$. Clearly, ${}^{q}y\in\lfloor\overline{0}\rfloor_{Y}^{=q}\cap\lfloor\overline{0}\rfloor_{X}^{\geq q}$ and ${}^{q}z\in\lfloor\overline{0}\rfloor_{Y}^{=q}\setminus\lfloor\overline{0}\rfloor_{X}^{\geq q}$. Furthermore, $\overline{0}\in\lfloor\overline{0}\rfloor_{X}^{\geq q}\setminus\lfloor\overline{0}\rfloor_{Y}^{=q}$ whence $\lfloor\overline{0}\rfloor_{X}^{\geq q}$ is not strong. Secondly, assume that $0\in Y$ and set $Z=Y\setminus\{0\}$. Since $|Y|>1$, we have $Z\neq\emptyset$ and it follows from

Theorem 2.9 that $\lfloor\overline{0}\rfloor_Z^{\geq q}$ is an interval of Π. Clearly, $\overline{0} \in \lfloor\overline{0}\rfloor_X^{\geq q} \cap \lfloor\overline{0}\rfloor_Z^{\geq q}$. Let $x \in (\{0\} \cup X) \setminus (\{0\} \cup Z)$ and $y \in (\{0\} \cup Z) \setminus (\{0\} \cup X)$. We have $x, y \in V \setminus \{0\}$ and thus ${}^q x \in \lfloor\overline{0}\rfloor_X^{\geq q} \setminus \lfloor\overline{0}\rfloor_Z^{\geq q}$ and ${}^q y \in \lfloor\overline{0}\rfloor_Z^{\geq q} \setminus \lfloor\overline{0}\rfloor_X^{\geq q}$. Consequently, $\lfloor\overline{0}\rfloor_X^{\geq q}$ is not a strong interval of Π in both cases.

Conversely, assume that $\{0\} \cup X$ is a strong interval of G and consider an interval I of Π such that $I \setminus \lfloor\overline{0}\rfloor_X^{\geq q}$ and $I \cap \lfloor\overline{0}\rfloor_X^{\geq q}$ are nonempty. We have to prove that $\lfloor\overline{0}\rfloor_X^{\geq q} \subseteq I$. So, assume that $I \neq {}^{\mathbb{Q}}V$. As $I \subseteq \lfloor\wedge I\rfloor$, we have $\lfloor\wedge I\rfloor \setminus \lfloor\overline{0}\rfloor_X^{\geq q}$ and $\lfloor\wedge I\rfloor \cap \lfloor\overline{0}\rfloor_X^{\geq q}$ are nonempty. It follows from Lemma 3.1 that $\lfloor\overline{0}\rfloor_X^{\geq q} \subseteq \lfloor\wedge I\rfloor$. In particular, $\overline{0} \in \lfloor\wedge I\rfloor$ and hence $\wedge I = \overline{0}$. For a contradiction, suppose that there is $r \in \mathbb{Q}$ such that $s(I^\star) = \{r\}$. Since $\wedge I = \overline{0}$, it follows from Theorem 2.18 that there exists an interval Y of G such that $I = \lfloor\overline{0}\rfloor_Y^{=r}$, where $0 \notin Y$ and $|Y| \geq 1$. Let $f \in \lfloor\overline{0}\rfloor_Y^{=r} \setminus \lfloor\overline{0}\rfloor_X^{\geq q}$ and $g \in \lfloor\overline{0}\rfloor_Y^{=r} \cap \lfloor\overline{0}\rfloor_X^{\geq q}$. As $\overline{0} \notin \lfloor\overline{0}\rfloor_Y^{=r}$, we have $f \neq \overline{0}$ and $g \neq \overline{0}$. We obtain that $s(f) = r \leq q$ and $s(g) = r \geq q$ so that $q = r$. Furthermore, $f(q) \in Y \setminus (\{0\} \cup X)$ and $g(q) \in Y \cap X = Y \cap (\{0\} \cup X)$. Since $\{0\} \cup X$ is assumed to be a strong interval of G, we should obtain that $\{0\} \cup X \subseteq Y$, which is impossible because $0 \notin Y$. Consequently, $|s(I^\star)| \neq 1$ and thus $|s(I^\star)| > 1$ because $|I| > 1$. By Theorem 2.9, one of the two cases below occurs.

- There is $r \in \mathbb{Q}$ such that $I = \lfloor\overline{0}\rfloor^{>r}$. By Lemma 3.2, I is a strong interval of Π and hence $\lfloor\overline{0}\rfloor_X^{\geq q} \subseteq I$.
- There is $r \in \mathbb{Q}$ and there is a nonempty subset Y of $V \setminus \{0\}$ such that $I = \lfloor\overline{0}\rfloor_Y^{\geq r}$ and $\{0\} \cup Y$ is an interval of G. Consider $f \in \lfloor\overline{0}\rfloor_Y^{\geq r} \setminus \lfloor\overline{0}\rfloor_X^{\geq q}$. We obtain that $r \leq s(f) \leq q$. Assume that $r < q$. Given $g \in \lfloor\overline{0}\rfloor_X^{\geq q}$, we have either $g = \overline{0}$ or $g \neq \overline{0}$ and $r < q \leq s(g)$. In both cases, $g \in \lfloor\overline{0}\rfloor_Y^{\geq r}$. Therefore, $\lfloor\overline{0}\rfloor_X^{\geq q} \subseteq \lfloor\overline{0}\rfloor_Y^{\geq r}$. Lastly, assume that $r = q$. We obtain that $s(f) = q$ and $f(q) \in Y \setminus X$ so that $(\{0\} \cup Y) \setminus (\{0\} \cup X) \neq \emptyset$. Since $0 \in (\{0\} \cup Y) \cap (\{0\} \cup X)$ and since $\{0\} \cup X$ is a strong interval of G, we have $\{0\} \cup X \subseteq \{0\} \cup Y$ and hence $X \subseteq Y$. Consequently, $\lfloor\overline{0}\rfloor_X^{\geq q} \subseteq I = \lfloor\overline{0}\rfloor_Y^{\geq r}$.

Lemma 3.4. *Let $q \in \mathbb{Q}$. Consider an interval X of G such that $|X| > 1$ and $X \subseteq V \setminus \{0\}$. We have $\lfloor\overline{0}\rfloor_X^{=q}$ is a strong interval of Π if and only if X is a strong interval of G.*

Proof : By Theorem 2.13, $\lfloor\overline{0}\rfloor_X^{=q}$ is an interval of Π. To commence, assume that X is not a strong interval of G. Since X is an interval of G, there exists an interval Y of G such that $X \cap Y$, $X \setminus Y$ and $Y \setminus X$ are all nonempty. Firstly, assume that $0 \notin Y$. By Theorem 2.13, $\lfloor\overline{0}\rfloor_Y^{=q}$ is an interval of Π. Let $x \in X \cap Y$, $y \in X \setminus Y$ and $z \in Y \setminus X$. Clearly, ${}^q x \in \lfloor\overline{0}\rfloor_X^{=q} \cap \lfloor\overline{0}\rfloor_Y^{=q}$, ${}^q y \in \lfloor\overline{0}\rfloor_X^{=q} \setminus \lfloor\overline{0}\rfloor_Y^{=q}$ and ${}^q z \in \lfloor\overline{0}\rfloor_Y^{=q} \setminus \lfloor\overline{0}\rfloor_X^{=q}$. Secondly, assume that $0 \in Y$ and set $Z = Y \setminus \{0\}$. By Theorem 2.9, $\lfloor\overline{0}\rfloor_Z^{\geq q}$ is an interval of Π. Clearly, $\overline{0} \in \lfloor\overline{0}\rfloor_Z^{\geq q} \setminus \lfloor\overline{0}\rfloor_X^{=q}$. Let $x \in X \cap Y$ and $y \in X \setminus Y$. As $x, y \in X$, we have $x, y \in V \setminus \{0\}$ and hence $x \in X \cap Z$. Then, ${}^q x \in \lfloor\overline{0}\rfloor_X^{=q} \cap \lfloor\overline{0}\rfloor_Z^{\geq q}$ and ${}^q y \in \lfloor\overline{0}\rfloor_X^{=q} \setminus \lfloor\overline{0}\rfloor_Z^{\geq q}$. Consequently, $\lfloor\overline{0}\rfloor_X^{=q}$ is not a strong interval of Π in both cases.

Conversely, assume that X is a strong interval of G. Consider an interval I of Π such that $I \setminus \lfloor\overline{0}\rfloor_X^{=q}$ and $I \cap \lfloor\overline{0}\rfloor_X^{=q}$ are nonempty. We have to establish that $\lfloor\overline{0}\rfloor_X^{=q} \subseteq I$. So, assume that $I \neq {}^{\mathbb{Q}}V$. As $I \subseteq \lfloor\wedge I\rfloor$, we obtain that $\lfloor\wedge I\rfloor \setminus \lfloor\overline{0}\rfloor_X^{=q}$ and $\lfloor\wedge I\rfloor \cap \lfloor\overline{0}\rfloor_X^{=q}$ are nonempty as well. By Lemma 3.1, $\lfloor\wedge I\rfloor$ is a strong interval of Π and thus $\lfloor\overline{0}\rfloor_X^{=q} \subseteq \lfloor\wedge I\rfloor$. Let x and y be distinct elements of X. Since ${}^q x \geq \wedge I$ and ${}^q y \geq \wedge I$, we have $\wedge I = \overline{0}$ because ${}^q x \wedge {}^q y = \overline{0}$. Firstly, assume that $|s(I^\star)| > 1$. As $I \neq {}^{\mathbb{Q}}V$, it follows from Theorem 2.9 that there is $q \in \mathbb{Q}$ such that either $I = \lfloor\overline{0}\rfloor^{>q}$ or $I = \lfloor\overline{0}\rfloor_Y^{\geq q}$, where $\emptyset \neq Y \subseteq V \setminus \{0\}$ and $\{0\} \cup Y$ is an interval of G. By Lemmas 3.2 and 3.3, I is a strong interval of Π and hence $\lfloor\overline{0}\rfloor_X^{=q} \subseteq I$. Secondly, assume that $|s(I^\star)| \leq 1$. As $|I| > 1$, there is $r \in \mathbb{Q}$ such that $s(I^\star) = \{r\}$. Since $\wedge I = \overline{0}$, it follows from Theorem 2.13 that there is an interval Z of G, with $|Z| > 1$ and $Z \subseteq V \setminus \{0\}$, such that $I = \lfloor \overline{0} \rfloor_Z^{=r}$. As $\lfloor \overline{0} \rfloor_Z^{=r} \cap \lfloor \overline{0} \rfloor_X^{=q} \neq \emptyset$, we have $q = r$ and $Z \cap X \neq \emptyset$. Furthermore, $\lfloor \overline{0} \rfloor_Z^{=q} \setminus \lfloor \overline{0} \rfloor_X^{=q} \neq \emptyset$ implies that $Z \setminus X \neq \emptyset$. Since X is a strong interval of G, we obtain that $X \subseteq Z$ and hence $\lfloor \overline{0} \rfloor_X^{=q} \subseteq \lfloor \overline{0} \rfloor_Z^{=q} = I$.

The next characterization of the strong intervals of Π follows from the four lemmas above by using Theorem 2.18 and Proposition 2.15.

Theorem 3.5. *Given a subset I of ${}^{\mathbb{Q}}V$ such that $|I| > 1$, I is a strong interval of Π if and only if there is $f \in {}^{\mathbb{Q}}V$, there is $\emptyset \neq X \subseteq V \setminus \{0\}$ and there is $q \in \mathbb{Q}$, with $q > S(f)$, such that one of the following is satisfied.*

1. *$I = \lfloor f \rfloor$.*
2. *$I = \lfloor f \rfloor^{>q}$.*
3. *$I = \lfloor f \rfloor_X^{\geq q}$ and $\{0\} \cup X$ is a strong interval of G.*
4. *$I = \lfloor f \rfloor_X^{=q}$, $|X| > 1$and X is a strong interval of G.*

Proof : By Theorem 2.18, we have only to consider the following subsets of ${}^{\mathbb{Q}}V$, where $f \in {}^{\mathbb{Q}}V$ and $q \in \mathbb{Q}$, with $q > S(f)$:

(i) $I = \lfloor f \rfloor$;

(ii) $I = \lfloor f \rfloor^{>q}$;

(iii) $I = \lfloor f \rfloor_X^{\geq q}$, where $\emptyset \neq X \subseteq V \setminus \{0\}$ and $\{0\} \cup X$ is an interval of G;

(iv) $I = \lfloor f \rfloor_X^{=q}$, where X is a non trivial interval of G contained in $V \setminus \{0\}$.

In the first case, Lemma 3.1 applies. Consider one of the other three. If $f = \overline{0}$, then it suffices to apply Lemma 3.2, Lemma 3.3 or Lemma 3.4. When $f \neq \overline{0}$, we conclude in the same way after using Proposition 2.15. Indeed, as $\lfloor f \rfloor$ is a strong interval of Π by Lemma 3.1 and as $I \subseteq \lfloor f \rfloor$, we have by Proposition 1.2 (B2.(ii)): I is a strong interval of Π if and only if I is a strong interval of $\Pi(\lfloor f \rfloor)$. By Proposition 2.15, we obtain: I is a strong interval of $\Pi(\lfloor f \rfloor)$ if and only if $\Theta_f(I)$ is a strong interval of Π. Lastly, it is sufficient to apply Lemma 3.2, Lemma 3.3 or Lemma 3.4 to $\Theta_f(I)$ because $\Theta_f(\lfloor f \rfloor^{>q}) = \lfloor \overline{0} \rfloor^{>(\theta_f)^{-1}(q)}$, $\Theta_f(\lfloor f \rfloor_X^{\geq q}) = \lfloor \overline{0} \rfloor_X^{\geq(\theta_f)^{-1}(q)}$ and $\Theta_f(\lfloor f \rfloor_X^{=q}) = \lfloor \overline{0} \rfloor_X^{=(\theta_f)^{-1}(q)}$.

Corollary 3.6. *Assume that G is indecomposable. Given a subset I of ${}^{\mathbb{Q}}V$ such that $|I| > 1$, I is a strong interval of Π if and only if there is $f \in {}^{\mathbb{Q}}V$ and there is $q \in \mathbb{Q}$, with $q > S(f)$, such that $I = \lfloor f \rfloor$, $\lfloor f \rfloor^{>q}$ or $\lfloor f \rfloor^{\geq q}$.*

4 The decomposition tree of Π

In the section, we utilize the partition $P(\Gamma)$ and the the decomposition tree $\mathcal{D}(\Gamma)$ associated with any graph Γ. Recall that they are introduced before Theorem 1.3 and after Lemma 1.4 respectively. We use the following notation, where $q \in \mathbb{Q}$:

- θ_q is an isomorphism from $\mathbb{Q}$ onto $\mathbb{Q}((q,+\infty))$;
- the function $\Theta_q : \lfloor\overline{0}\rfloor^{>q} \longrightarrow {}^{\mathbb{Q}}V$ is defined by $\Theta_q(f) = (f_{/(q,+\infty)}) \circ \theta_q$ for $f \in \lfloor\overline{0}\rfloor^{>q}$.
- for every function $g : (q,+\infty) \longrightarrow V$ such that $\{r > q : g(r) \neq 0\}$ is finite, $\varepsilon_q(g)$ is the element of ${}^{\mathbb{Q}}V$ defined by $\varepsilon_q(g)(r) = 0$ if $r \leq q$ and $\varepsilon_q(g)(r) = g(r)$ if $r > q$. The function $\Omega_q : {}^{\mathbb{Q}}V \longrightarrow \lfloor\overline{0}\rfloor^{>q}$ is defined by $\Omega_q(g) = \varepsilon_q(g \circ (\theta_q)^{-1})$.

Lemma 4.1. *For every $q \in \mathbb{Q}$, Θ_q realizes an isomorphism from $\Pi(\lfloor\overline{0}\rfloor^{>q})$ onto Π and $(\Theta_q)^{-1} = \Omega_q$.*

Proof : Given $f \in \lfloor\overline{0}\rfloor^{>q}$, we have

$$(\Omega_q \circ \Theta_q)(f) = \Omega_q((f_{/(q,+\infty)}) \circ \theta_q) = \varepsilon_q(((f_{/(q,+\infty)}) \circ \theta_q) \circ (\theta_q)^{-1}) = \varepsilon_q(f_{/(q,+\infty)})$$

and $\varepsilon_q(f_{/(q,+\infty)}) = f$ because $f \in \lfloor\overline{0}\rfloor^{>q}$. Conversely, given $g \in \Pi$, we have

$$(\Theta_q \circ \Omega_q)(g) = \Theta_q(\varepsilon_q(g \circ (\theta_q)^{-1})) = (\varepsilon_q(g \circ (\theta_q)^{-1}))_{(q,+\infty)} \circ \theta_q.$$

But, $(\varepsilon_q(g \circ (\theta_q)^{-1}))_{(q,+\infty)} = g \circ (\theta_q)^{-1}$ and hence

$$(\varepsilon_q(g \circ (\theta_q)^{-1}))_{(q,+\infty)} \circ \theta_q = (g \circ (\theta_q)^{-1}) \circ \theta_q = g.$$

Consequently, Θ_q is bijective and $(\Theta_q)^{-1} = \Omega_q$.

Let f and f' be distinct elements of $\lfloor\overline{0}\rfloor^{>q}$. Clearly, $\delta(f,f') > q$. For every $r < (\theta_q)^{-1}(\delta(f,f'))$, we have $\theta_q(r) < \delta(f,f')$ and hence $\Theta_q(f)(r) = f(\theta_q(r)) = f'(\theta_q(r)) = \Theta_q(f')(r)$. Furthermore, we have

$$\Theta_q(f)((\theta_q)^{-1}(\delta(f,f'))) = f(\delta(f,f')) \neq f'(\delta(f,f')) = \Theta_q(f)((\theta_q)^{-1}(\delta(f,f'))).$$

It follows that $\delta(\Theta_q(f),\Theta_q(f')) = (\theta_q)^{-1}(\delta(f,f'))$. Clearly,

$$\begin{aligned}\Theta_q(f)(\delta(\Theta_q(f),\Theta_q(f'))) &= f(\theta_q(\delta(\Theta_q(f),\Theta_q(f'))))\\ &= f(\theta_q((\theta_q)^{-1}(\delta(f,f')))) = f(\delta(f,f'))\end{aligned}$$

and $\Theta_q(f')(\delta(\Theta_q(f),\Theta_q(f'))) = f'(\delta(f,f'))$ as well. Therefore,

$$\begin{aligned}[\Theta_q(f),\Theta_q(f')]_\Pi &= [\Theta_q(f)(\delta(\Theta_q(f),\Theta_q(f'))),\Theta_q(f')(\delta(\Theta_q(f),\Theta_q(f')))]_G\\ &= [f(\delta(f,f')),f'(\delta(f,f'))]_G = [f,f']_\Pi.\end{aligned}$$

Consequently, Θ_q realizes an isomorphism from $\Pi(\lfloor\overline{0}\rfloor^{>q})$ onto Π.

The next result is an immediate consequence of the preceding lemma and of Proposition 2.15.

Corollary 4.2. *For every $f \in ({}^{\mathbb{Q}}V)^\star$ and for every $q > S(f)$, $\Theta_{(\theta_f)^{-1}(q)} \circ ((\Theta_f)_{/\lfloor f\rfloor^{>q}})$ is an isomorphism from $\Pi(\lfloor f\rfloor^{>q})$ onto Π.*

Proof : We have $\Theta_f(\lfloor f\rfloor^{>q}) = \lfloor\overline{0}\rfloor^{>(\theta_f)^{-1}(q)}$. Consequently, $(\Theta_f)_{/\lfloor f\rfloor^{>q}}$ is an isomorphism from $\Pi(\lfloor f\rfloor^{>q})$ onto $\Pi(\lfloor\overline{0}\rfloor^{>(\theta_f)^{-1}(q)})$.

We obtain the following characterization of the strong intervals of Π which are not limit.

Theorem 4.3. *Given a subset I of ${}^{\mathbb{Q}}V$ such that $|I| > 1$, $I \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$ if and only if there is $f \in {}^{\mathbb{Q}}V$, there is $\emptyset \neq X \subseteq V \setminus \{0\}$ and there is $q \in \mathbb{Q}$, with $q > S(f)$, such that one of the following is satisfied.*

1. $I = \lfloor f \rfloor_X^{\geq q}$ *and* $\{0\} \cup X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$.
2. $I = \lfloor f \rfloor_X^{=q}$, $|X| > 1$*and* $X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$.

Proof : By Theorem 3.5, we have only to consider the following subsets of ${}^{\mathbb{Q}}V$, where $f \in {}^{\mathbb{Q}}V$ and $q \in \mathbb{Q}$, with $q > S(f)$:

(i) $I = \lfloor f \rfloor$;

(ii) $I = \lfloor f \rfloor^{>q}$;

(iii) $I = \lfloor f \rfloor_X^{\geq q}$, where $\emptyset \neq X \subseteq V \setminus \{0\}$ and $\{0\} \cup X$ is a strong interval of G;

(iv) $I = \lfloor f \rfloor_X^{=q}$, where X is a non trivial strong interval of G contained in $V \setminus \{0\}$.

Let $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence of rational numbers such that $(q_n)_{n\in\mathbb{N}} \searrow_{-\infty}$ when $n \nearrow^{+\infty}$. By Theorem 3.5, $\lfloor \overline{0} \rfloor^{>q_n}$ is a strong interval of Π for each $n \in \mathbb{N}$. Therefore, $(\lfloor \overline{0} \rfloor^{>q_n})_{n\in\mathbb{N}}$ is a sequence of strong intervals of Π increasing under inclusion such that $(\lfloor \overline{0} \rfloor^{>q_n})_{n\in\mathbb{N}} \nearrow {}^{\mathbb{Q}}V$ when $n \nearrow^{+\infty}$. Consequently, ${}^{\mathbb{Q}}V$ is limit. Given $q \in \mathbb{Q}$, $\lfloor \overline{0} \rfloor^{>q}$ is a strong interval of Π by Lemma 3.2. By Lemma 4.1, $\lfloor \overline{0} \rfloor^{>q}$ is limit also. Given $f \in ({}^{\mathbb{Q}}V)^{\star}$, consider $q \in \mathbb{Q}$ such that $q > S(f)$. By Theorem 3.5, $\lfloor f \rfloor$ and $\lfloor f \rfloor^{>q}$ are strong intervals of Π. It follows from Proposition 2.15 and Corollary 4.2 that $\lfloor f \rfloor$ and $\lfloor f \rfloor^{>q}$ are limit as well. To continue, consider a nonempty subset X of $V \setminus \{0\}$ such that $\{0\} \cup X \in \mathcal{L}(G)$. By Theorem 3.5, $\lfloor f \rfloor_X^{\geq q} \in \mathcal{S}(\Pi)$. As $\{0\} \cup X$ is limit, there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ of strong intervals of G increasing under inclusion such that $(Y_n)_{n\in\mathbb{N}} \nearrow \bigcup_{n\in\mathbb{N}} Y_n = \{0\} \cup X$ when $n \nearrow^{+\infty}$. There is $p \in \mathbb{N}$ such that $0 \in Y_n$ for $n \geq p$. Set $X_m = Y_{p+m} \setminus \{0\}$ for each $m \in \mathbb{N}$. For $m \in \mathbb{N}$, we have $\emptyset \neq X_m \subseteq X \subseteq V \setminus \{0\}$. It follows from Theorem 3.5 that $\lfloor f \rfloor_{X_m}^{\geq q} \in \mathcal{S}(\Pi)$ for every $m \in \mathbb{N}$. We obtain a sequence $(\lfloor f \rfloor_{X_m}^{\geq q})_{m\in\mathbb{N}}$ of strong intervals of Π increasing under inclusion such that $(\lfloor f \rfloor_{X_m}^{\geq q})_{m\in\mathbb{N}} \nearrow \bigcup_{m\in\mathbb{N}}(\lfloor f \rfloor_{X_m}^{\geq q}) = \lfloor f \rfloor_X^{\geq q}$ when $m \nearrow^{+\infty}$. Consequently, $\lfloor f \rfloor_X^{\geq q} \in \mathcal{L}(\Pi)$. Finally, consider $X \in \mathcal{L}(G)$ such that $|X| \geq 2$ and $X \subseteq V \setminus \{0\}$. By Theorem 3.5, $\lfloor f \rfloor_X^{=q} \in \mathcal{S}(\Pi)$. Since X is limit, there is a sequence $(X_n)_{n\in\mathbb{N}}$ of non trivial strong intervals of G increasing under inclusion such that $(X_n)_{n\in\mathbb{N}} \nearrow \bigcup_{n\in\mathbb{N}}(X_n) = X$ when $n \nearrow^{+\infty}$. For each $n \in \mathbb{N}$, we have $X_n \subseteq X \subseteq V \setminus \{0\}$ so that $\lfloor f \rfloor_{X_n}^{=q} \in \mathcal{S}(\Pi)$ by Theorem 3.5. We obtain a sequence $(\lfloor f \rfloor_{X_n}^{=q})_{n\in\mathbb{N}}$ of strong intervals of Π increasing under inclusion such that $(\lfloor f \rfloor_{X_n}^{=q})_{n\in\mathbb{N}} \nearrow \lfloor f \rfloor_X^{=q}$ when $n \nearrow^{+\infty}$. Consequently, $\lfloor f \rfloor_X^{=q} \in \mathcal{L}(\Pi)$.

Conversely, we begin verifying for $q \in \mathbb{Q}$ and for $\emptyset \neq X \subseteq V \setminus \{0\}$ the following:

(a) if $|X| > 1$ and $X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then $\lfloor {}^{q}X \rfloor \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$;

(b) if $\{0\} \cup X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then $\lfloor \overline{0} \rfloor_X^{\geq q} \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$.

By using Theorem 3.5, with each nonempty strong interval Y of G, we associate the strong interval I_Y of Π defined as follows:

- if $Y = \{0\}$, then $I_Y = \lfloor \overline{0} \rfloor^{>q}$;

- if $Y = \{y\}$ and $y \in V \setminus \{0\}$, then $I_Y = \lfloor^q y \rfloor$;
- if $0 \in Y$ and $|Y| > 1$, then $I_Y = \lfloor \overline{0} \rfloor^{\geq q}_{Y \setminus \{0\}}$;
- if $0 \notin Y$ and $|Y| > 1$, then $I_Y = \lfloor^q Y \rfloor$.

Firstly, assume that $|X| > 1$ and $X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$. By Theorem 3.5, $\lfloor \overline{0} \rfloor^{=q}_X \in \mathcal{S}(\Pi)$. Consider an element Y of $P(G(X))$ and a strong interval $\mathcal{J}$ of Π such that $I_Y \subset \mathcal{J} \subseteq \lfloor^q X \rfloor$. As $\mathcal{J} \neq \emptyset$ and $\mathcal{J} \subseteq \lfloor^q X \rfloor$, we have $\overline{0} \notin \mathcal{J}$ and $s(\mathcal{J}) = \{q\}$. Since $I_Y = \lfloor^q Y \rfloor$ or $\lfloor^q y \rfloor$, when $Y = \{y\}$, we obtain that $\lfloor^q y \rfloor \subset \mathcal{J}$ for $y \in Y$. Let $h \in \mathcal{J} \setminus \lfloor^q y \rfloor$, where $y \in Y$. As $s(\mathcal{J}) = \{q\}$, we have $s(h) = q$ and $h(q) \neq y$ because $h \notin \lfloor^q y \rfloor$. Therefore, y and $h(q)$ are distinct elements of $\mathcal{J}\downarrow$. It follows from Theorem 3.5 that $\mathcal{J} = \lfloor^q Z \rfloor$, where Z is a strong interval of G such that $Z \subseteq V \setminus \{0\}$ and $|Z| \geq 2$. Obviously, $Y \subset Z \subseteq X$ because $I_Y \subset \mathcal{J} \subseteq \lfloor^q X \rfloor$. Since $Y \in P(G(X))$, we obtain that $Z = X$ and hence $\mathcal{J} = \lfloor^q X \rfloor$. Consequently, $I_Y \in P(\Pi(\lfloor^q X \rfloor))$ for every $Y \in P(G(X))$. We obtain that $P(\Pi(\lfloor^q X \rfloor)) = \{I_Y ; Y \in P(G(X))\}$ and hence $\lfloor^q X \rfloor \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$.

Secondly, assume that $\{0\} \cup X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$. By Theorem 3.5, $\lfloor \overline{0} \rfloor^{\geq q}_X \in \mathcal{S}(\Pi)$. Consider an element Y of $P(G(\{0\} \cup X))$ and a strong interval $\mathcal{J}$ of Π such that $I_Y \subset \mathcal{J} \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$.

To begin, assume that $0 \notin Y$ so that $I_Y = \lfloor^q Y \rfloor$ or $\lfloor^q y \rfloor$, when $Y = \{y\}$. In both cases, $q \in s(\mathcal{J}^\star)$. For a contradiction, suppose that $s(\mathcal{J}^\star) = \{q\}$. As previously shown, we obtain that $|\mathcal{J}\downarrow| > 1$. Then, by Theorem 3.5, $\mathcal{J} = \lfloor^q Z \rfloor$, where Z is a strong interval of G such that $Z \subseteq V \setminus \{0\}$ and $|Z| \geq 2$. Since $I_Y \subset \mathcal{J} \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$, we would obtain that $Y \subset Z \subset \{0\} \cup X$, which contradicts $Y \in P(G(\{0\} \cup X))$. Consequently, $|s(\mathcal{J}^\star)| > 1$. By Theorem 3.5, $\mathcal{J} = \lfloor \overline{0} \rfloor^{>r}$ or $\lfloor \overline{0} \rfloor^{\geq r}_Z$, where $r \in \mathbb{Q}$ and Z is a nonempty subset Z of $V \setminus \{0\}$ such that $\{0\} \cup Z$ is a strong interval of G. If $\mathcal{J} = \lfloor \overline{0} \rfloor^{>r}$, then $q > r$ because $q \in s(\mathcal{J}^\star)$. But, given $v \in V \setminus \{0\}$ and $r' \in \mathbb{Q}$ such that $r < r' < q$, we would have ${}^{(r')}v \in \mathcal{J} \setminus \lfloor \overline{0} \rfloor^{\geq q}_X$. Thus, $\mathcal{J} = \lfloor \overline{0} \rfloor^{\geq r}_Z$. Since $q \in s(\mathcal{J}^\star)$, $r \leq q$ and $r \geq q$ because ${}^r z \in \lfloor \overline{0} \rfloor^{\geq r}_Z \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$ for $z \in Z$. Therefore, we have $I_Y \subset \mathcal{J} = \lfloor \overline{0} \rfloor^{\geq q}_Z \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$ and hence $Y \subset \{0\} \cup Z \subseteq \{0\} \cup X$. As $Y \in P(G(\{0\} \cup X))$, $\{0\} \cup Z = \{0\} \cup X$ and $\mathcal{J} = \lfloor \overline{0} \rfloor^{\geq q}_X$. It follows that $I_Y \in P(\Pi(\lfloor \overline{0} \rfloor^{\geq q}_X))$ for every $Y \in P(G(\{0\} \cup X))$ such that $0 \notin Y$.

To continue, assume that $0 \in Y$ so that $I_Y = \lfloor \overline{0} \rfloor^{\geq q}_{Y \setminus \{0\}}$ or $\lfloor \overline{0} \rfloor^{>q}$, when $Y = \{0\}$. In both cases, we obtain that $\lfloor \overline{0} \rfloor^{>q} \subset \mathcal{J} \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$. Given $h \in \mathcal{J} \setminus \lfloor \overline{0} \rfloor^{>q}$, we have $s(h) = q$ and $h(q) \in X$ because $h \in \lfloor \overline{0} \rfloor^{\geq q}_X$. Thus, $s(\mathcal{J}^\star) = [q, +\infty)$ because $\lfloor \overline{0} \rfloor^{>q} \subset \mathcal{J}$. By Theorem 3.5, $\mathcal{J} = \lfloor \overline{0} \rfloor^{\geq q}_Z$, where Z is a nonempty subset of $V \setminus \{0\}$ such that $\{0\} \cup Z$ is a strong interval of G. We have $Y \subset \{0\} \cup Z \subseteq \{0\} \cup X$ because $I_Y \subset \mathcal{J} \subseteq \lfloor \overline{0} \rfloor^{\geq q}_X$. Since $Y \in P(G(\{0\} \cup X))$, we obtain that $\{0\} \cup Z = \{0\} \cup X$ and thus $\mathcal{J} = \lfloor \overline{0} \rfloor^{\geq q}_X$. Therefore, $I_Y \in P(\Pi(\lfloor \overline{0} \rfloor^{\geq q}_X))$ for every $Y \in P(G(\{0\} \cup X))$ such that $0 \in Y$. As we established the same whenever $0 \notin Y$, we obtain that $P(\Pi(\lfloor \overline{0} \rfloor^{\geq q}_X)) = \{I_Y ; Y \in P(G(\{0\} \cup X))\}$ and thus $\lfloor \overline{0} \rfloor^{\geq q}_X \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$.

To conclude, consider $f \in ({}^{\mathbb{Q}}V)^\star$, $q \in \mathbb{Q}$ such that $q > S(f)$ and $\emptyset \neq X \subseteq V \setminus \{0\}$. By Proposition 2.15, Ω_f is an isomorphism from Π onto $\Pi(\lfloor f \rfloor)$. We demonstrated that if $|X| > 1$ and $X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then $\lfloor^{(\theta_f)^{-1}(q)} X \rfloor \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$. Thus, $\Omega_f(\lfloor^{(\theta_f)^{-1}(q)} X \rfloor) = \lfloor f \rfloor^{=q}_X \in \mathcal{S}(\Pi(\lfloor f \rfloor)) \setminus \mathcal{L}(\Pi(\lfloor f \rfloor))$. It follows from Proposition 1.2 (B2.(ii)) that $\lfloor f \rfloor^{=q}_X \in \mathcal{S}(\Pi) \setminus \mathcal{L}(\Pi)$. If $\{0\} \cup X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then we conclude similarly because $\Omega_f(\lfloor \overline{0} \rfloor^{\geq (\theta_f)^{-1}(q)}_X) = \lfloor f \rfloor^{\geq q}_X$.

The next result follows from the preceding theorem and from the last part of its demonstration.

Corollary 4.4. *Consider $f \in {}^{\mathbb{Q}}V$, $\emptyset \neq X \subseteq V \setminus \{0\}$ and $q \in \mathbb{Q}$, with $q > S(f)$.*

1. *If $\{0\} \cup X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then $P(\Pi(\lfloor f \rfloor_X^{\geq q}))$ contains the following elements:*
    - *$\lfloor^q f_x \rfloor$ for $\{y\} \in P(G(\{0\} \cup X))$ and $y \neq 0$,*
    - *$\lfloor f \rfloor_Y^{=q}$ for $Y \in P(G(\{0\} \cup X))$ such that $|Y| \geq 2$ and $Y \subseteq V \setminus \{0\}$,*
    - *$\lfloor f \rfloor^{>q}$ when $\{0\} \in P(G(\{0\} \cup X))$,*
    - *$\lfloor f \rfloor_Y^{\geq q}$ when $\{0\} \cup Y \in P(G(\{0\} \cup X))$ and $Y \neq \emptyset$.*
2. *If $|X| > 1$ and $X \in \mathcal{S}(G) \setminus \mathcal{L}(G)$, then $P(\Pi(\lfloor f \rfloor_X^{=q}))$ contains the following elements:*
    - *$\lfloor^q f_x \rfloor$ for $\{y\} \in P(G(X))$,*
    - *$\lfloor f \rfloor_Y^{=q}$ for $Y \in P(G(X))$, with $|Y| \geq 2$.*

When G is indecomposable, we obtain the following:

Corollary 4.5. *Assume that G is indecomposable.*

1. *Given a strong interval I of Π such that $|I| > 1$, I is not limit if and only if there is $f \in {}^{\mathbb{Q}}V$ and there is $q \in \mathbb{Q}$, with $q > S(f)$, such that $I = \lfloor f \rfloor^{\geq q}$.*
2. *Consider $f \in {}^{\mathbb{Q}}V$ and $q \in \mathbb{Q}$, with $q > S(f)$.*
    (a) *$P(\Pi(\lfloor f \rfloor^{\geq q}) = \{\lfloor f \rfloor^{>q}\} \cup \{\lfloor^q f_x \rfloor; x \in V \setminus \{0\}\}$.*
    (b) *The function $V \longrightarrow P(\Pi(\lfloor f \rfloor^{\geq q}))$, defined by $0 \mapsto \lfloor f \rfloor^{>q}$ and $x \mapsto \lfloor^q f_x \rfloor$ for $x \in V \setminus \{0\}$, realizes an isomorphism from G onto the quotient $\Pi(\lfloor f \rfloor^{\geq q})/P(\Pi(\lfloor f \rfloor^{\geq q}))$.*
    (c) *For every $I \in P(\Pi(\lfloor f \rfloor^{\geq q}))$, $\Pi(I)$ is isomorphic to Π.*
3. *$\mathcal{D}(\Pi)$ contains $\lfloor f \rfloor^{\geq q}$, $\lfloor f \rfloor^{>q}$ and $\lfloor^q f_x \rfloor$ for $f \in {}^{\mathbb{Q}}V$, $q \in \mathbb{Q}$, with $q > S(f)$, and $x \in V \setminus \{0\}$.*
4. *For every $f \in {}^{\mathbb{Q}}V$, $\mathcal{D}(\Pi)(\{I \in \mathcal{D}(\Pi) : f \in I\})$ is isomorphic to the lexicographic product $\mathbb{Q}[2]$.*

Proof : The first three assertions follow from Proposition 2.15, Theorem 4.3 and Corollaries 9 and 10. Concerning the fourth, the result is clear when $f = \overline{0}$ since $\{I \in \mathcal{D}(\Pi) : \overline{0} \in I\}) = \{\lfloor \overline{0} \rfloor^{>q}, \lfloor \overline{0} \rfloor^{\geq q}\}_{q \in \mathbb{Q}}$. Now, consider $f \in ({}^{\mathbb{Q}}V)^{\star}$. For convenience, set $n = n(f)$, $q_i = q_i^f$ and $f(q_i) = x_i$ for $i \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, n\}$, we consider the element f_i of ${}^{\mathbb{Q}}V$ defined by $\sigma(f_i) = \{q_1, \ldots, q_i\}$ and $f_i(q_j) = f(q_j)$ for $j \in \{1, \ldots, i\}$. Lastly, set $f_0 = \overline{0}$. Obviously, $f \in \lfloor f \rfloor^{>q} \subset \lfloor f \rfloor^{\geq q}$ for $q > S(f)$. We have

$$\lfloor f \rfloor^{>q} \searrow \Big(\bigcap_{q \nearrow +\infty} \lfloor f \rfloor^{>q} \Big) = \{f\} \text{ when } q \nearrow +\infty$$

and

$$\lfloor f \rfloor^{\geq q} \nearrow (\bigcup_{q \searrow S(f)} \lfloor f \rfloor^{\geq q}) = \lfloor f \rfloor \text{ when } q \searrow_{S(f)} .$$

But, $f =^{q_n} (f_{n-1})_{x_n}$ and $\lfloor^{q_n}(f_{n-1})_{x_n}\rfloor \subset \lfloor f_{n-1} \rfloor^{\geq q_n}$. Assume that $n \geq 2$ and consider $0 \leq i \leq n-2$. We have $\lfloor^{q_{n-i}}(f_{n-i-1})_{x_{n-i}}\rfloor \subset \lfloor f_{n-i-1} \rfloor^{\geq q_{n-i}}$. Consider any $q \in \mathbb{Q}$ such that $q_{n-i-1} < q < q_{n-i}$. We have

$$\lfloor f_{n-i-1} \rfloor^{>q} \searrow (\bigcap_{q \nearrow^{q_{n-i}}} \lfloor f_{n-i-1} \rfloor^{>q}) = \lfloor f_{n-i-1} \rfloor^{\geq q_{n-i}} \text{ when } q \nearrow^{q_{n-i}}$$

and

$$\lfloor f_{n-i-1} \rfloor^{\geq q} \nearrow (\bigcup_{q \searrow q_{n-i-1}} \lfloor f_{n-i-1} \rfloor^{\geq q}) = \lfloor f_{n-i-1} \rfloor \text{ when } q \searrow_{q_{n-i-1}} .$$

Similarly, $f_{n-i-1} =^{q_{n-i-1}} (f_{n-i-2})_{x_{n-i-1}}$ and we have

$$\lfloor^{q_{n-i-1}}(f_{n-i-2})_{x_{n-i-1}}\rfloor \subset \lfloor f_{n-i-2} \rfloor^{\geq q_{n-i-1}}.$$

Finally, when $i = n-2$, we obtain that $\lfloor f_{n-i-2} \rfloor^{\geq q_{n-i-1}} = \lfloor \overline{0} \rfloor^{\geq q_1}$. Consider any $q \in \mathbb{Q}$ such that $q < q_1$. We have

$$\lfloor \overline{0} \rfloor^{>q} \searrow (\bigcap_{q \nearrow^{q_1}} \lfloor \overline{0} \rfloor^{>q}) = \lfloor \overline{0} \rfloor^{\geq q_1} \text{ when } q \nearrow^{q_1}$$

and

$$\lfloor \overline{0} \rfloor^{\geq q} \nearrow (\bigcup_{q \searrow -\infty} \lfloor \overline{0} \rfloor^{\geq q}) = \lfloor \overline{0} \rfloor =^{\mathbb{Q}} V \text{ when } q \searrow_{-\infty} .$$

We define a function $\varphi : \mathbb{Q} \times \{0,1\} \longrightarrow \mathcal{D}(\Pi)(\{I \in \mathcal{D}(\Pi) : f \in I\})$ as follows:

- for $1 \leq i \leq n$, $(q_i, 0) \mapsto \lfloor f_{i-1} \rfloor^{\geq q_i}$ and $(q_i, 1) \mapsto \lfloor f_i \rfloor$;
- for $q < q_1$, $(q, 0) \mapsto \lfloor \overline{0} \rfloor^{\geq q}$ and $(q, 1) \mapsto \lfloor \overline{0} \rfloor^{>q}$;
- for $1 \leq i \leq n-1$ and for $q_i < q < q_{i+1}$, $(q, 0) \mapsto \lfloor f_i \rfloor^{\geq q}$ and $(q, 1) \mapsto \lfloor f_i \rfloor^{>q}$;
- for $q > q_n$, $(q, 0) \mapsto \lfloor f \rfloor^{\geq q}$ and $(q, 1) \mapsto \lfloor f \rfloor^{>q}$.

Clearly, φ realizes an isomorphism from $\mathbb{Q}[2]$ onto the dual $(\mathcal{D}(\Pi)(\{I \in \mathcal{D}(\Pi) : f \in I\}))^d$ of $\mathcal{D}(\Pi)(\{I \in \mathcal{D}(\Pi) : f \in I\})$. Thus, φ is an isomorphism from the dual $(\mathbb{Q}[2])^d$ of $\mathbb{Q}[2]$ onto $\mathcal{D}(\Pi)(\{I \in \mathcal{D}(\Pi) : f \in I\})$ (see Figure 1). To conclude, recall that $\mathbb{Q}$ and its dual $\mathbb{Q}^d$ are isomorphic and hence $\mathbb{Q}[2]$ and $(\mathbb{Q}[2])^d$ are also.

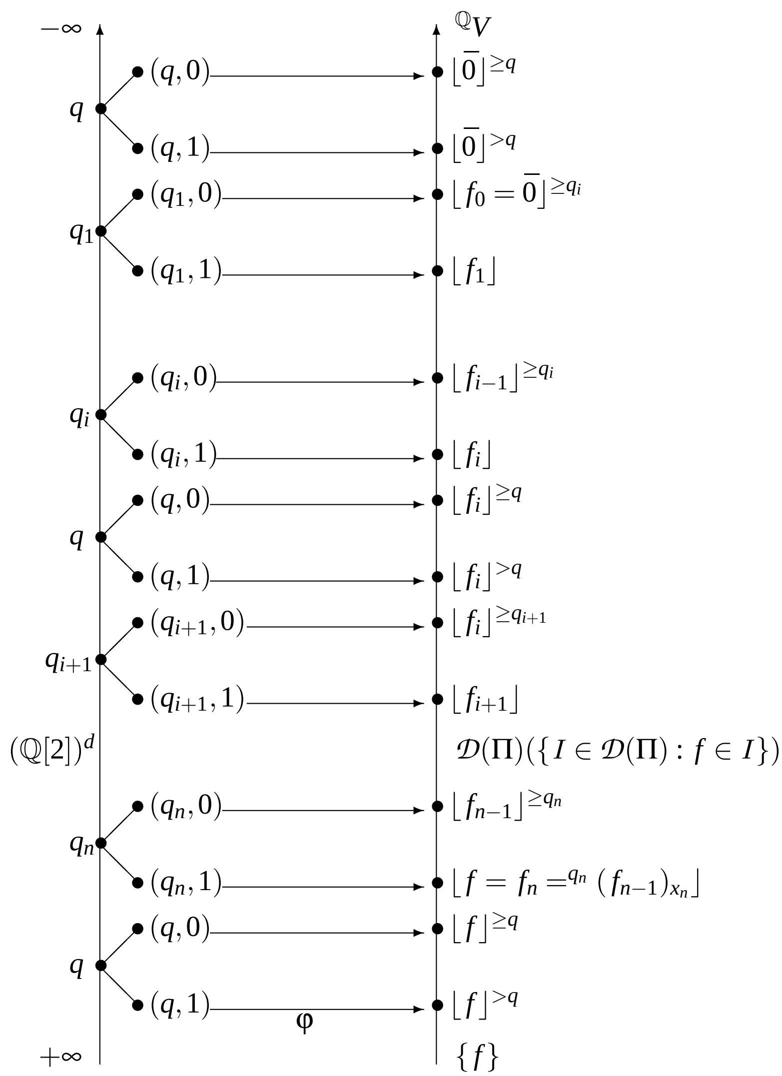

Figure 1. φ is an isomorphism from $(\mathbb{Q}[2])^d$ onto $\mathcal{D}(\Pi)(\{I\in\mathcal{D}(\Pi):f\in I\})$.

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