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BEST APPROXIMATION FOR WEIERSTRASS TRANSFORM CONNECTED WITH RIEMANN -LIOUVILLE OPERATOR

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Abstract

Using reproducing kernels for Hilbert spaces, we give best approximation for Weierstrass transform associated with Riemann-Liouville operator. Also, estimates of extremal functions are checked.

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1 Introduction

In [3], we consider the following singular partial differential operators

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}; \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \qquad (r,x) \in]0,+\infty[\times\mathbb{R}, \qquad \alpha \geqslant 0. \end{cases}$$

We associate to Δ_1 and Δ_2 the so-called Riemann-Liouville transform $\mathscr{R}_\alpha$ defined on the space $\mathcal{C}_*(\mathbb{R}^2)$ formed by the continuous functions on $\mathbb{R}^2$, even with respect to the first

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variable, by

$$\mathscr{R}_\alpha(f)(r,x)=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2},x+rt\right)\\ \qquad\times\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2},x+rt\right)\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0.\end{cases}$$

The operator $\mathscr{R}_\alpha$ generalizes the mean operator $\mathscr{R}_0$ defined by

$$\mathscr{R}_0(f)(r,x)=\frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta,x+r\cos\theta)d\theta.$$

The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ defined by

$${}^t\mathscr{R}_0(r,x)=\frac{1}{\pi}\int_{\mathbb{R}} g\left(\sqrt{r^2+(x-y)^2},y\right)dy,$$

play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [8, 9] or in the linearized inverse scattering problem in acoustics [6].

The operators $\mathscr{R}_0$ and ${}^t\mathscr{R}_0$ have been studied by many authors and from many points of view ([1, 12, 13]).

We define the Fourier transform connected with the operator $\mathscr{R}_\alpha$ by

$$\mathscr{F}_\alpha(f)(\mu,\lambda)=\int_{\mathbb{R}}\int_0^{+\infty} f(r,x)j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x}dm_\alpha(r,x),$$

where j_α is the modified Bessel function of the first kind with index α, and $dm_\alpha(r,x)$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$dm_\alpha(r,x)=\frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}r^{2\alpha+1}dr\otimes dx.$$

We have constructed the harmonic analysis related to the Fourier transform $\mathscr{F}_\alpha$ (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem...).

Our investigation in the present paper consists to define and study the Weierstrass transform $\mathscr{W}_{\alpha,t}$ associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$. This transform is defined by

$$\mathscr{W}_{\alpha,t}(f)(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} \mathscr{E}_t((r,x),(s,y))f(s,-y)dm_\alpha(s,y),$$

where $\mathscr{E}_t((r,x),(s,y)),\ t>0$; is the heat kernel associated with the Riemann-Liouville operator which will be defined later. This integral transform which generalized the usual Weierstrass transform [11, 14, 15], solves the heat equation

$$\Delta u((r,x),t)=(2\Delta_1^2+\Delta_2)u((r,x),t)=\frac{\partial}{\partial t}u((r,x),t),$$

with the initial condition $u((r,x),0) = f(r,x)$.

Building on the ideas of Saitoh, Matsuura, Fujiwara and Yamada [7, 14, 15, 16, 20], and using the theory of reproducing kernels [2], we give best approximation of this transform and nice estimates of the associated extremal function.

Let $L^2(dm_\alpha)$ be the Hilbert space of square integrable functions on $[0,+\infty[\times\mathbb{R}$ with respect to the measure dm_α and $\langle ./. \rangle_{m_\alpha}$ its inner product.

For $\nu \in \mathbb{R}$, we consider the Sobolev type space's $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$, consisting of functions $f \in L^2(dm_\alpha)$ such that the function

$$(\mu,\lambda) \longmapsto (1+\mu^2+2\lambda^2)^{\nu/2}\mathcal{F}_\alpha(f)(\mu,\lambda),$$

is square integrable on the set

$$\Gamma^+ = [0,+\infty[\times\mathbb{R} \cup \left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ 0\leqslant\mu\leqslant|\lambda|\right\},$$

with respect to the measure $d\gamma_\alpha$, defined by

$$\int\int_{\Gamma^+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}\Big\{\int_{\mathbb{R}}\int_0^{+\infty} f(\mu,\lambda)\left(\mu^2+\lambda^2\right)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)\left(\lambda^2-\mu^2\right)^\alpha \mu d\mu d\lambda\Big\}.$$

Then, for $\nu > \frac{2\alpha+3}{2}$, $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ is the Hilbert space when endowed with the inner product

$$\langle f/g \rangle_\nu = \int\int_{\Gamma^+} (1+\mu^2+2\lambda^2)^\nu \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\mathcal{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

Moreover, the kernel

$$\mathcal{K}_\nu\big((r,x),(s,y)\big) = \int\int_{\Gamma^+} \frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2\lambda^2)^\nu}d\gamma_\alpha(\mu,\lambda);$$

is a reproducing kernel of the space $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$, where $\varphi_{\mu,\lambda}$ is the eigenfunction that will be defined in the first section. Using the properties of the Fourier transform $\mathcal{F}_\alpha$ and its connection with the convolution product, we show that the Weierstrass transform $\mathcal{W}_{\alpha,t}$ is a bounded linear operator from $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ into $L^2(dm_\alpha)$ and that for all $f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$;

$$\|\mathcal{W}_{\alpha,t}(f)\|_{2,m_\alpha} \leqslant ||f||_\nu.$$

Next, for $\rho > 0$, we define on the space $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$, the new inner product by setting

$$\langle f/g \rangle_{\nu,\rho} = \rho\langle f/g\rangle_\nu + \langle \mathcal{W}_{\alpha,t}(f)/\mathcal{W}_{\alpha,t}(g)\rangle_{m_\alpha}.$$

We show that $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ is an Hilbert space when equipped with the inner product $\langle f/g\rangle_{\nu,\rho}$, and we exhibit a reproducing kernel, that is

$$\mathcal{K}_{\nu,\rho}\big((r,x),(s,y)\big) = \int\int_{\Gamma^+} \frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}}d\gamma_\alpha(\mu,\lambda).$$

The last section of this paper is devoted to study the extremal function. More precisely, for all $\nu > \alpha + \frac{3}{2}$, $\rho > 0$ and $g \in L^2(dm_\alpha)$; the infimum of

$$\left\{\rho||f||_\nu^2 + ||g - \mathscr{W}_{\alpha,t}(f)||_{2,m_\alpha}^2;\ \ f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})\right\},$$

is attained at one function $f^*_{\rho,g}$, called the extremal function. We establish also, the following estimates.

For all $f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ and $g = \mathscr{W}_{\alpha,t}(f)$,

$$\lim_{\rho\to o^+} \left\| f^*_{\rho,g} - f \right\|_\nu = 0.$$

For all $f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ and $g = \mathscr{W}_{\alpha,t}(f)$,

$$\lim_{\rho\to o^+} f^*_{\rho,g}(r,x) = f(r,x), \text{ uniformly.}$$

2 Fourier Transform Associated with Riemann-Liouville Operator

In [3], we showed that for all $(\mu,\lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) = -i\lambda u(r,x); \\ \Delta_2 u(r,x) = -\mu^2 u(r,x); \\ u(0,0) = 1, \qquad \dfrac{\partial u}{\partial r}(0,x) = 0; \qquad \forall x \in \mathbb{R}; \end{cases}$$

admits a unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp\left(-i\lambda x\right),$$

where j_α is the modified Bessel function defined by

$$\begin{aligned} j_\alpha(s) &= 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k} \\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}e^{-its}dt; \end{aligned}$$

and J_α is the Bessel function of first kind and index α [4, 5, 10, 19].

For all $(\mu,\lambda) \in \Gamma$, we have

$$\sup_{(r,x)\in\mathbb{R}^2}\left|\varphi_{\mu,\lambda}(r,x)\right| = 1 \tag{2.1}$$

where Γ is the set given by

$$\Gamma = \mathbb{R}^2 \cup \left\{(i\mu,\lambda);\ (\mu,\lambda) \in \mathbb{R}^2,\ |\mu| \leqslant |\lambda|\right\}. \tag{2.2}$$

Moreover, the function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x)=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\left(\mu rs\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right)\\ \qquad\qquad\times\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\mu\sqrt{1-t^2}\right)\exp(-i\lambda(x+rt))\\ \qquad\qquad\times\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0.\end{cases}\tag{2.3}$$

This integral representation allows us to define the Riemann-Liouville transform $\mathcal{R}_\alpha$, connected with the operators Δ_1 and Δ_2. More precisely; we have

Definition 2.1. The Riemann-Liouville transform $\mathcal{R}_\alpha$ associated with the operators Δ_1 and Δ_2 is defined for a continuous function f on $\mathbb{R}^2$, even with respect to the first variable by

$$\mathcal{R}_\alpha(f)(r,x)=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}f\left(rs\sqrt{1-t^2},x+rt\right)\\ \qquad\times\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}f\left(r\sqrt{1-t^2},x+rt\right)\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0.\end{cases}\tag{2.4}$$

Remark 2.2. i) From the relations (2.3), and (2.4), we have

$$\varphi_{\mu,\lambda}(r,x)=\mathcal{R}_\alpha\left(\cos(\mu.)\exp(-i\lambda.)\right)(r,x).$$

ii) By a simple change of variables, we have

$$\mathcal{R}_0(f)(r,x)=\frac{1}{2\pi}\int_0^{2\pi}f(r\sin\theta,x+r\cos\theta)d\theta,$$

which means that $\mathcal{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and with radius r.

In the sequel, we denote by
$dm_\alpha(r,x)$ the measure defined on $[0,+\infty[\times\mathbb{R}$, by

$$dm_\alpha(r,x)=\frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}r^{2\alpha+1}dr\otimes dx,$$

$L^p(dm_\alpha)$, $p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, satisfying

$$\|f\|_{p,m_\alpha}=\left(\int_{\mathbb{R}}\int_0^{+\infty}|f(r,x)|^p dm_\alpha(r,x)\right)^{\frac{1}{p}}<+\infty,\quad \text{if } p\in[1,+\infty[;$$
$$\|f\|_{\infty,m_\alpha}=\underset{(r,x)\in[0,+\infty[\times\mathbb{R}}{ess\ sup}|f(r,x)|<+\infty,\qquad \text{if } p=+\infty.$$

$\langle ./.\rangle_{m_\alpha}$ the inner product on $L^2(dm_\alpha)$ defined by

$$\langle f/g\rangle_{m_\alpha}=\int_{\mathbb{R}}\int_0^{+\infty}f(r,x)\overline{g(r,x)}dm_\alpha(r,x).$$

Γ^+ the subset of Γ given by

$$\Gamma^+ = [0,+\infty[\times\mathbb{R}\cup\left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ 0\leqslant\mu\leqslant|\lambda|\right\}.$$

$\mathcal{B}_{\Gamma^+}$ the σ-algebra on Γ^+;

$$\mathcal{B}_{\Gamma^+} = \theta^{-1}(\mathcal{B}_{[0,+\infty[\times\mathbb{R}}),$$

where θ is the bijective function defined on Γ^+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda). \tag{2.5}$$

$d\gamma_\alpha(\mu,\lambda)$ the measure defined on Γ^+ by

$$\gamma_\alpha(A) = m_\alpha(\theta(A));\ \ A\in\mathcal{B}_{\Gamma^+}.$$

$L^p(d\gamma_\alpha),\ p\in[1,+\infty]$, the space of measurable functions on Γ^+ satisfying

$$\|f\|_{p,\gamma_\alpha} = \left(\int\int_{\Gamma^+}|f(\mu,\lambda)|^p\,d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{p}} < +\infty, \quad \text{if } p\in[1,+\infty[,$$

$$\|f\|_{\infty,\gamma_\alpha} = \underset{(\mu,\lambda)\in\Gamma^+}{ess\ sup}\ |f(\mu,\lambda)| < +\infty, \quad \text{if } p=+\infty.$$

$\langle ./.\rangle_{\gamma_\alpha}$ the inner product on $L^2(d\gamma_\alpha)$ defined by

$$\langle f/g\rangle_{\gamma_\alpha} = \int\int_{\Gamma^+} f(\mu,\lambda)\overline{g(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

Then, we have the following properties

Proposition 2.3. *i) For all non negative measurable functions f on Γ^+ (respectively integrable on Γ^+ with respect to the measure $d\gamma_\alpha$), we have*

$$\int\int_{\Gamma^+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}\Big\{\int_{\mathbb{R}}\int_0^{+\infty} f(\mu,\lambda)\left(\mu^2+\lambda^2\right)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)\left(\lambda^2-\mu^2\right)^\alpha \mu d\mu d\lambda\Big\}.$$

ii) For all non negative measurable functions g on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure dm_α), we have

$$\int_{\mathbb{R}}\int_0^\infty g(r,x)dm_\alpha(r,x) = \int\int_{\Gamma^+} g\circ\theta(\mu,\lambda)d\gamma_\alpha(\mu,\lambda). \tag{2.6}$$

In the following, we shall define the translation operators and the convolution product associated with the Riemann-Liouville transform. For this, we use the product formula for the function $\varphi_{\mu,\lambda}$;

for all $(r,x),(s,y)\in\mathbb{R}^2$;

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi \varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{2\alpha}\theta d\theta. \tag{2.7}$$

Definition 2.4. i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(dm_\alpha)$ by, $\forall\ (r,x),\ (s,y) \in [0,+\infty[\times\mathbb{R}$;

$$\tau_{(r,x)} f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta.$$

ii)The convolution product of $f,g \in L^1(dm_\alpha)$ is defined, by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}; \qquad f*g(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} \tau_{(r,-x)}\check{f}(s,y)g(s,y)dm_\alpha(s,y),$$

where $\check{f}(s,y) = f(s,-y)$.

We have the following properties.

The relation (2.7) can be written: $\tau_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\ \varphi_{\mu,\lambda}(s,y)$.

If $f \in L^p(dm_\alpha),\ 1 \leqslant p \leqslant +\infty$, then for all $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function $\tau_{(s,y)}f$ belongs to $L^p(dm_\alpha)$, and we have

$$\left\|\tau_{(s,y)}f\right\|_{p,m_\alpha} \leqslant \|f\|_{p,m_\alpha}. \tag{2.8}$$

In particular, for all $f \in L^1(dm_\alpha)$ and $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function $\tau_{(s,y)}f$ belongs to $L^1(dm_\alpha)$ and we have

$$\int_{\mathbb{R}}\int_0^{+\infty} \tau_{(s,y)} f(r,x)dm_\alpha(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)dm_\alpha(r,x). \tag{2.9}$$

For $f \in L^1(dm_\alpha),\ g \in L^p(dm_\alpha),\ 1 \leqslant p < +\infty$, the function $f*g \in L^p(dm_\alpha)$, and we have

$$\|f*g\|_{p,m_\alpha} \leqslant \|f\|_{1,m_\alpha}\|g\|_{p,m_\alpha}. \tag{2.10}$$

In the sequel, we shall define the Fourier transform $\mathcal{F}_\alpha$ associated with $\mathcal{R}_\alpha$ and we recall some properties that we need in the next section.

Definition 2.5. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(dm_\alpha)$, by

$$\forall (\mu,\lambda) \in \Gamma; \qquad \mathcal{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\varphi_{\mu,\lambda}(r,x)dm_\alpha(r,x),$$

where Γ is the set defined by the relation (2.2).

The Fourier transform $\mathcal{F}_\alpha$ satisfies the properties

For every f in $L^1(dm_\alpha)$ and $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$\forall (\mu,\lambda) \in \Gamma,\ \ \mathcal{F}_\alpha\left(\tau_{(r,-x)}f\right)(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathcal{F}_\alpha(f)(\mu,\lambda). \tag{2.11}$$

For $f,g \in L^1(dm_\alpha)$, we have

$$\forall (\mu,\lambda) \in \Gamma,\ \ \mathcal{F}_\alpha(f*g)(\mu,\lambda) = \mathcal{F}_\alpha(f)(\mu,\lambda)\mathcal{F}_\alpha(g)(\mu,\lambda).$$

For $f \in L^1(dm_\alpha)$, we have

$$\forall(\mu,\lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathcal{F}}_\alpha(f) \circ \theta(\mu,\lambda), \tag{2.12}$$

where

$$\forall(\mu,\lambda) \in \mathbb{R}^2, \quad \widetilde{\mathcal{F}}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x) j_\alpha(r\mu)\exp(-i\lambda x)dm_\alpha(r,x),$$

and θ is the function defined by (2.5).

Theorem 2.6. *(Inversion formula for $\mathcal{F}_\alpha$) Let $f \in L^1(dm_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(d\gamma_\alpha)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have*

$$f(r,x) = \int\int_{\Gamma^+} \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

It's well known ([17, 18]) that the transform $\widetilde{\mathcal{F}}_\alpha$ is an isometric isomorphism from $L^2(dm_\alpha)$ onto itself. Then, using the relations (2.6), (2.12) and the fact that the function θ defined by (2.5), is bijective from Γ^+ in $[0,+\infty[\times\mathbb{R}$; we have the following result

Theorem 2.7. *(Plancherel theorem) The transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(dm_\alpha)$ onto $L^2(d\gamma_\alpha)$. In particular, we have the Parseval's equality; for all $f,g \in L^2(dm_\alpha)$*

$$\int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\overline{g(r,x)}dm_\alpha(r,x) = \int\int_{\Gamma^+} \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\mathcal{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

Remark 2.8. i) From the relation (2.1), it follows that the Fourier transform $\mathcal{F}_\alpha$ is a bounded linear operator from $L^1(dm_\alpha)$ into $L^\infty(d\gamma_\alpha)$ and that for all $f \in L^1(dm_\alpha)$,

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \leqslant \|f\|_{1,m_\alpha}.$$

ii) Let $f \in L^1(dm_\alpha)$ and $g \in L^2(dm_\alpha)$, by the relation (2.9), the function $f*g$ belongs to $L^2(dm_\alpha)$; moreover

$$\mathcal{F}_\alpha(f*g) = \mathcal{F}_\alpha(f).\mathcal{F}_\alpha(g). \tag{2.13}$$

iii) For all $f,g \in L^2(dm_\alpha)$; the function $f*g$ belongs to the space $\mathcal{C}_{*,0}(\mathbb{R}^2)$ consisting of continuous functions h on $\mathbb{R}^2$; even with respect to the first variable and such that

$$\lim_{r^2+x^2\to+\infty} h(r,x) = 0.$$

Moreover,

$$f*g = \mathcal{F}_\alpha^{-1}\big(\mathcal{F}_\alpha(f).\mathcal{F}_\alpha(g)\big). \tag{2.14}$$

3 Weierstrass Transform Associated with the Riemann-Liouville Operator

In this section, we will define and study the Weierstrass transform associated with $\mathcal{R}_\alpha$. For this we define some Hilbert spaces and we exhibit their reproducing kernels.

Let ν be a real number, $\nu > \alpha + \frac{3}{2}$. We denote by

$\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ the subspace of $L^2(dm_\alpha)$ formed by the functions f, such that the application

$$(\mu,\lambda) \longmapsto (1+\mu^2+2\lambda^2)^{\nu/2}\mathcal{F}_\alpha(f)(\mu,\lambda)$$

belongs to $L^2(d\gamma_\alpha)$.

$\langle ./. \rangle_\nu$ the inner product on $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ defined by

$$\langle f/g \rangle_\nu = \int\int_{\Gamma^+} (1+\mu^2+2\lambda^2)^\nu \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\mathcal{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

$||.||_\nu$ the norm of $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ defined by

$$||f||_\nu = \sqrt{\langle f/f \rangle_\nu}.$$

Remark 3.1. For $\nu > \alpha + \frac{3}{2}$; the function

$$(\mu,\lambda) \longmapsto \frac{1}{(1+\mu^2+2\lambda^2)^{\nu/2}}$$

belongs to $L^2(d\gamma_\alpha)$. Hence, for all $f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$, $\mathcal{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$ and for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$f(r,x) = \int\int_{\Gamma^+} \mathcal{F}_\alpha(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

Proposition 3.2. *For $\nu > \alpha + \frac{3}{2}$; the function $\mathcal{K}_\nu$ defined on $([0,+\infty[\times\mathbb{R})^2$ by*

$$\mathcal{K}_\nu\big((r,x),(s,y)\big) = \int\int_{\Gamma^+} \frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2\lambda^2)^\nu}d\gamma_\alpha(\mu,\lambda);$$

is a reproducing kernel of the space $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$, that is

i) For all $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function

$$(r,x) \longmapsto \mathcal{K}_\nu\big((r,x),(s,y)\big),$$

belongs to $\mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$.

ii) (The reproducing property) For all $f \in \mathbf{H}_\nu^\alpha([0,+\infty[\times\mathbb{R})$ and $(s,y) \in [0,+\infty[\times\mathbb{R}$,

$$\langle f/\mathcal{K}_\nu\big((.,.),(s,y)\big)\rangle_\nu = f(s,y).$$

Proof. i) From remark 3.1 and the relation (2.1), we deduce that for all $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function

$$(\mu,\lambda) \longmapsto \frac{\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2\lambda^2)^\nu},$$

belongs to $L^1(d\gamma_\alpha)\cap L^2(d\gamma_\alpha)$. Then, the function $\mathcal{K}_\nu$ is well defined and by theorem 2.6, we have

$$\mathcal{K}_\nu\big((r,x),(s,y)\big)=\mathcal{F}_\alpha^{-1}\Big(\frac{\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2\lambda^2)^\nu}\Big)(r,x).$$

By Plancherel theorem, it follows that for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{K}_\nu\big((.,.),(s,y)\big)$ belongs to $L^2(dm_\alpha)$, and we have

$$\mathcal{F}_\alpha\Big(\mathcal{K}_\nu\big((.,.),(s,y)\big)\Big)(\mu,\lambda)=\frac{\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2\lambda^2)^\nu}. \tag{3.1}$$

Again, by the relation (2.1) and remark 3.1, it follows that the function

$$(\mu,\lambda)\longmapsto(1+\mu^2+2\lambda^2)^{\nu/2}\mathcal{F}_\alpha(\mathcal{K}_\nu\big((.,.),(s,y)\big))(\mu,\lambda),$$

belongs to $L^2(d\gamma_\alpha)$.

ii) Let f be in $\mathbf{H}^\alpha_\nu([0,+\infty[\times\mathbb{R})$. For every $(s,y)\in[0,+\infty[\times\mathbb{R}$, we have

$$\begin{aligned}\langle f/\mathcal{K}_\nu\big((.,.),(s,y)\big)\rangle_\nu=\int\int_{\Gamma^+}&(1+\mu^2+2\lambda^2)^\nu\mathcal{F}_\alpha(f)(\mu,\lambda)\times\\&\overline{\mathcal{F}_\alpha(\mathcal{K}_\nu\big((.,.),(s,y)\big))(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda),\end{aligned}$$

and by the relation (3.1), we get

$$\langle f/\mathcal{K}_\nu\big((.,.),(s,y)\big)\rangle_\nu=\int\int_{\Gamma^+}\mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}d\gamma_\alpha(\mu,\lambda).$$

The result follows from remark 3.1. □

The heat equation associated with the Riemann-Liouville operator is

$$\Delta u((r,x),t)=\frac{\partial}{\partial t}u((r,x),t), \tag{3.2}$$

where

$$\Delta=2\Delta_1^2+\Delta_2.$$

Let E be the kernel defined by

$$\begin{aligned}E((r,x),t)) &= \int\int_{\Gamma^+}\exp[-t(\mu^2+2\lambda^2)]\varphi_{\mu,-\lambda}(r,x)d\gamma_\alpha(\mu,\lambda) \\ &= \frac{1}{(2t)^{\alpha+3/2}}\exp(-\frac{r^2+x^2}{4t}).\end{aligned} \tag{3.3}$$

Then, the kernel E solves the equation (3.2).

Definition 3.3. The heat kernel associated with the Riemann-Liouville transform is defined by

$$\begin{aligned}\mathcal{E}_t((r,x),(s,y))&=\tau^\alpha_{(r,x)}\Big(E((.,.),t)\Big)(s,y)\\&=\frac{1}{(2t)^{\alpha+3/2}}\exp(-\frac{r^2+s^2}{4t})\exp(-\frac{(x+y)^2}{4t})j_\alpha(\frac{irs}{2t}).\end{aligned} \tag{3.4}$$

Then, we have the following properties

i) For all $t>0$, $\mathcal{E}_t \geqslant 0$.

ii) From the relations (2.8),(2.11),(3.3) and (3.4), for all $t>0$, $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{E}_t((r,x),(.,.))$ belongs to $L^1(dm_\alpha)$ and for all $(\mu,\lambda)\in\Gamma$, we have

$$\mathcal{F}_\alpha\big(\mathcal{E}_t((r,x),(.,.))\big)(\mu,\lambda)=\exp\big(-t(\mu^2+2\lambda^2)\big)\varphi_{\mu,-\lambda}(r,x).$$

iii) From the relations (2.8), (2.9), (3.3) and (3.4), for all $t>0$ and $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{E}_t((.,.),(s,y))$ belongs to $L^1(dm_\alpha)$ and we have

$$\int_{\mathbb{R}}\int_0^{+\infty}\mathcal{E}_t((r,x),(s,y))dm_\alpha(r,x)=\int_{\mathbb{R}}\int_0^{+\infty}E((r,x),t)dm_\alpha(r,x)=1.$$

iv) For all $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function

$$\big((r,x),t\big)\longmapsto\mathcal{E}_t((r,x),(s,y))$$

solves the heat equation (3.2).

In the following, we shall define the Weierstrass transform associated with the Riemann-Liouville operator and we establish some properties that we use in the next section.

Definition 3.4. The Weierstrass transform associated with the Riemann-Liouville operator is defined on $L^2(dm_\alpha)$, by

$$\begin{aligned}\mathcal{W}_{\alpha,t}(f)(r,x) &= \big(E((.,.),t)*f\big)(r,x) \qquad (3.5)\\ &= \int_{\mathbb{R}}\int_0^{+\infty}\mathcal{E}_t((r,x),(s,y))f(s,-y)dm_\alpha(s,y).\end{aligned}$$

For the classical Weierstrass transform, one can see [11, 14, 15].

Proposition 3.5. *i) For all $f\in L^2(dm_\alpha)$, the function $\mathcal{W}_{\alpha,t}(f)$ solves the heat equation (3.2), with the initial condition*

$$\lim_{t\to0^+}\mathcal{W}_{\alpha,t}(f)=f;\quad \text{in } L^2(dm_\alpha).$$

ii) For all $t>0$ and $\nu>\alpha+3/2$, the transform $\mathcal{W}_{\alpha,t}$ is a bounded linear operator from $\mathbf{H}^\alpha_\nu([0,+\infty[\times\mathbb{R})$ into $L^2(dm_\alpha)$ and we have

$$\big\|\mathcal{W}_{\alpha,t}(f)\big\|_{2,m_\alpha}\leqslant||f||_\nu.$$

Proof. i) From the relations (3.4), (3.5), the derivative's theorem and the fact that for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function $\big((r,x),t\big)\longmapsto\mathcal{E}_t((r,x),(s,y))$ solves the heat equation (3.2), we deduce that the function $\mathcal{W}_{\alpha,t}(f)$ is a solution of (3.2).

The family $\big(E((.,.),t)\big)_{t>0}$ is an approximate identity; in particular for all $f\in L^2(dm_\alpha)$

$$\lim_{t\to0}E((.,.),t)*f=f \qquad \text{in} \quad L^2(dm_\alpha).$$

ii) From relations (2.10) and (3.5); for all $f\in L^2(dm_\alpha)$, we have

$$\begin{aligned}\big\|\mathcal{W}_{\alpha,t}(f)\big\|_{2,m_\alpha}&=\|E((.,.),t)*f\|_{2,m_\alpha}\\ &\leqslant\|E((.,.),t)\|_{1,m_\alpha}\|f\|_{2,m_\alpha}\\ &=\|f\|_{2,m_\alpha}=\|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}\leqslant\|f\|_\nu.\end{aligned}$$

□

Notations. For all positive real numbers ρ, t and for $\nu > \alpha + 3/2$, we denote by $\langle ./. \rangle_{\nu,\rho}$ the inner product defined on the space $\mathbf{H}^{\alpha}_{\nu}([0,+\infty[\times\mathbb{R})$ by

$$\langle f/g \rangle_{\nu,\rho} = \rho \langle f/g \rangle_{\nu} + \langle \mathcal{W}_{\alpha,t}(f)/\mathcal{W}_{\alpha,t}(g) \rangle_{m_\alpha}.$$

$\mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$ the space $\mathbf{H}^{\alpha}_{\nu}([0,+\infty[\times\mathbb{R})$ equipped with the inner product $\langle ./. \rangle_{\nu,\rho}$ and the norm

$$||f||^2_{\nu,\rho} = \rho ||f||^2_{\nu} + ||\mathcal{W}_{\alpha,t}(f)||^2_{2,m_\alpha}.$$

Then, we have the following main result [11, 15]

Theorem 3.6. *For all* $\rho, t > 0$ *and* $\nu > \alpha + 3/2$*, the Hilbert space* $\mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$ *possesses the following reproducing kernel*

$$\mathcal{K}_{\nu,\rho}\big((r,x),(s,y)\big) = \int\int_{\Gamma^+} \frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^{\nu} + e^{-2t(\mu^2+2\lambda^2)}} d\gamma_\alpha(\mu,\lambda);$$

that is

i) For all $(s,y) \in [0,+\infty[\times\mathbb{R}$*, the function* $(r,x) \longmapsto \mathcal{K}_{\nu,\rho}\big((r,x),(s,y)\big)$ *belongs to* $\mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$.

ii) (The reproducing property.) For all $f \in \mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$ *and* $(s,y) \in [0,+\infty[\times\mathbb{R}$*,*

$$\langle f/\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big) \rangle_{\nu,\rho} = f(s,y).$$

Proof. i) Let $(s,y) \in [0,+\infty[\times\mathbb{R}$. From the inequality (2.1), we have

$$\frac{\big|\varphi_{\mu,\lambda}(s,y)\big|}{\rho(1+\mu^2+2\lambda^2)^{\nu} + e^{-2t(\mu^2+2\lambda^2)}} \leqslant \frac{1}{\rho(1+\mu^2+2\lambda^2)^{\nu}}.$$

Then by, the hypothesis $\nu > \alpha + 3/2$, we deduce that for all $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function

$$(\mu,\lambda) \longmapsto \frac{\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^{\nu} + e^{-2t(\mu^2+2\lambda^2)}},$$

belongs to $L^1(d\gamma_\alpha) \cap L^2(d\gamma_\alpha)$, and by Plancherel theorem, the function

$$(r,x) \mapsto \mathcal{K}_{\nu,\rho}\big((r,x),(s,y)\big) = \mathcal{F}_\alpha^{-1}\Big(\frac{\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^{\nu} + e^{-2t(\mu^2+2\lambda^2)}}\Big)(r,x) \tag{3.6}$$

belongs to $L^2(dm_\alpha)$, moreover the function

$$\begin{aligned}(\mu,\lambda) \longmapsto &(1+\mu^2+2\lambda^2)^{\nu/2}\mathcal{F}_\alpha\Big(\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)(\mu,\lambda)\\ &= \frac{(1+\mu^2+2\lambda^2)^{\nu/2}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^{\nu} + e^{-2t(\mu^2+2\lambda^2)}}\end{aligned}$$

belongs to $L^2(d\gamma_\alpha)$.

This proves that for all $(s,y) \in [0,+\infty[\times\mathbb{R}$, the function $\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)$ belongs to the space $\mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$.

ii) Let f be in $\mathbf{H}^{\alpha}_{\nu,\rho}([0,+\infty[\times\mathbb{R})$. By the relation (3.6), we get

$$\langle f/\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\rangle_{\nu} = \int\int_{\Gamma^+}(1+\mu^2+2\lambda^2)^{\nu}\mathcal{F}_{\alpha}(f)(\mu,\lambda)\times \overline{\left(\frac{\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^{\nu}+e^{-2t(\mu^2+2\lambda^2)}}\right)}d\gamma_{\alpha}(\mu,\lambda). \tag{3.7}$$

On the other hand, we have

$$\mathcal{W}_{\alpha,t}\Big(\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)(r,x) = \Big(E\big((.,.),t\big)*\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)(r,x),$$

and by the relations (2.14), (3.3) and (3.6), we get

$$\mathcal{W}_{\alpha,t}\Big(\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)(r,x) = \mathcal{F}_{\alpha}^{-1}\Big(\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^{\nu}+e^{-2t(\mu^2+2\lambda^2)}}\Big)(r,x). \tag{3.8}$$

By the same way

$$\mathcal{W}_{\alpha,t}(f)(r,x) \quad = \quad \mathcal{F}_{\alpha}^{-1}\Big(e^{-t(\mu^2+2\lambda^2)}\mathcal{F}_{\alpha}(f)\Big)(r,x). \tag{3.9}$$

Thus,

$$\langle \mathcal{W}_{\alpha,t}(f)/\mathcal{W}_{\alpha,t}\Big(\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)\rangle_{m_{\alpha}} = \langle \mathcal{F}_{\alpha}^{-1}\Big(e^{-t(\mu^2+2\lambda^2)}\mathcal{F}_{\alpha}(f)\Big)/\mathcal{F}_{\alpha}^{-1}\Big(\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^{\nu}+e^{-2t(\mu^2+2\lambda^2)}}\Big)\rangle_{m_{\alpha}}.$$

Using the Parseval formula, we get

$$\langle \mathcal{W}_{\alpha,t}(f)/\mathcal{W}_{\alpha,t}\Big(\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)\rangle_{m_{\alpha}} = \langle e^{-t(\mu^2+2\lambda^2)}\mathcal{F}_{\alpha}(f)/\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^{\nu}+e^{-2t(\mu^2+2\lambda^2)}}\rangle_{\gamma_{\alpha}}. \tag{3.10}$$

Combining the relations (3.7) and (3.10), we obtain

$$\langle f/\mathcal{K}_{\nu,\rho}\big((.,.),(s,y)\big)\rangle_{\nu,\rho} = \int\int_{\Gamma^+}\mathcal{F}_{\alpha}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}d\gamma_{\alpha}(\mu,\lambda).$$

The desired result arises from remark 3.1. □

4 The Extremal Function

This section contains the main result of this paper, that is the existence and unicity of the extremal function related to the generalized Weierstrass transform studied in the previous section.

Theorem 4.1. *Let* $\nu > \alpha + \frac{3}{2}$, $\rho > 0$ *and* $g \in L^2(dm_\alpha)$. *Then, there exists a unique function* $f^*_{\rho,g} \in \mathbf{H}^\alpha_\nu([0,+\infty[\times\mathbb{R})$, *where the infimum of*

$$\Big\{\rho||f||^2_\nu + ||g - \mathscr{W}_{\alpha,t}(f)||^2_{2,m_\alpha}, \quad f \in \mathbf{H}^\alpha_\nu([0,+\infty[\times\mathbb{R})\Big\}$$

is attained. Moreover, the extremal function $f^*_{\rho,g}$, *is given by*

$$f^*_{\rho,g}(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} g(s,y)\mathbf{Q}_\rho((r,x),(s,y))dm_\alpha(s,y); \tag{4.1}$$

where,

$$\mathbf{Q}_\rho((r,x),(s,y)) = \int\int_{\Gamma^+} \frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}} d\gamma_\alpha(\mu,\lambda). \tag{4.2}$$

Proof. The existence and unicity of the extremal function $f^*_{\rho,g}$ is given by [11, 14, 15]. On the other hand, we have

$$f^*_{\rho,g}(s,y) = \langle g/\mathscr{W}_{\alpha,t}\Big(\mathscr{K}_{\nu,\rho}\big((.,.),(s,y)\big)\Big)\rangle_{m_\alpha},$$

and by (3.8), we obtain

$$\begin{aligned}
f^*_{\rho,g}(s,y) &= \langle g/\mathscr{F}^{-1}_\alpha\Big(\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}}\Big)\rangle_{m_\alpha} \\
&= \int_{\mathbb{R}}\int_0^{+\infty} g(r,x)\Big(\int\int_{\Gamma^+} \frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,-\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}} d\gamma_\alpha(\mu,\lambda)\Big)dm_\alpha(r,x) \\
&= \int_{\mathbb{R}}\int_0^{+\infty} g(r,x)\overline{\mathbf{Q}_\rho((r,x),(s,y))}dm_\alpha(r,x).
\end{aligned}$$

□

Corollary 4.2. *Let* $\nu > \alpha + \frac{3}{2}$, $\rho > 0$ *and* $g \in L^2(dm_\alpha)$. *The extremal function* $f^*_{\rho,g}$, *satisfies the following inequality*

$$\Big\|f^*_{\rho,g}\Big\|^2_{2,m_\alpha} \leqslant \frac{\Gamma(\nu-\alpha-3/2)}{\rho 2^{2\alpha+5}\Gamma(\nu)} \int_{\mathbb{R}}\int_0^{+\infty} e^{r^2+x^2}|g(r,x)|^2 dm_\alpha(r,x);$$

Proof. We have

$$f^*_{\rho,g}(s,y) = \int_{\mathbb{R}}\int_0^{+\infty} e^{-\frac{r^2+x^2}{2}} e^{\frac{r^2+x^2}{2}} g(r,x)\overline{\mathbf{Q}_\rho((r,x),(s,y))}dm_\alpha(r,x).$$

By Hölder's inequality, we get

$$\begin{aligned}
|f^*_{\rho,g}(s,y)|^2 \leqslant &\Big(\int_{\mathbb{R}}\int_0^{+\infty} e^{-(r^2+x^2)}dm_\alpha(r,x)\Big)\times \\
&\Big(\int_{\mathbb{R}}\int_0^{+\infty} e^{r^2+x^2}|g(r,x)|^2\Big|\mathbf{Q}_\rho((r,x),(s,y))\Big|^2 dm_\alpha(r,x)\Big).
\end{aligned}$$

Integrating over $[0,+\infty[\times\mathbb{R}$, with respect to the measure $dm_\alpha(s,y)$, we obtain

$$\begin{aligned}\left\|f^*_{\rho,g}\right\|^2_{2,m_\alpha} \leqslant &\Big(\int_{\mathbb{R}}\int_0^{+\infty} e^{-(r^2+x^2)}dm_\alpha(r,x)\Big)\times\\ &\Big(\int_{\mathbb{R}}\int_0^{+\infty} e^{r^2+x^2}\,|g(r,x)|^2\,\left\|\mathbf{Q}_\rho((r,x),(.,.))\right\|^2_{2,m_\alpha} dm_\alpha(r,x)\Big). \qquad (4.3)\end{aligned}$$

However, by (4.2)

$$\mathbf{Q}_\rho((r,x),(s,y)) = \mathcal{F}_\alpha^{-1}\Big(\frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,\lambda}(r,x)}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}\Big)(s,y);$$

then, by Plancherel theorem

$$\left\|\mathbf{Q}_\rho((r,x),(.,.))\right\|^2_{2,m_\alpha} = \int\int_{\Gamma^+}\frac{e^{-2t(\mu^2+2\lambda^2)}\left|\varphi_{\mu,\lambda}(r,x)\right|^2}{\left|\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}\right|^2}d\gamma_\alpha(\mu,\lambda).$$

Since, $a^2+b^2\geqslant 2ab;\ \ a,b\geqslant 0$, and in virtue of (2.1); it follows that

$$\left\|\mathbf{Q}_\rho((r,x),(.,.))\right\|^2_{2,m_\alpha} \leqslant \frac{1}{4\rho}\int\int_{\Gamma^+}\frac{d\gamma_\alpha(\mu,\lambda)}{(1+\mu^2+2\lambda^2)^\nu};$$

and by the relation (2.6),

$$\left\|\mathbf{Q}_\rho((r,x),(.,.))\right\|^2_{2,m_\alpha} \leqslant \frac{1}{4\rho}\int_{\mathbb{R}}\int_0^{+\infty}\frac{dm_\alpha(u,v)}{(1+u^2+v^2)^\nu}. \qquad (4.4)$$

We complete the proof by using the relations (4.3), (4.4) and the fact that

$$\int_{\mathbb{R}}\int_0^{+\infty} e^{-(r^2+x^2)}dm_\alpha(r,x) = \frac{1}{2^{\alpha+3/2}};$$

and

$$\int_{\mathbb{R}}\int_0^{+\infty}\frac{dm_\alpha(u,v)}{(1+u^2+v^2)^\nu} = \frac{\Gamma(\nu-\alpha-3/2)}{2^{\alpha+\frac{3}{2}}\Gamma(\nu)}.$$

□

Corollary 4.3. *Let* $\nu>\alpha+\frac{3}{2}$. *For all* $g_1,g_2\in L^2(dm_\alpha)$, *we have*

$$\left\|f^*_{\rho,g_1}-f^*_{\rho,g_2}\right\|_\nu \leqslant \frac{\|g_1-g_2\|_{2,m_\alpha}}{2\sqrt{\rho}}.$$

Proof. Let $\nu>\alpha+\frac{3}{2}$. For all $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function

$$(\mu,\lambda)\longmapsto\frac{e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}\varphi_{\mu,\lambda}(r,x)$$

belongs to $L^1(d\gamma_\alpha)\cap L^2(d\gamma_\alpha)$.

From the relation (4.1) and the fact that

$$\mathbf{Q}_\rho((r,x),(s,y)) = \overline{\mathcal{F}_\alpha^{-1}\Big(\frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}\Big)(r,x)};$$

we deduce that for all $g \in L^2(dm_\alpha)$ and $(s,y) \in [0,+\infty[\times\mathbb{R}$, we have

$$f^*_{\rho,g}(s,y) = \int_{\mathbb{R}}\int_0^{+\infty} g(r,x)\overline{\mathcal{F}_\alpha^{-1}\Big(\frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}\Big)(r,x)}dm_\alpha(r,x).$$

Applying Parseval's equality, we get

$$\begin{aligned} f^*_{\rho,g}(s,y) &= \int\int_{\Gamma^+} \mathcal{F}_\alpha(g)(\mu,\lambda)\frac{e^{-t(\mu^2+2\lambda^2)}\varphi_{\mu,-\lambda}(s,y)}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}d\gamma_\alpha(\mu,\lambda) \\ &= \mathcal{F}_\alpha^{-1}\Big(\mathcal{F}_\alpha(g)\frac{e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}\Big)(s,y), \end{aligned}$$

which implies that

$$\mathcal{F}_\alpha(f^*_{\rho,g})(\mu,\lambda) = \mathcal{F}_\alpha(g)(\mu,\lambda)\frac{e^{-t(\mu^2+2\lambda^2)}}{\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}}, \quad (4.5)$$

then, for all $g_1, g_2 \in L^2(dm_\alpha)$,

$$\Big\| f^*_{\rho,g_1} - f^*_{\rho,g_2}\Big\|_\nu^2 = \int\int_{\Gamma^+} \frac{(1+\mu^2+2\lambda^2)^\nu e^{-2t(\mu^2+2\lambda^2)}\Big|\mathcal{F}_\alpha(g_1-g_2)(\mu,\lambda)\Big|^2}{\Big(\rho(1+\mu^2+2\lambda^2)^\nu+e^{-2t(\mu^2+2\lambda^2)}\Big)^2}d\gamma_\alpha(\mu,\lambda).$$

Applying again the fact that $a^2+b^2 \geqslant 2ab$; $a,b \geqslant 0$; we obtain

$$\begin{aligned} \Big\| f^*_{\rho,g_1} - f^*_{\rho,g_2}\Big\|_\nu^2 &\leqslant \frac{1}{4\rho}\int\int_{\Gamma^+} |\mathcal{F}_\alpha(g_1-g_2)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \\ &= \frac{1}{4\rho}\|g_1-g_2\|^2_{2,m_\alpha}. \end{aligned}$$

□

Corollary 4.4. *Let* $\nu > \alpha + \frac{3}{2}$. *For every* $f \in \mathbf{H}^\alpha_\nu([0,+\infty[\times\mathbb{R})$ *and* $g = \mathcal{W}_{\alpha,t}(f)$, *we have*

$$\lim_{\rho\to o^+}\Big\| f^*_{\rho,g} - f\Big\|_\nu = 0$$

Moreover, $(f^*_{\rho,g})_{\rho>0}$ *converges uniformly to* f *as* $\rho \to 0$.

Proof. Let $f \in \mathbf{H}^{\alpha}_{\nu}([0,+\infty[\times\mathbb{R})$ and $g = \mathscr{W}_{\alpha,t}(f)$. From proposition 3.5; the function g belongs to $L^2(dm_\alpha)$. Applying the relations (3.9) and (4.5), we obtain

$$\mathscr{F}_\alpha(f^*_{\rho,g} - f)(\mu,\lambda) = \frac{-\rho(1+\mu^2+2\lambda^2)^\nu \mathscr{F}_\alpha(f)(\mu,\lambda)}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}}. \tag{4.6}$$

Consequently,

$$\left\| f^*_{\rho,g} - f \right\|^2_\nu = \int\int_{\Gamma+} \frac{\rho^2(1+\mu^2+2\lambda^2)^{2\nu}}{\left(\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}\right)^2} \times (1+\mu^2+2\lambda^2)^\nu \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^2 d\gamma_\alpha(\mu,\lambda).$$

Using the dominate convergence theorem and the fact that

$$\frac{\rho^2(1+\mu^2+2\lambda^2)^{3\nu}\left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^2}{\left(\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}\right)^2} \leqslant (1+\mu^2+2\lambda^2)^\nu \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^2,$$

and $f \in \mathbf{H}^{\alpha}_{\nu}([0,+\infty[\times\mathbb{R})$, we deduce that

$$\lim_{\rho \to o^+} \left\| f^*_{\rho,g} - f \right\|_\nu = 0.$$

From remark 3.1, the function $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha) \cap L^2(d\gamma_\alpha)$, then by inversion formula and relation (4.6), we get

$$(f^*_{\rho,g} - f)(r,x) = \int\int_{\Gamma^+} \frac{-\rho(1+\mu^2+2\lambda^2)^\nu \mathscr{F}_\alpha(f)(\mu,\lambda)}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}} \varphi_{\mu,-\lambda}(r,x) d\gamma_\alpha(\mu,\lambda).$$

So, for all $(r,x) \in [0,+\infty[\times\mathbb{R}$;

$$\left|(f^*_{\rho,g} - f)(r,x)\right| \leqslant \int\int_{\Gamma^+} \frac{\rho(1+\mu^2+2\lambda^2)^\nu \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}} d\gamma_\alpha(\mu,\lambda).$$

Again, by dominate convergence theorem and the fact that

$$\frac{\rho(1+\mu^2+2\lambda^2)^\nu \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|}{\rho(1+\mu^2+2\lambda^2)^\nu + e^{-2t(\mu^2+2\lambda^2)}} \leqslant \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|,$$

we deduce that

$$\sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} \left|(f^*_{\rho,g} - f)(r,x)\right| \longrightarrow 0, \ as\ \rho \longrightarrow 0^+.$$

□

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References

[1] L. E. Andersson, On the determination of a function from spherical averages. *SIAM. J. Math. Anal.*, **19** (1988), pp. 214-234.

[2] N. Aronszajn, Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, **68** (1948), pp. 337-404.

[3] C. Baccar, N. B. Hamadi and L. T. Rachdi, Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators. *Internat. J. Math. Math. Sci.*, **2006**, Article ID 86238, (2006), pp 1-26.

[4] A. Erdely and all, *Higher transcendental functions,* vol.I, Mc Graw-Hill Book Compagny, New York, (1953).

[5] A. Erdely and all, *Tables of integral transforms,* vol.II, Mc Graw-Hill Book Compagny, New York, (1954).

[6] J. A. Fawcett, Inversion of N-dimensional spherical means. *SIAM. J. Appl. Math.*, **45** (1985), pp 336-341.

[7] H. Fujiwara, T. Matsuura and S. Saitoh, Numerical real inversion formulas of the Laplace transform. Announcement.

[8] H. Helesten and L. E. Anderson, An inverse method for the processing of synthetic aperture radar data. *Inv. Prob.*, **3** (1987), pp 111-124.

[9] M. Herberthson, A numerical implementation of an inverse formula for CARABAS raw data. Internal Report D 30430-3.2, National Defense Research Institute, Linköping, Sweden, (1986).

[10] N. N. Lebedev, *Special functions and their applications.* Dover publications, Inc. New-York, (1972).

[11] T. Matsuura, S. Saitoh and D. D. Trong. Inversion formulas in heat conduction multidimensional spaces. *J. Inv. Ill-posed Problems*, **13** (2005), pp 479-493.

[12] M. M. Nessibi, L. T. Rachdi and K. Trimèche, Ranges and inversion formulas for spherical mean operator and its dual. *J. Math. Anal. Appl.*, **196** (1995), pp 861-884.

[13] L. T. Rachdi and K. Trimèche, Weyl transforms associated with the spherical mean operator. *Anal. Appl.*, **1** No. 2 (2003), pp 141-164.

[14] S. Saitoh, The Weierstrass transform and isometry in the heat equation. *Applicable Anal.*, **16** (1983), pp 1-6.

[15] S. Saitoh, Approximate real inversion formulas of the gaussian convolution. *Applicable Anal.*, **83**, No. 7 (2004), pp 727-733.

[16] S. Saitoh, Applications of Tikhonov regularization to inverse problems using reproducing kernels. *J. Phys.:Conf. Ser.*, **88** (2007), 012019.

[17] K. Trimèche, Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentiel singulier sur $(0, +\infty)$. *J. Math. Pures Appl.*, **60** (1981), pp 51-98.

[18] K. Trimèche, Inversion of the Lions translation operator using generalized wavelets. *Appl. Comput. Harmonic Anal.*, **4** (1997), pp 97-112.

[19] G. N. Watson, *A treatise on the theory of Bessel functions* 2nd ed. Cambridge Univ. Press. London/New-York, (1966).

[20] M. Yamada, T. Matsuura and S. Saitoh, Representations of inverse functions by the integral transform with the sign kernel. *Frac. Calc. Appl. Anal.*, **2** (2007), pp 161-168.