Communications in Mathematical Analysis
Volume 3, Number 2, pp. 54–68, 2007
Procceedings of the 15th Symposium of
The Tunisian Mathematical Society
held in Sousse, March 19-22, 2007
www.commun-math-anal.org
ISSN 1938-9787

GENERALIZED BOCHNER-HECKE THEOREMS AND APPLICATION TO HOMOGENEOUS DISTRIBUTIONS IN DUNKL'S THEORY

KHALIFA TRIMÈCHE *
Department of Mathematics
Faculty of Sciences of Tunis
1060 Tunis, Tunisia

(Communicated by Said Zarati)

Abstract

In this paper we prove generalized Bochner-Hecke theorems for the Dunkl transform on $\mathbb{R}^d$, and we give an application of these theorems to homogeneous distributions .

AMS Subject Classification: 33C80, 51F15, 44A15.

Keywords: Dunkl transform, Bochner-Hecke theorems, Dunkl transform of homogeneous distributions.

1 Introduction

We consider the differential-difference operators T_j, $j = 1,...,d$, on $\mathbb{R}^d$ introduced by C. F. Dunkl in [6] and called Dunkl operators in the literature. These opertors are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions with root systems (see [3][11]), and they are closely related to certain representations of degenerate affine Hecke algebras [2][18], moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional spaces (see [12][15][16]).

C.F. Dunkl has studied in [5] the spherical harmonics associated with the Dunkl operators, and with the aid of the Dunkl kernel which is an eigenfunction of these operators, he has introduced in [7] an integral transform on $\mathbb{R}^d$ called the Dunkl transform.

In this paper we give first an other proof of the analogue in the Dunkl's theory, of the Funk-Hecke formula associated with the classical spherical harmonics on $\mathbb{R}^d$. (see [10] p. 29). This formula has been established in a general form by Y. Xu in [22] (see also [8] p. 191).

*E-mail address: khlifa.trimeche@fst.rnu.tn

Next we prove generalized Bochner-Hecke theorems for the Dunkl transform. In the case of the classical Fourier transform on $\mathbb{R}^d$ these theorems are given in [10] p. 30-31 and [9] p. 66-70.

As application of generalized Bochner-Hecke theorems we determine the Dunkl transform of some homogeneous distributions on $\mathbb{R}^d$. The same application has been studied in [9] p. 88-93, in the case of the calssical Fourier transform on $\mathbb{R}^d$.

2 The eigenfunction of the Dunkl operators

In this section we collect some notations and results on Dunkl operators and Dunkl kernel (see [6][7][13]).

2.1 Reflection groups, root systems and multiplicity functions

We consider $\mathbb{R}^d$ with the euclidean scalar product $\langle .,. \rangle$ and $\|x\| = \sqrt{\langle x,x \rangle}$. On $\mathbb{C}^d$, $\|.\|$ denotes also the standard Hermitian norm, while $\langle z,w \rangle = \sum_{j=1}^{d} z_j \bar{w}_j$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, le σ_α be the relfection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\|\alpha\|^2}\, \alpha. \tag{1.1}$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R}^d.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $\|\alpha\|^2 = 2$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d_{reg} = \mathbb{R}^d \backslash U_{\alpha \in R} H_\alpha$,we fix the positive subsystem $R_+ = \{\alpha \in R, \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \to \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W . For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha \in R_+} k(\alpha) \tag{1.2}$$

Moreover, let w_k denotes the weight function

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \tag{1.3}$$

which is W-invariant and homogenous of degree 2γ. In this paper we suppose that k is nonnegative.

For $d = 1$ and $W = \mathbb{Z}_2$, the multiplicity function k is a single parameter denoted by $\gamma \geq 0$ and

$$\forall\, x \in \mathbb{R},\ w_k(x) = |x|^{2\gamma}. \tag{1.4}$$

We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} \exp(-\|x\|^2) w_k(x) dx \right)^{-1} \tag{1.5}$$

which is known for all Coxeter groups W. (See [6]).

For an integrable function on $\mathbb{R}^d$ with respect to the measure $w_k(x)dx$ we have the relation

$$\int_{\mathbb{R}^d} f(x) w_k(x) dx = \int_0^{+\infty} \left(\int_{S^{d-1}} f(r\beta) w_k(\beta) d\sigma(\beta) \right) r^{2\gamma+d-1} dr, \tag{1.6}$$

where $d\sigma$ is the surface measure on the unit sphere S^{d-1} of $\mathbb{R}^d$. We have

$$\Omega_{d-1} = \sigma(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \,. \tag{1.7}$$

2.2 Dunkl operators and Dunkl kernel

The Dunkl operators T_j, $j = 1, ...d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given for a function f of class C^1 on $\mathbb{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \, \frac{f(x) - f(\sigma_n(x))}{\langle \alpha, x \rangle}. \tag{1.8}$$

In the case $k = 0$, the T_j, $j = 1, ..., d$, reduce to the corresponding partial derivatives.

The Dunkl Laplacian Δ_k is defined by

$$\Delta_k f = \sum_{j=1}^{d} T_j^2 f = \Delta f + 2 \sum_{\alpha \in R_+} k(\alpha) \delta_\alpha f, \tag{1.9}$$

with

$$\delta_\alpha(f)(x) = \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}$$

where f is a function of class C^2 on $\mathbb{R}^d$, and Δ, ∇ are respectively the Laplacian and the gradian on $\mathbb{R}^d$.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) & = y_j u(x,y), \quad j = 1, ..., d, \\ u(0,y) & = 1, \end{cases} \tag{1.10}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called the Dunkl kernel.

This kernel has an unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. For all $x, z \in \mathbb{C}^d$ it satisfies the following relations

- $K(x,z) = K(z,x)$. (1.11)
- $K(x,0) = 1$. (1.12)
- $K(\lambda x, z) = K(x, \lambda z)$, for all $\lambda \in \mathbb{C}$. (1.13)
- $\forall\, x, y \in \mathbb{R}^d$, $|K(-ix,y)| \leq 1$. (1.14)

When $d = 1$ and $W = \mathbb{Z}_2$ the Dunkl kernel is given by

$$K(x,z) = j_{\gamma-\frac{1}{2}}(ixz) + \frac{xz}{2\gamma+1} j_{\gamma+\frac{1}{2}}(ixz), \quad x,z \in \mathbb{C}, \tag{1.15}$$

where j_α with $\alpha \geq -\frac{1}{2}$, is the normalized Bessel function of first kind, defined by

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(x)}{x^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{2n}}{n!\Gamma(\alpha+n+1)}\ , \tag{1.16}$$

with J_α the Bessel function of first kind and index α. (See [7]).

3 The Dunkl transform

The Dunkl kernel gives rise to an integral transform on $\mathbb{R}^d$ called the Dunkl transform which was first introduced by C.F. Dunkl in [7] and further studied in [14].

Notations We denote by

- $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n.

- H_n^k the space of Dunkl harmonic homogeneous polynomials of degree n. It is defined by

$$H_n^k = (ker\Delta_k) \cap \mathcal{P}_n^d.$$

- $\mathcal{D}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ with compact support .

$$\mathcal{D}(\mathbb{R}^d) = \cup_{a>0} \mathcal{D}_a(\mathbb{R}^d),$$

where $\mathcal{D}_a(\mathbb{R}^d)$ is the space of C^∞-functions on $\mathbb{R}^d$ with support in the closed ball $B(0,a)$ of center 0 and radius $a > 0$.

The topology on $\mathcal{D}_a(\mathbb{R}^d)$ is defined by the seminorms

$$p_n(\varphi) = \sup_{\substack{|\mu|\leq n \\ x\in B(0,a)}} |D^\mu \varphi(x)|, \quad n \in \mathbb{N},$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \ldots \partial x_d^{\mu_n}},\ \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d,\ |\mu| = \mu_1 + \ldots + \mu_d.$$

The space $\mathcal{D}(\mathbb{R}^d)$ is equipped with the inductive limit topology.

- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of C^∞-functions on $\mathbb{R}^d$, rapidly decreasing together with their partial derivatives.

The topology on $\mathcal{S}(\mathbb{R}^d)$ is defined by the seminorms $P_{\ell,m}$, $(\ell,m) \in \mathbb{N}^2$, given by

$$P_{\ell,m}(\varphi) = \sup_{\substack{|\mu|\leq m \\ x\in \mathbb{R}^d}} (1+\|x\|^2)^\ell |D^\mu \varphi(x)|.$$

- $L_k^p(\mathbb{R}^d), p \in [0,+\infty]$, the space of measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{k,p} = \left(\int_{\mathbb{R}^d} |f(x)|^p\, w_k(x) dx \right)^{1/p} < +\infty, \text{ if } 1 \leq p < +\infty$$

$$\|f\|_{k,\infty} = ess \sup_{x\in\mathbb{R}^d} |f(x)| < +\infty.$$

- $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{D}(\mathbb{R}^d)$.
- $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

The Dunkl transform of a function f in $\mathcal{D}(\mathbb{R}^d)$ is given by

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(-ix,y)w_k(x)dx. \tag{2.1}$$

This transform has the following properties

i) (Riemann-Lebesgue Lemma). For f in $L^1_k(\mathbb{R}^d)$ the function $\mathcal{F}_D(f)$ is continuous on $\mathbb{R}^d$ and vanish at infinity.

ii) For f in $L^1_k(\mathbb{R}^d)$ we have

$$\|\mathcal{F}_D(f)\|_{k,\infty} \le \|f\|_{k,1} \tag{2.2}$$

iii) Let f be in $\mathcal{D}(\mathbb{R}^d)$. If for $x\in\mathbb{R}^d$ and $g\in W$ we have $f^-(x) = f(-x)$ and $f_g(x) = f(gx)$, then for all $y\in\mathbb{R}^d$ we have

$$\mathcal{F}'_D(f^-)(y) = \overline{\mathcal{F}_D(f)(y)} \quad \text{and} \quad \mathcal{F}_D(f_g)(y) = \mathcal{F}_D(f)(gy). \tag{2.3}$$

iv) For f in $\mathcal{S}(\mathbb{R}^d)$, we have for all $y\in\mathbb{R}^d$ and $j = 1,\dots,d$

$$\mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y). \tag{2.4}$$

$$T_j(\mathcal{F}_D(f))(y) = -\mathcal{F}_D(ix_j f)(y). \tag{2.5}$$

The following theorems are proved in [7][14].

Theorem 2.1 The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. The inverse transform $\mathcal{F}_D^{-1}$ is given by

$$\forall\, x\in\mathbb{R}^d,\ \mathcal{F}_D^{-1}(f)(x) = 2^{-2\gamma-d}c_k^2\mathcal{F}_D(f)(-x). \tag{2.6}$$

Theorem 2.2 For f in $L^1_k(\mathbb{R}^d)$ such that $\mathcal{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$, we have the following inversion formula for the transform $\mathcal{F}_D$:

$$f(x) = \frac{c_k^2}{2^{2\gamma+d}}\int_{\mathbb{R}^d}\mathcal{F}_D(f)(y)K(ix,y)w_k(y)dy. \tag{2.7}$$

Theorem 2.3

i) Plancherel formula for $\mathcal{F}_D$.
For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d}|f(x)|^2w_k(x)dx = \frac{c_k^2}{2^{2\gamma+d}}\int_{\mathbb{R}^d}|\mathcal{F}_D(f)(y)|^2w_k(y)dy \tag{2.8}$$

ii) Plancherel theorem for $\mathcal{F}_D$.
The renormalized Dunkl transform $f \to 2^{-\gamma-\frac{d}{2}} c_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.

Next we show that the Dunkl transform of radial functions in $L_k^1(\mathbb{R}^d)$ are again radial, and that the Dunkl transform can be computed via the Fourier-Bessel transform (see [20]). More precisely we have the following results (see [19] p. 585-586).

Theorem 2.4. Let f be a radial function in $L_k^1(\mathbb{R}^d)$. Then the function f_0 defined on $[0,+\infty[$ by

$$\forall\, x \in \mathbb{R}^d,\ f(x) = f_0(\|x\|) = f_0(r), \quad \text{with } r = \|x\|,$$

is integrable on $[0,+\infty[$ with respect to the measure $r^{2\gamma+d-1}dr$, and we have

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(f)(y) = 2^{\gamma+\frac{d}{2}} c_k^{-1} \mathcal{F}_B^{\gamma+\frac{d}{2}-1}(f_0)(\|y\|), \tag{2.9}$$

where $\mathcal{F}_B^{\gamma+\frac{d}{2}-1}$ is the Fourier-Bessel transform of order $\gamma+\dfrac{d}{2}-1$, given by

$$\mathcal{F}_B^{\gamma+\frac{d}{2}-1}(h)(\lambda) = \int_0^\infty h(r)\, j_{\gamma+\frac{d}{2}-1}(\lambda r)\, \frac{r^{2\gamma+d-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}\, dr,\ \lambda \geq 0. \tag{2.10}$$

The following theorem gives the Dunkl transform of the functions $P(x)e^{-\frac{\|x\|^2}{2}}$ and $P(x)e^{-\frac{\lambda\|x\|^2}{2}}$ with P in H_n^k and $\lambda > 0$.

Theorem 2.5 We have the following relations

$$\int_{\mathbb{R}^+} P(x)K(-ix,y)e^{-\frac{\|x\|^2}{2}} w_k(x)dx = \frac{i^{-n}}{c_k} e^{-\frac{\|y\|^2}{2}} P(y). \tag{2.11}$$

$$\int_{\mathbb{R}^d} P(x)K(-ix,y)e^{-\frac{\lambda\|x\|^2}{2}} w_k(x)dx = \frac{i^{-n}}{c_k} \lambda^{-(n+\gamma+\frac{d}{2})} e^{-\frac{\|y\|^2}{2\lambda}} P(y). \tag{2.12}$$

Proof. Using the fact that P is Dunkl harmonic, we deduce from Proposition 2.1 of [7] p. 127 (see also [8] p. 216) the relation (2.11). We obtain (2.12) by change of variables.

The Dunkl transform of a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(S), \varphi\rangle = \langle S, \mathcal{F}_D(\varphi)\rangle,\ \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.13}$$

Theorem 2.6. The Dunkl transform is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^d)$ onto itself. The inverse transform is given by

$$\langle \mathcal{F}_D^{-1}(S),\ \varphi\rangle = \langle S, \mathcal{F}_D^{-1}(\varphi)\rangle,\ \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.14}$$

4 The generalized Funk-Hecke formula and generalized Bochner-Hecke theorems

4.1 The generalized Funk-Hecke formula

Y. Xu has established in [22] a Funk-Hecke formula of general form in the Dunkl's theory. In this subsection we give an other proof of a particular case of this formula. Our formula

is very useful for the proof of the results of the following subsection.
Theorem 3.1. Let P be in H_n^k. Then

$$\forall\, y \in \mathbb{R}^d,\ \int_{S^{d-1}} P(u)K(-iu,y)w_k(u)d\sigma(u) = a_{\gamma,n}\, j_{n+\gamma+\frac{d}{2}-1}(\|y\|)P(y), \tag{3.1}$$

with

$$a_{\gamma,n} = \frac{i^{-n}2^{-n-\gamma-\frac{d}{2}+1}}{c_k\Gamma(n+\gamma+\frac{d}{2})}\ , \tag{3.2}$$

and $j_{n+\gamma+\frac{d}{2}-1}$ the normalized Bessel function of first kind.

Proof. By using in (2.11) the spherical coordinates $x = \sqrt{2s}u$ with $s \in]0,+\infty[, u \in S^{d-1}$ and by puting $y = \sqrt{2\tau}v$ with $\tau \in]0,+\infty[, v \in S^{d-1}$, then Fubini's theorem implies

$$\int_0^\infty e^{-\lambda s}\left[\int_{S^{d-1}} P(u)K(-iu,2\sqrt{s\tau}v)w_k(u)d\sigma(u)\right] s^{\frac{n}{2}+\gamma+\frac{d}{2}-1}ds =$$

$$\frac{i^{-n}2^{-\frac{n}{2}-\gamma-\frac{d}{2}+1}}{c_k}\,\lambda^{-(n+\gamma+\frac{d}{2})}e^{-\frac{\tau}{\lambda}}P(\sqrt{2\tau}v). \tag{3.3}$$

But from formula (4) of [21] p.394 we have

$$\lambda^{-(n+\gamma+\frac{d}{2})}e^{-\frac{\tau}{\lambda}} = \frac{1}{\Gamma(n+\gamma+\frac{d}{2})}\int_0^\infty e^{-\lambda s} j_{n+\gamma+\frac{d}{2}-1}(2\sqrt{s\tau})s^{n+\gamma+\frac{d}{2}-1}ds.$$

By using this relation in (3.3) we obtain

$$\int_0^\infty e^{-\lambda s}\left[s^{-\frac{n}{2}}\int_{S^{d-1}} P(u)K(-iu,2\sqrt{s\tau}v)w_k(u)d\sigma(u)\right] s^{n+\gamma+\frac{d}{2}-1}ds =$$

$$\frac{i^{-n}2^{-\frac{n}{2}-\gamma-\frac{d}{2}+1}}{c_k\Gamma(n+\gamma+\frac{d}{2})}P(\sqrt{2\tau}v)\int_0^\infty e^{-\lambda s} j_{n+\gamma+\frac{d}{2}-1}(2\sqrt{s\tau})s^{n+\gamma+\frac{d}{2}-1}ds.$$

The injectivity of the Laplace transform implies $\forall\, s > 0$,

$$\int_{S^{d-1}} P(u)K(-iu,2\sqrt{s\tau}v)w_k(u)d\sigma(u) = \frac{i^{-n}2^{-\frac{n}{2}-\gamma-\frac{d}{2}+1}s^{\frac{n}{2}}}{c_k\Gamma(n+\gamma+\frac{d}{2})} j_{n+\gamma+\frac{d}{2}-1}(2\sqrt{s\tau})P(\sqrt{2\tau}v).$$

We obtain (3.1) by taking $s = \dfrac{1}{2}$.

Remark. In [17] the authors give the analogue of Theorem 3.1 for the Dunkl-Bessel Laplace operator.

4.2 Generalized Bochner-Hecke theorems

We give in this subsection the analogue in Dunkl's theory of the classical Bochner-Hecke theorems (see [10] p. 30-31 and [9] p. 66-70).
Theorem 3.2. Let P be in H_n^k and f a measurable function on $[0,+\infty[$ such that

$$\int_0^\infty |f(x)|x^{n+2\gamma+d-1}dx < +\infty. \tag{3.4}$$

Then the function $F(x) = f(\|x\|)P(x)$ belongs to $L^1_k(\mathbb{R}^d)$ and its Dunkl transform is given by

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(F)(y) = b_{\gamma,n}\mathcal{F}_B^{\frac{n}{2}+\gamma+\frac{d}{2}-1}(f)(\|y\|)P(y), \tag{3.5}$$

where

$$b_{\gamma,n} = 2^{\frac{n}{2}+\gamma+\frac{d}{2}-1}\Gamma(\frac{n}{2}+\gamma+\frac{d}{2})a_{\gamma,n}. \tag{3.6}$$

Proof. The spherical coordinates and Fubini-Tonelli's theorem imply that the function F belongs to $L^1_k(\mathbb{R}^d)$.

We have

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(F)(y) = \int_{\mathbb{R}^d} f(\|x\|)P(x)K(-ix,y)w_k(x)dx.$$

By using spherical coordinates, Fubini's theorem and Theorem 3.1, we obtain

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(F)(y) = a_{n,\gamma}\left(\int_0^\infty f(r)\,j_{\frac{n}{2}+\gamma+\frac{d}{2}-1}(r\|y\|)r^{n+2\gamma+d-1}dr\right)P(y).$$

The definition (2.10) of the Fourier-Bessel transform $\mathcal{F}_B^{\frac{n}{2}+\gamma+\frac{d}{2}-1}$ implies that this relation can also be written in the form (3.5).

Remark. S. Ben Said has used the theory of representations to obtain in [1] the analogue of Theorem 3.2 for functions of $\mathcal{S}(\mathbb{R}^d)$.

To state and prove the second generalized Bochner Hecke theorem we need the following Notations and Lemmas.

Notations. Let $n \in \mathbb{N}$ and P in H^k_n. We denote by

- $L^p_{(n,\gamma)}([0,+\infty[), p = 1,2$, the space of measurable functions f on $[0,+\infty[$ such that

$$\|f\|_{(n,\gamma)),p} = \left(\int_0^\infty |f(r)|^p r^{2n+2\gamma+d-1}dr\right)^{1/p} < +\infty.$$

- $L^2_{k,P}(\mathbb{R}^d) = \{f(\|x\|)P(x)$ in $L^2_k(\mathbb{R}^d)$, with f defined $(a,e.)$ in $[0,+\infty[\}$.

Lemma 3.1. The application T_P from $L^2_{(n,\gamma)}([0,+\infty[)$ into $L^2_{k,P}(\mathbb{R}^d)$ defined by

$$T_P(f)(x) = f(\|x\|)P(x), \tag{3.7}$$

satisfies

$$\|T_P(f)\|_{k,2} = C\|f\|_{(n,\gamma),2}\ , \tag{3.8}$$

with

$$C = \left(\int_{S^{d-1}} |P(u)|^2 w_k(u)d\sigma(u)\right)^{1/2}.$$

Proof. We obtain the relation (3.8) by using spherical coordinates.

Lemma 3.2. The set of linear combination of the functions $r \to e^{-\lambda r^2/2}, \lambda > 0$, is dense in $L^2_{(n,\gamma)}([0,+\infty[)$.

Proof. We must prove that if the function φ in $L^2_{(n,\gamma)}([0,+\infty[)$ is such that for all $\mu > 0$:

$$\int_0^\infty \varphi(r) e^{-\mu r^2} r^{2n+2\gamma+d-1} dr = 0 \tag{3.9}$$

then

$$\varphi = 0.$$

We consider the function

$$\psi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \varphi(\sqrt{x}) x^{n+\gamma+\frac{d}{2}-1} e^{-x/2} & \text{if } x > 0. \end{cases}$$

By using the change of variables $x = r^2$ and Schwartz inequality we obtain

$$\begin{aligned} \int_0^\infty |\psi(x)| dx &= 2\int_0^\infty |\varphi(r)| e^{-r^2/2} r^{2n+2\gamma+d-1} dr \\ &\leq 2\Big(\int_0^\infty |\varphi(r)|^2 r^{2n+2\gamma+d-1} dr\Big)\Big(\int_0^\infty e^{-r^2} r^{2n+2\gamma+d-1} dr\Big) < +\infty, \end{aligned}$$

as supp $\psi \subset [0,+\infty[$, then the function ψ is integrable on $\mathbb{R}$ with respect to the Lebesgue measure.
On the other hand for all $s > 0$, the change of variables $x = r^2$ implies

$$\int_0^\infty \psi(x) e^{-sx} dx = 2\int_0^\infty \varphi(r) e^{-(s+\frac{1}{2})r^2} r^{2n+2\gamma+d-1} dr.$$

From this relation and (3.9) we deduce that

$$\int_0^\infty \psi(x) e^{-sx} dx = 0.$$

The injectivity of the Laplace transform implies that $\psi = 0$, and then $\varphi = 0$.
Theorem 3.3. Let f be in $L^2_{(n,\gamma)}([0,+\infty[)$. Then

i) the function $F(x) = f(\|x\|)P(x)$ belongs to $L^2_k(\mathbb{R}^d)$, and its Dunkl transform is of the form

$$\mathcal{F}_D(F)(y) = g(\|y\|)P(y), \quad y \in \mathbb{R}^d, \tag{3.10}$$

with g in $L^2_{(n,\gamma)}([0,+\infty[)$.

ii) If moreover f belongs to $L^1_{(n,\gamma)}([0,+\infty[)$ then we have

$$\forall\, r \geq 0,\; g(r) = b_{\gamma,n} \mathcal{F}_B^{\frac{n}{2}+\gamma+\frac{d}{2}-1}(f)(r), \tag{3.11}$$

with $b_{\gamma,n}$ the constant given by (3.6).

Proof.

i) It is clear that from Lemma 3.1 the function $F(x) = f(\|x\|)\ P(x)$ belongs to $L^2_{(n,\gamma)}([0,+\infty[)$. Also from this Lemma, up to a constant of normalization, the application $\mathcal{F}_D\, o\, T_P$ is an isometric from $L^2_{(n,\gamma)}([0,+\infty[)$ into $L^2_k(\mathbb{R}^d)$. From the relation (2.11) this isometric apply all function of the type $e^{-\frac{\lambda\|x\|^2}{2}}, \lambda > 0$, in the space $L^2_P(\mathbb{R}^d)$. Then by using Lemma 3.2, we deduce that the space $L^2_P(\mathbb{R}^d)$ is invariant under the Dunkl transform. Thus

$$\mathcal{F}_D(F)(y) = g(\|y\|)P(y), \quad y \in \mathbb{R}^d,$$

with g in $L^2_{(n,\gamma)}([0,+\infty[)$.

ii) If moreover the function f belongs to $L^1_{(n,\gamma)}([0,+\infty[)$, then we have

$$\int_0^\infty |f(r)|r^{n+2\gamma+d-1}dr = \int_0^1 (|f(r)|r^n)r^{2\gamma+d-1}dr + \int_1^\infty |f(r)|r^{n+2\gamma+d-1}dr.$$

By applying Schwartz inequality to the first integral and by replacing r^n by r^{2n} in the second integral, we obtain

$$\int_0^\infty |f(r)|r^{n+2\gamma+d-1}dr \le \frac{1}{2\gamma+d}\|f\|^2_{(n,\gamma),2} + \|f\|_{(n,\gamma),1} < +\infty.$$

Thus the function f satisfies the condition (3.4) of Theorem 3.2. We obtain (3.11) from (3.10) and (3.5).

5 Application to homogeneous distributions

In this section we shall use the generalized Bochner-Hucke theorems to obtain the Dunkl transform of some homogenous distributions on $\mathbb{R}^d$.

Let $\beta \in \mathbb{R}$. A function f defined on $\mathbb{R}^d$ is homogeneous of degree β if for all $\lambda > 0$ we have

$$f(\lambda x) = \lambda^\beta f(x), \tag{4.1}$$

Let f be a locally integrable function on $\mathbb{R}^d$ with respect to the Lebesgue measure, and which is homogeneous of degree β. We consider the distribution T_{fw_k} of $\mathcal{D}'(\mathbb{R}^d)$ given by the function fw_k. For all φ in $\mathcal{D}(\mathbb{R}^d)$ and $\lambda > 0$ we have

$$\langle T_{fw_k}, \varphi_\lambda\rangle = \lambda^{-d-2\gamma-\beta}\langle T_{fw_k}, \varphi\rangle, \tag{4.2}$$

where

$$\forall\, x \in \mathbb{R}^d,\ \varphi_\lambda(x) = \varphi(\lambda x).$$

The relation (4.2) implies that in the Dunkl theory's, we say that a distribution S in $\mathcal{D}'(\mathbb{R}^d)$ is homogeneous of degree β if for all φ in $\mathcal{D}(\mathbb{R}^d)$ and $\lambda > 0$, we have

$$\langle S, \varphi_\lambda \rangle = \lambda^{-d-2\gamma-\beta} \langle S, \varphi \rangle. \tag{4.3}$$

Remark. All distribution in $\mathcal{D}'(\mathbb{R}^d)$ homogeneous, belongs to $S'(\mathbb{R}^d)$ (See [4] p. 154).

Proposition 4.1. Let S be in $\mathcal{D}'(\mathbb{R}^d)$ homogeneous of degree β. Then its Dunkl transform is homogeneous of degree $-d-2\gamma-\beta$.

Proof. We have

$$\forall\, y \in \mathbb{R}^d\,, \quad \mathcal{F}_D(\varphi_\lambda)(y) = \int_{\mathbb{R}^d} \varphi(\lambda x) K(-ix, y) w_k(x) dx.$$

By the change of variables $t = \lambda x$, we obtain

$$\forall\, y \in \mathbb{R}^d, \;\; \mathcal{F}_D(\varphi_\lambda)(y) = \lambda^{-d-2\gamma} \int_{\mathbb{R}^d} \varphi(t) K(-it, \frac{y}{\lambda}) w_k(t) dt.$$

Thus

$$\forall\, y \in \mathbb{R}^d, \;\; \mathcal{F}_D(\varphi_\lambda)(y) = \lambda^{-d-2\gamma}\, \mathcal{F}_D(\varphi)(\frac{y}{\lambda}). \tag{4.4}$$

From this relation and (4.3) we obtain

$$\begin{aligned} \langle \mathcal{F}_D(S), \varphi_\lambda \rangle &= \langle S, \mathcal{F}_D(\varphi_\lambda) \rangle \\ &= \lambda^{-d-2\gamma} \langle S_y, \mathcal{F}_D(\varphi)(\frac{y}{\lambda}) \rangle \\ &= \lambda^{-d-2\gamma-(-d-2\gamma-\beta)}. \end{aligned}$$

Proposition 4.2. Let P be in H_n^k and $s \in \mathbb{C}$. Then the function $G_s(x) = \dfrac{P(x)}{\|x\|^s}$ is homogeneous of degree $n-s$.

Theorem 4.1. The Dunkl transform of the function G_s, with $n < Res < n + 2\gamma + d$, is given by

$$\mathcal{F}_D(G_s)(y) = M_{\gamma,n,s} \frac{P(y)}{\|y\|^{2n+2\gamma+d-s}}\,, \quad y \in \mathbb{R}^d\,, \tag{4.5}$$

where

$$M_{\gamma,n,s} = \frac{2^{n+\gamma+\frac{d}{2}} i^{-n}}{c_k} \, \frac{\Gamma(\frac{n+2\gamma+d-s}{2})}{\Gamma(\frac{s}{2})} \tag{4.6}$$

Proof. We suppose first that : $n + \gamma + \dfrac{d}{2} < Res < n + 2\gamma + d$.

We write G_s in the form

$$G_s(x) = G_s(x) \mathbf{1}_{B(0,1)}(x) + G_s(x) \mathbf{1}_{{}^cB(0,1)}(x)\,,$$

where $B(0,1)$ is the closed unit ball of $\mathbb{R}^d$ and ${}^cB(0,1)$ its complementary, and $\mathbf{1}_{B(0,1)}$, $\mathbf{1}_{{}^cB(0,1)}$ their characteristic functions . It is clear that $G_s(x)\mathbf{1}_{B(0,1)}(x)$ is in $L_k^1(\mathbb{R}^d)$ and $G_s(x)\mathbf{1}_{{}^cB(0,1)}$ is in $L_k^2(\mathbb{R}^d)$. By applying to these functions respectively Theorems 3.2 and 3.3, we deduce that

$$\mathcal{F}_D(G_s)(y) = g(\|y\|) P(y), \quad y \in \mathbb{R}^d, \tag{4.7}$$

with g a function defined (a.e.) on $[0,+\infty[$.
As from Propositions 4.1, 4.2, the function $\mathcal{F}_D(G_s)$ is homogeneous of degree $-d-2\gamma-n+s$, then the function g is homogeneous of degree $-d-2\gamma-2n+s$. Thus it is necessary of the form

$$g(\|y\|) = \frac{M_{n,\gamma,s}}{\|y\|^{2n+2\gamma+d-s}} \tag{4.8}$$

where $M_{n,\gamma,s}$ is a constant.
On the other hand from (4.7), (4.8), for all φ in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \frac{P(x)}{\|x\|^s}\, \mathcal{F}_D(\varphi)(x) w_k(x)dx = M_{n,\gamma,s} \int_{\mathbb{R}^+} \frac{P(y)\varphi(y)w_k(y)}{\|y\|^{2n+2\gamma+d-s}}\, dy. \tag{4.9}$$

To obtain the value of $M_{n,\gamma,s}$ we consider the function $\varphi(x) = e^{-\frac{\|x\|^2}{2}} P(x)$. Then from (2.11) the relation (4.9) takes the form

$$\frac{i^{-n}}{c_k} \int_{\mathbb{R}^d} \frac{P^2(x)}{\|x\|^s}\, e^{-\frac{\|x\|^2}{2}} w_k(x)dx = M_{n,\gamma,s} \int_{\mathbb{R}^d} \frac{P^2(y)e^{\frac{-\|y\|^2}{2}}}{\|y\|^{2n+2\gamma+d-s}}\, w_k(y)dy.$$

By using spherical coordinates and Fubini's theorem, we deduce that

$$\frac{i^{-n}}{c_k} \int_0^\infty e^{-\frac{r^2}{2}} r^{2n+2\gamma+d-s-1} dr = M_{n,\gamma,s} \int_0^\infty e^{-\frac{r^2}{2}} r^{s-1} dr.$$

The definition of the function gamma implies the relation (4.6).

We have proved the relation (4.9) in the case $n+\gamma+\dfrac{d}{2} < Re s < n+2\gamma+d$. But the two members of this relation are analytic functions of the complex variable s in the band $n < Re s < n+2\gamma+d$. The identity (4.9) is then true in this band.
This completes the proof of the theorem.

We consider now the function

$$G(x) = \frac{P(x)}{\|x\|^{n+2\gamma+d}}\ , \tag{4.10}$$

where P is in H_n^k, with $n \geq 1$.

Lemma 4.1. The distribution denoted also by G, defined by the relation

$$\begin{aligned} \langle G, \varphi \rangle &= vp \int_{\mathbb{R}^d} G(x)\varphi(x)w_k(x)dx \\ &= \lim_{\varepsilon\to 0} \int_{\|x\|\geq\varepsilon>0} G(x)\varphi(x)w_k(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \end{aligned} \tag{4.11}$$

belongs to $\mathcal{S}'(\mathbb{R}^d)$.

Proof. We have

$$\int_{\mathbb{R}^d} G(x)\varphi(x)w_k(x)dx = \int_{B(0,1)} G(x)\varphi(x)w_k(x)dx + \int_{{}^cB(0,1)} G(x)\varphi(x)w_k(x)dx, \tag{4.12}$$

where $B(0,1)$ is the unit closed ball of $\mathbb{R}^d$ and ${}^cB(0,1)$ its complementary. As the function $G(x)\mathbf{1}_{{}^cB(0,1)}(x)$, with $\mathbf{1}_{{}^cB(0,1)}$ the characteristic function of ${}^cB(0,1)$, belongs to $L^2_k(\mathbb{R}^d)$, then from Schwartz inequality we deduce that there exists a positive constant C_1 such that

$$\left|\int_{{}^cB(0,1)} G(x)\varphi(x)w_k(x)dx\right| \leq C_1 \sup_{x\in\mathbb{R}^d} |\varphi(x)|. \tag{4.13}$$

On the other hand as the degree of P is greater than one, then by using spherical coordinates, Fubini's theorem and the orthogonality of the polynomials P (see Theorem 5.1.6 of [8] p. 177) we obtain

$$\int_{\varepsilon\leq\|x\|\leq 1} G(x)w_k(x)dx = \int_\varepsilon^1 \frac{1}{r}\left(\int_{S^{d-1}} P(u)w_k(u)d\sigma(u)\right)dr = 0.$$

Thus

$$\int_{\varepsilon\leq\|x\|\leq 1} G(x)\varphi(x)w_k(x)dx = \int_{\varepsilon\leq\|x\|\leq 1} G(x)[\varphi(x)-\varphi(0)]w_k(x)dx.$$

From Taylor formula we deduce that

$$|\varphi(x)-\varphi(0)| \leq \|x\| \sup_{x\in\mathbb{R}^d} |\frac{\partial}{\partial x_1}\varphi(x)+\ldots+\frac{\partial}{\partial x_d}\varphi(x)|.$$

As the function $\|x\|G(x)\mathbf{1}_{B(0,1)}(x)$, with $\mathbf{1}_{B(0,1)}$ the characteristic function of $B(0,1)$, belongs to $L^1_k(\mathbb{R}^d)$, then

$$\int_{\varepsilon\leq\|x\|\leq 1} |G(x)||\varphi(x)-\varphi(0)|w_k(x)dx \leq C_2 \sup_{x\in\mathbb{R}^d} |\frac{\partial}{\partial x_1}\varphi(x)+\ldots+\frac{\partial}{\partial x_d}\varphi(x)|, \tag{4.14}$$

with

$$C_2 = \int_{B(0,1)} \|x\|G(x)w_k(x)dx.$$

Using (4.11) and (4.12) (4.13), (4.14), we deduce that there exists a positive constant C such that

$$|\langle G,\varphi\rangle| \leq CP_{0,1}(\varphi).$$

Thus the distribution G belongs to $\mathcal{S}'(\mathbb{R}^d)$.

Theorem 4.2. The Dunkl transform of the distribution G given by (4.11) is the distribution T_{Fw_k} of $\mathcal{S}'(\mathbb{R}^d)$ given by the function Fw_k with

$$F(y) = M^0_{n,\gamma}\frac{P(y)}{\|y\|^n}, \quad y\in\mathbb{R}^d, \tag{4.15}$$

where

$$M^0_{n,\gamma} = \frac{i^{-n}2^{-\gamma-\frac{d}{2}}}{c_k}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+2\gamma+d}{2})} \tag{4.16}$$

Proof. We shall see that to obtain (4.15) it suffices to make $s = n+2\gamma+d$ in Theorem 4.1.

In the proof of Theorem 4.1 we have shown that for $n < Res < n+2\gamma+d$, we have

$$M_{n,\gamma,s}\int_{\mathbb{R}^d}\frac{P(y)\varphi(y)w_k(y)}{\|y\|^{2n+2\gamma+d-s}}dy = \int_{\mathbb{R}^d}\frac{P(x)}{\|x\|^s}\,\mathcal{F}_D(\varphi)(x)w_k(x)dx,\ \ \varphi\in\mathcal{S}(\mathbb{R}) \tag{4.17}$$

It is clear that in the left member, when s tends to $n+2\gamma+d$, we obtain $M^0_{n,\gamma}\int_{\mathbb{R}^d}\frac{P(y)}{\|y\|^n}w_k(y)dy$, with $M^0_{n,\gamma}$ given by (4.16).
On the other hand using the fact that

$$\int_{S^{d-1}}P(u)w_k(u)d\sigma(u) = 0,$$

and by considering the function $\psi = \mathcal{F}_D(\varphi)$ in the right member of (4.17), we obtain

$$\begin{aligned}
&\lim_{s\to n+2\gamma+d}\int_{\mathbb{R}^d}\frac{P(x)}{\|x\|^s}\mathcal{F}_D(\varphi)(x)w_k(x)dx\\
&= \lim_{s\to n+2\gamma+d}\left\{\int_{B(0,1)}\frac{P(x)}{\|x\|^s}[\psi(x)-\psi(0)]w_k(x)dx + \int_{^cB(0,1)}\frac{P(x)}{\|x\|^s}\psi(x)w_k(x)dx\right.\\
&= \int_{B(0,1)}G(x)[\psi(x)-\psi(0)]w_k(x)dx + \int_{^cB(0,1)}G(x)\psi(x)w_k(x)dx\\
&= vp\int_{\mathbb{R}^d}G(x)\psi(x)w_k(x)dx\\
&= \langle G,\psi\rangle.
\end{aligned}$$

Thus we obtain (4.15).
This completes the proof of the theorem.

References

[1] S. Ben Said, On the integrability of a representation of $s\ell(2,\mathbb{R})$. *Preprint*. 2006.

[2] L. Cherednik, A unification of the Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras. *Invent. Math* **106**, 1991, p. 411.-432.

[3] J. F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement. *Comm. Math. Phys.* **188**, 1997, p. 467-497.

[4] W. F. Donoghue, Distributions and Fourier transforms. *Academic Press*, 1969.

[5] C. F. Dunkl, Reflection groups and orthogonal polynomials on the sphere. *Math. Z.* **197**, 1988, p. 33-60.

[6] C. F. Dunkl, Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* **311**, 1989, p. 167-183.

[7] C. F. Dunkl, Hankel transforms associated to finite reflection groups. *Contemp. Math.* **138**, 1992, p. 123-138.

[8] C. F. Dunkl and Y. Xu, Orthogonal polynomials of several variables. *Cambridge Univ. Press.* 2001.

[9] P. Eymard, Analyse de Fourier Euclidienne. Ecole d'Été "Analyse Harmonique". CIMPA. Université de Nancy I, 1980.

[10] J. Faraut et K. Harzallah, Deux cours d'Analyse Harmonique. École d'Été d'Analyse Harmonique de Tunis, 1984. Birkhäuser 1987.

[11] G. J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam. *Invent. Math.* **103**, 1991, p. 341-350.

[12] K. Hikami, Dunkl operators formalism for quantum many body problems associated with classical root systems. *J. Phys. Soc. Japan*, **65**, 1996, p. 394-401.

[13] J. E. Humphreys, Reflection groups and Coexter groups. Cambridge Univ. Press. Cambridge, England, 1990.

[14] M. F. E de Jeu, The Dunkl transform. *Invent. Math.* **113**, 1993, p. 147-162.

[15] S. Kakei, Common algebraic structure for the Calogero-Sutherland models. *J. Phys.* **A 29**, 1996, p. 619-624.

[16] L. Lapointe and L. Vinet, Exact operator solution of the Calogero-Sutherland model. *Comm. Math. Phys.* **178**, 1996, p. 425-452.

[17] H. Mejjaoli and K. Trimèche, Harmonic analysis asosciated with the Dunkl-Bessel Laplace operator and a mean value property. *Frac. Calculus and Appl. Analy.* **4**, No 4, 2001, p. 443-480.

[18] E. M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* **175**, 1995, p. 75-121.

[19] M. Rösler and M. Voit, Markov Processes related with Dunkl operators. *Adv. Appl. Math.* **21**, 1998, p. 575-643.

[20] K. Trimèche, Generalized harmonic analysis and wavelet packets. *Gordon and Breach Sciences Publishers*, 2001

[21] G. N. Watson, A treatise on the theory of Bessel functions. 2nd Ed. Cambridge Univ. Press, London, New York, 1966.

[22] Y. Xu, Funk-Hecke formula for orthogonal polynomials on spheres and on balls. *Bull. London Math. Soc.* **32**, 2000, p. 447-457.