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TAUBERIAN THEOREMS FOR THE OPERATOR OF WEIGHTED MEANS

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Abstract

In this paper we start with the Hardy's well-known Tauberian theorem for Cesàro means formulated as follows, if the sequence $X = (x_n)_n$ satisfies $\lim C_1 X = L$ and $\Delta x_n = O(1/n)$ then $\lim X = L$. We give here some extensions to the cases when the Cesàro operator is replaced by the operator of weighted means $\overline{N}_q$ defined by Hardy and by the operator $C(\lambda)$. We show among other things that under some conditions if the sequence $X = (x_n)_n$ satisfies $\lim \overline{N}_q X = L_1$ and $\lim Q_n \Delta q_n x_n = L_2$ then $\lim X = L_1$. Similarly under other conditions it is also shown that $\lim X = L_1$ if X satisfies the conditions $\lim \overline{N}_q X = L_1$ and $\lim q_n^{-1} Q_n \Delta x_n = L_2$. Next we determine sequences μ for which the conditions $[C(\lambda) X]_n$ and $\mu_n \Delta x_n$ are convergent imply together that X is convergent. This result is an extension of Hardy's Tauberian theorem and the other results extend in a certain sense some Tauberian theorems given by Hardy and Littlewood.

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1 Introduction and preliminary results

In this paper we start from results on Hardy's *Tauberian* theorem for *Cesàro means*. This one was formulated as follows, if the sequence $X = (x_n)_n$ satisfies $\lim_{n\to\infty} C_1 X = L$ and $\Delta x_n = O(1/n)$ then $\lim_{n\to\infty} X = L$. It was shown by Fridy and Khan [2] that the hypothesis $\lim_{n\to\infty} C_1 X = L$ can be replaced by the weaker assumption of the *statistical limit* $st-\lim C_1 X = L$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |[C_1 X]_k - L| \ge \varepsilon\}| = 0.$$

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Here our aim is to show that the Hardy's *Tauberian* theorem for *Cesàro means* can be extended to the cases when C_1 is successively replaced by the operator of weighted means $\overline{N}_q$ defined in [3] and by $C(\lambda)$. In this way in Theorem 2 we show that under some conditions, if $X = (x_n)_n$ satisfies $\lim_{n\to\infty} \overline{N}_q X = L_1$ and $\lim_{n\to\infty} Q_n \Delta q_n x_n = L_2$ then

$$\lim_{n\to\infty} X = L_1. \tag{1.1}$$

Similarly in Theorem 6 we show that under some other conditions we have (1.1) for X satisfying

$$\lim_{n\to\infty} \overline{N}_q X = L_1 \text{ and } \lim_{n\to\infty} \frac{Q_n}{q_n} \Delta x_n = L_2.$$

The result stated in Proposition 8 is an extension of Hardy's Tauberian theorem, there we consider the case when C_1 is replaced by $C(\lambda)$ and we determine sequences μ for which the conditions $[C(\lambda) X]_n$ and $\mu_n \Delta x_n$ are convergent imply together that X is convergent.

Now recall some definitions and results that are useful in the following. For given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ of complex numbers we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m \tag{1.2}$$

where $X = (x_n)_{n\geq 1}$ is a complex sequence. The series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1, 2, \ldots \tag{1.3}$$

where $B = (b_n)_{n\geq 1}$ is a *one-column matrix and X the unknown one-column matrix.* The system (1.3) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. To simplify we will write $[A]_{nm} = a_{nm}$. By s we will denote the set of all complex sequences, and by c, c_0 and l_∞ we will denote the sets of all convergent, convergent to zero and bounded sequences respectively. For given subsets E and F of s we will denote (E, F) for the *set of all infinite matrices mapping E to F.* We will write $e = (1, 1, \ldots, 1, \ldots)$ and S_1 is the set of all infinite matrices $A = (a_{nm})_{n,m\geq 1}$ with $\|A\|_{S_1} = \sup_{n\geq 1} \sum_{m=1}^{\infty} |a_{nm}| < \infty$. It is well known that S_1 is a *Banach algebra with norm* $\|A\|_{S_1}$.

Recall the characterization of (c, c) that we will use in all that follows.

Lemma 1.1. $A = (a_{nm})_{n,m\geq 1} \in (c, c)$ *if and only if*

i) $A \in S_1$,

ii) $\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{nm} = l$ *for some* $l \in \mathbb{C}$

iii) $\lim_{n\to\infty} a_{nm} = l_m$ *for some* $l_m \in \mathbb{C}$ *and for all* $m \geq 1$.

A matrix transformation $A = (a_{nm})_{n,m\geq 1} \in (c, c)$ is said to be *regular* if $A = (a_{nm})_{n,m\geq 1} \in (c, c)$ and $x_n \to l$ $(n \to \infty)$ implies $A_n(x) = \sum_{m=1}^{\infty} a_{nm} x_m$ is convergent for all n and converges to the *same limit*. We will write $x_n \to l$ implies $A_n(x) \to l$ $(n \to \infty)$. Recall that A is *regular* if and only if A satisfies Lemma 1.1 i), $\lim_{n\to\infty} A_n(e) = 1$ and $\lim_{n\to\infty} a_{nm} = 0$ for all $m \geq 1$.

In the following we will use triangles. Recall that the infinite matrix T is a triangle if $T_{nm} = 0$ for $m > n$, for all n, m and $T_{nn} \neq 0$ for all n. Let £ be the set of all triangles T, the set £ with the product of matrices is a group.

Let $q=(q_n)_{n\geq 1}$ be a positive sequence, Q be the sequence defined by $Q_n=\sum_{m=1}^{n} q_m$ for all $n\geq 1$. The operator of weighted means $\overline{N}_q \in £$ is defined by

$$\left[\overline{N}_q\right]_{nm} = \begin{cases} \dfrac{q_m}{Q_n} & \text{for } m\leq n, \\ 0 & \text{otherwise.} \end{cases}$$

In all that follows we write $x_n=0$ for any term of sequence with negative subscript. For given $\alpha\in s$ we write $D_\alpha=(\alpha_n\delta_{nm})_{n,m\geq 1}$ for the diagonal matrix, (where $\delta_{nn}=1$ for all n and $\delta_{nm}=0$ for $n\neq m$). We write Σ for the infinite matrix defined by $\Sigma_{nm}=1$ for $m\leq n$ and $\Sigma_{nm}=0$ otherwise. By Δ we denote the matrix with $\Delta_{nn}=1$, $\Delta_{n,n-1}=-1$ for all n and $\Delta_{nm}=0$ otherwise. It can be shown that $\Delta\Sigma=\Sigma\Delta=I$ in the set $£$. With these notations it can easily be seen that $\overline{N}_q=D_{1/Q}\Sigma D_q$.

We will use the matrix transformation $C(\lambda)$ where $\lambda\in s$ and $\lambda_n\neq 0$ for all n, see [1], recall that $C(\lambda)\in £$ and $[C(\lambda)]_{nm}=1/\lambda_n$ for $m\leq n$, $n=1,\ 2,\ldots.$ For $\lambda=e$ we have $C(\lambda)=\Sigma$. In the set $£$ we have

$$[C(\lambda)]^{-1}=\left(D_{1/\lambda}\Sigma\right)^{-1}=\Delta D_\lambda.$$

So putting $\Delta(\lambda)=\Delta D_\lambda$ we conclude $[C(\lambda)]^{-1}=\Delta(\lambda)$. For $\lambda=(n)_{n\geq 1}$, $C(\lambda)=C_1$ is the *Cesàro operator*.

2 Tauberian theorems

In this section we will give two versions of Tauberian theorems concerning the operator of weighted means $\overline{N}_q$. Then we will deal with the operator $C(\lambda)$.

2.1 First version of Tauberian theorem for $\overline{N}_q$

State the next result.

Theorem 2.1. *(i) The following statements are equivalent.*

(a)

$$\frac{Q}{q}\in l_\infty,$$

(b) for any given sequence $(x_n)_{n\geq 1}$

$$\lim_{n\to\infty}\frac{q_1x_1+\ldots+q_nx_n}{Q_n}=L_1 \text{ if and only if } \lim_{n\to\infty}x_n=L_1$$

for some $L_1\in\mathbb{C}$.

(ii) Assume

$$\lim_{n\to\infty}\frac{1}{nq_n}\sum_{m=1}^{n}\frac{m}{Q_m} \quad = \quad L \tag{2.1}$$

and

$$\lim_{n\to\infty}\frac{Q_n}{nq_n}=L'\neq 0 \tag{2.2}$$

for some scalars L and L'. Then for any given sequence $(x_n)_{n\geq 1}$ the conditions

$$\lim_{n\to\infty}\frac{q_1x_1+\ldots+q_nx_n}{Q_n}=L_1 \text{ and } \lim_{n\to\infty}Q_n\left(q_nx_n-q_{n-1}x_{n-1}\right)=L_2$$

for some $L_1, L_2\in\mathbb{C}$ imply together $\lim_{n\to\infty}x_n=L_1$.

Proof. (i) In the set £ we have $\overline{N}_q^{-1}=\left(D_{1/Q}\Sigma D_q\right)^{-1}=D_{1/q}\Delta D_Q$ that is $\left[\overline{N}_q^{-1}\right]_{n,n-1}=-Q_{n-1}/q_n$, $\left[\overline{N}_q^{-1}\right]_{nn}=Q_n/q_n$ for all $n\geq 1$ (with the convention $Q_0=0$) and $\left[\overline{N}_q^{-1}\right]_{nn}=0$ otherwise. Since Q is increasing and $Q/q\in l_\infty$ we have

$$\left\|\overline{N}_q^{-1}\right\|_{S_1}=\sup_n\left(\frac{Q_n+Q_{n-1}}{q_n}\right)\leq 2\sup_n\frac{Q_n}{q_n}<\infty.$$

Then $\lim_{n\to\infty}\left(Q_n-Q_{n-1}\right)/q_n=1$ and we conclude that $\overline{N}_q^{-1}$ is regular. This shows (a) holds if and only if $\overline{N}_q^{-1}$ is regular. So (a) means that for any $Y=(y_n)_{n\geq 1}$ the condition $y_n=\left[\overline{N}_qX\right]_n\to L_1$ implies

$$x_n=\left[\overline{N}_q^{-1}Y\right]_n\to L_1\ \ (n\to\infty).$$

Now $\overline{N}_q$ is regular so it is trivial that

$$x_n\to L_1 \text{ implies } y_n\to L_1\ \ (n\to\infty).$$

This completes the proof of (i).

(ii) Let $X=(x_n)_{n\geq 1}\in s$ and put $Y=(y_n)_{n\geq 1}=\overline{N}_qX$. Writing $Z=(z_n)_{n\geq 1}=\left(Q_n\left(q_nx_n-q_{n-1}x_{n-1}\right)\right)_{n\geq 1}$ we easily see that

$$Z=D_Q\Delta D_qX. \tag{2.3}$$

In the set £ we have $\left(D_Q\Delta D_q\right)^{-1}=D_{1/q}\Sigma D_{1/Q}$ and by (2.3) we get

$$X=\left(D_Q\Delta D_q\right)^{-1}Z=D_{1/q}\Sigma D_{1/Q}Z.$$

Then

$$\begin{aligned} Y &= \overline{N}_qX=\overline{N}_qD_{1/q}\Sigma D_{1/Q}Z \\ &= D_{1/Q}\Sigma D_qD_{1/q}\Sigma D_{1/Q}Z=D_{1/Q}\Sigma^2D_{1/Q}Z \end{aligned}$$

and the infinite matrix $\Sigma^2\in$ £ is the triangle defined by $\left[\Sigma^2\right]_{nm}=n+1-m$ for $m\leq n$ and $\left[\Sigma^2\right]_{nm}=0$ otherwise. So we easily get

$$y_n=\frac{1}{Q_n}\sum_{m=1}^{n}\frac{n+1-m}{Q_m}z_m=\frac{n+1}{Q_n}\sum_{m=1}^{n}\frac{z_m}{Q_m}-\frac{1}{Q_n}\sum_{m=1}^{n}\frac{m}{Q_m}z_m.$$

Since

$$x_n = \left[D_{1/q}\Sigma D_{1/Q}Z\right]_n = \frac{1}{q_n}\sum_{m=1}^{n}\frac{z_m}{Q_m}$$

we get

$$y_n = \frac{n+1}{Q_n}q_n x_n - \frac{1}{Q_n}\sum_{m=1}^{n}\frac{m}{Q_m}z_m.$$

Consider now the triangle

$$\widehat{Q} = \begin{pmatrix} \cdot & & & \\ \cdot & \cdot & 0 & \\ \cdot & \frac{1}{(n+1)q_n}\frac{m}{Q_m} & \cdot & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Condition (2.2) implies $1/nq_n \sim L'/Q_n$ $(n \to \infty)$ and since Q_n is increasing we have

$$0 < \lim_{n\to\infty} Q_n \leq \infty$$

and $(1/nq_n)_{n\geq 1} \in c$. So $\left[\widehat{Q}\right]_{nm}$ tends to a limit as n tends to infinity for all m. This and condition (2.1) imply $\widehat{Q} \in (c,c)$ and since $Z \in c$ we have

$$\frac{1}{(n+1)\,q_n}\sum_{m=1}^{n}\frac{m}{Q_m}z_m \to l \ (n\to\infty) \text{ for some } l \in \mathbb{C}.$$

Using (2.2) we deduce that if $y_n \to L_1$ and $z_n \to L_2$ $(n\to\infty)$ then

$$x_n = \frac{Q_n}{(n+1)\,q_n}y_n + \frac{1}{(n+1)\,q_n}\sum_{m=1}^{n}\frac{m}{Q_m}z_m \to L'L_1 + l$$

and $X \in c$. Now since $\overline{N}_q$ is regular and $y_n = \left[\overline{N}_q X\right]_n$ we have

$$y_n \to L_1 = L'L_1 + l \ (n\to\infty).$$

We conclude $x_n \to L_1$ $(n\to\infty)$. □

Concerning Theorem 2.1 (i) we have the next result.

Corollary 2.2. *Let $X = (x_n)_{n\geq 1}$ be any given sequence. The condition*

$$\left[\overline{N}_q X\right]_n \to L \text{ implies } x_n \to L \ (n\to\infty) \tag{2.4}$$

for some $L \in \mathbb{C}$ implies there are $\gamma > 1$ and $K > 0$ such that

$$q_n \geq K\gamma^n \text{ for all } n.$$

Proof. Condition (2.4) implies $\overline{N}_q^{-1}$ is regular, that is

$$\frac{Q_n + Q_{n-1}}{q_n} = O(1) \ (n\to\infty).$$

and $Q/q \in l_\infty$. Then $q \in \widehat{C}_1$ where $\widehat{C}_1$ is the set of all sequences X with $\left(\left(\sum_{k=1}^{n} x_k\right)/x_n\right)_{n\geq 1} \in l_\infty$, (cf. [1]). We conclude by [1, Proposition 2.1, pp. 1786]. □

As a direct consequence of Theorem 2.1 (ii) we get

Corollary 2.3. *Let* $\alpha \geq 0$ *and let* $(x_n)_{n\geq 1}$ *be a sequence with*

$$\frac{x_1 + 2^\alpha x_2 + \ldots + n^\alpha x_n}{\sum_{m=1}^n m^\alpha} \to L_1 \text{ and } \left(\sum_{m=1}^n m^\alpha\right)\left(n^\alpha x_n - (n-1)^\alpha x_{n-1}\right) \to L_2$$

for some $L_1, L_2 \in \mathbb{C}$. *Then* $x_n \to L_1$ $(n \to \infty)$.

Proof. If $\alpha = 0$ (2.1) and (2.2) are trivially satisfied. Now put $q_n = n^\alpha$ with $\alpha > 0$ and $\alpha \neq 1$. We obtain

$$\frac{1}{n^{\alpha+1}} \int_0^n x^\alpha dx \leq \frac{Q_n}{nq_n} = \frac{\sum_{m=1}^n m^\alpha}{n^{\alpha+1}} \leq \frac{1}{n^{\alpha+1}} \int_1^{n+1} x^\alpha dx;$$

and since the sequences $\left(1/n^{\alpha+1}\right) \int_0^n x^\alpha dx$ and $\left(1/n^{\alpha+1}\right) \int_1^{n+1} x^\alpha dx$ tends to the same limit $1/(\alpha+1)$ as n tends to infinity we conclude $\lim_{n\to\infty} Q_n/nq_n \to 1/(\alpha+1)$ and (2.2) holds. Now we need to verify (2.1). For this note that for every $m \geq 2$

$$\frac{m}{Q_m} \leq \frac{m}{\int_0^m x^\alpha dx} = \frac{\alpha+1}{m^\alpha}.$$

Then

$$\begin{aligned}\sum_{m=1}^n \frac{m}{Q_m} &\leq 1 + (\alpha+1) \sum_{m=2}^n \frac{1}{m^\alpha} \\ &\leq 1 + (\alpha+1) \int_1^n \frac{dx}{x^\alpha} \\ &\leq 1 + \frac{\alpha+1}{1-\alpha}\left(n^{1-\alpha} - 1\right).\end{aligned}$$

Thus

$$\frac{1}{n^{\alpha+1}} \sum_{m=1}^n \frac{m}{Q_m} \leq \frac{1+\alpha}{1-\alpha}\left(\frac{1}{n^{2\alpha}} - \frac{1}{n^{\alpha+1}}\right) + \frac{1}{n^{\alpha+1}}.$$

and

$$\frac{1}{n^{\alpha+1}} \sum_{m=1}^n \frac{m}{Q_m} \to 0 \ (n \to \infty).$$

We conclude applying Theorem 2.1.

For $\alpha = 1$ we get

$$\frac{1}{nq_n} \sum_{m=1}^n \frac{m}{Q_m} = \frac{1}{n^2} \sum_{m=1}^n \frac{2}{(m+1)} \leq \frac{2}{n^2} \int_0^n \frac{dx}{x+1} = \frac{2}{n^2} \ln(n+1)$$

and

$$\frac{1}{nq_n} \sum_{m=1}^n \frac{m}{Q_m} \to 0 \ (n \to \infty).$$

Since (2.2) trivially holds with $L' = 1/2$ we can apply Theorem 2.1 and conclude that $x_n \to L_1$ $(n \to \infty)$. □

We immediatly deduce from the previous proof the following.

Corollary 2.4. *Let $(x_n)_{n\geq 1}$ be any sequence. If*

$$\frac{x_1+2x_2+\ldots+nx_n}{n^2} \to L_1 \text{ and } n^2\left(nx_n-(n-1)x_{n-1}\right) \to L_2$$

then $x_n \to 2L_1$ $(n\to\infty)$.

2.2 Second version of Tauberian theorem for matrix of weighted means

In this subsection we consider a second statement of Tauberian theorem where conditions (2.1) and (2.2) in Theorem 2.1 are replaced by the convergence of $(1/Q_n)\sum_{m=2}^n q_m Q_{m-1}/Q_m$ and the condition on $Q_n(q_n x_n - q_{n-1}x_{n-1})$ is replaced by a similar condition on another sequence defined by $Q_n(x_n-x_{n-1})/q_n$.

Theorem 2.5. *Assume*

$$\lim_{n\to\infty}\frac{1}{Q_n}\sum_{m=2}^{n} q_m\frac{Q_{m-1}}{Q_m}=L \tag{2.5}$$

for some scalar L. For any given sequence $(x_n)_{n\geq 1}$ the conditions

$$\lim_{n\to\infty}\frac{q_1x_1+\ldots+q_nx_n}{Q_n}=L_1 \text{ and } \lim_{n\to\infty}\frac{Q_n}{q_n}(x_n-x_{n-1})=L_2 \tag{2.6}$$

for some $L_1, L_2\in\mathbb{C}$ imply $\lim_{n\to\infty}x_n=L_1$.

Proof. Put

$$Y=(y_n)_{n\geq 1}=\overline{N}_q X \tag{2.7}$$

and $Z=D_{Q/q}\Delta X$. Then

$$X=\Sigma D_{q/Q}Z \tag{2.8}$$

and

$$Y=\overline{N}_q\Sigma D_{q/Q}Z=D_{1/Q}\Sigma D_q\Sigma D_{q/Q}Z.$$

We have

$$[\Sigma D_q\Sigma]_{nm}=\begin{cases}\sum_{i=m}^n q_i & \text{for } m\leq n,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}
y_n &= \frac{1}{Q_n}\sum_{m=1}^{n}\left(\sum_{i=m}^{n}q_i\right)\frac{q_m}{Q_m}z_m\\
&= \frac{1}{Q_n}\sum_{m=1}^{n}\left(Q_n-Q_{m-1}\right)\frac{q_m}{Q_m}z_m\\
&= \sum_{m=1}^{n}\frac{q_m}{Q_m}z_m-\frac{1}{Q_n}\sum_{m=1}^{n}\frac{Q_{m-1}}{Q_m}q_mz_m.
\end{aligned}$$

Using (2.8) we deduce

$$y_n=x_n-\frac{1}{Q_n}\sum_{m=1}^{n}\frac{Q_{m-1}}{Q_m}q_mz_m$$

and

$$x_n = y_n + \frac{1}{Q_n}\sum_{m=1}^{n}\frac{Q_{m-1}}{Q_m}q_m z_m.$$

Now consider the matrix $\widetilde{Q}$ with $\left[\widetilde{Q}\right]_{nm} = q_m Q_{m-1}/Q_n Q_m$ for $2 \le m \le n$ and $\left[\widetilde{Q}\right]_{nm} = 0$ otherwise. Since Q is increasing we have $1/Q \in c$ and $\left[\widetilde{Q}\right]_{nm}$ tends to a limit as n tends to infinity. This and conditions (2.5) imply together $\widetilde{Q} \in (c,c)$. Consider now the sequence w defined by

$$w_n = \frac{1}{Q_n}\sum_{m=1}^{n}\frac{Q_{m-1}}{Q_m}q_m z_m.$$

The conditions given in (2.6) mean that $y_n \to L_1$ and $z_n \to L_2$ $(n \to \infty)$ and since $\widetilde{Q} \in (c,c)$ we have

$$x_n = y_n + w_n = y_n + \left[\widetilde{Q}Z\right]_n \to L_1 + l \text{ for some } l \in \mathbb{C}.$$

To complete the proof we need to show that $l = 0$. For this it is enough to see that since $\overline{N}_q$ is regular if $x_n \to L_1 + l$ then

$$y_n = \left[\overline{N}_q X\right]_n \to L_1 + l = L_1 \ (n \to \infty)$$

and so $x_n \to L_1$ $(n \to \infty)$. This concludes the proof. □

This result leads to the next corollary.

Corollary 2.6. *Let* $(x_n)_{n\ge 1}$ *be a sequence with*

$$\lim_{n\to\infty}\frac{1}{\ln n}\left(x_1 + \frac{1}{2}x_2 + \dots + \frac{1}{n}x_n\right) = L_1 \text{ and } \lim_{n\to\infty} n\ln n\,(x_n - x_{n-1}) = L_2.$$

Then $\lim_{n\to\infty} x_n = L_1$.

Proof. We have $q_n = 1/n$ for all n and $Q_n = \sum_{m=1}^{n} 1/m$ and

$$u_n = \frac{1}{Q_n}\sum_{m=2}^{n} q_m \frac{Q_{m-1}}{Q_m} = \frac{1}{Q_n}\sum_{m=2}^{n}\frac{1}{m} - \sigma_n$$

with

$$\sigma_n = \frac{1}{Q_n}\sum_{m=2}^{n}\frac{1}{m^2 Q_m}.$$

Since Q_n tends to infinity as n tends to infinity and $\sigma_n \le (1/Q_n)\sum_{m=2}^{n} 1/m^2$ we have σ_n tends to zero. Then u_n tends to 1 as n tends to infinity and condition (2.5) in Theorem 2.5 is verified. Finally since $Q_n \sim \ln n$ we have $Q_n/q_n \sim n\ln n$ $(n \to \infty)$ and we conclude by Theorem 2.5.

Remark 2.7. We can see that Theorem 2.1 and Theorem 2.5 are not embedded the one into the other. Indeed, consider the case when $q_n = n$ for all n. Then the sequence $X = e$ verifies Theorem 2.5 since (2.5) is satisfied with $L = 1$ and $(Q_n/q_n)(x_n - x_{n-1}) = 0$ for all n but

$$Q_n\left(q_n x_n - q_{n-1}x_{n-1}\right) \sim n^2/2 \ (n \to \infty)$$

so $Q_n(q_n x_n - q_{n-1}x_{n-1}) \to \infty$ $(n \to \infty)$ and Theorem 2.1 cannot be verified.

Furthermore in the case when $q_n = 1/n$ we have seen in Corollary 2.6 that condition (2.5) in Theorem 2.5 is verified but (2.2) in Theorem 2.1 is not satisfied.

2.3 Case of the operator $C(\lambda)$

In this subsection we will consider the case when $\overline{N}_q$ is replaced by $C(\lambda)$. We will obtain some results that extend some of those given in the previous subsections.

Reasoning as in Theorem 2.1 and Theorem 2.5 we can state another result where $\overline{N}_q$ is replaced by $C(\lambda)$. We will see that in the case $\lambda = \mu$ the sequence λ plays the role of Q with $q = e$. In the following we will write U^+ for the set of all sequences $(x_n)_{n\geq 1}$ with $x_n > 0$ for all n.

Proposition 2.8. *Let λ, $\mu \in U^+$ and assume*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \frac{m}{\mu_m} = L \tag{2.9}$$

and

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = L'$$

for some scalars L and L'.

Then for any given sequence $(x_n)_{n\geq 1} \in s$ conditions

$$\frac{x_1 + \ldots + x_n}{\lambda_n} \to l \text{ and } \mu_n (x_n - x_{n-1}) \to l' \ \ (n \to \infty) \tag{2.10}$$

for some l, $l' \in \mathbb{C}$ imply together $(x_n)_{n\geq 1}$ is convergent and

$$x_n \to L'l \ \ (n \to \infty).$$

Proof. Put $y_n = (x_1 + \ldots + x_n)/\lambda_n$ and $z_n = \mu_n(x_n - x_{n-1})$. We have $Y = C(\lambda)X$ and $Z = D_\mu \Delta X$ and then $Y = C(\lambda)\Sigma D_{1/\mu}Z$. Since $C(\lambda) = D_{1/\lambda}\Sigma$ and $(D_\mu\Delta)^{-1} = \Sigma D_{1/\mu}$ we get $Y = D_{1/\lambda}\Sigma^2 D_{1/\mu}Z$ and $X = \Sigma D_{1/\mu}Z$. As we have seen in the proof of Theorem 2.5 the explicit calculation of $D_{1/\lambda}\Sigma^2 D_{1/\mu}$ permits us to obtain

$$y_n = \frac{1}{\lambda_n}\sum_{m=1}^{n} \frac{n-m+1}{\mu_m} z_m.$$

Similarly we have

$$x_n = \left[\Sigma D_{1/\mu} Z\right]_n = \sum_{m=1}^{n} \frac{z_m}{\mu_m}.$$

So we successively get

$$y_n = \frac{n+1}{\lambda_n} x_n - \frac{1}{\lambda_n}\sum_{m=1}^{n} \frac{m}{\mu_m} z_m, \tag{2.11}$$

and

$$x_n = \frac{\lambda_n}{n+1} y_n + \frac{1}{n+1}\sum_{m=1}^{n} \frac{m}{\mu_m} z_m. \tag{2.12}$$

Now let Λ be the triangle defined by $\Lambda_{nm} = m/((n+1)\mu_m)$ for $2 \leq m \leq n$, $\Lambda_{nm} = 0$ for $m = 1$ or $m > n$ and for all n, m. By (2.9) we have $\Lambda \in (c,c)$. Now conditions given in (2.10) mean that $y_n \to l$ and $z_n \to l'$ $(n \to \infty)$ and since $\Lambda \in (c,c)$ we deduce that

$$x_n = \frac{\lambda_n}{n+1} y_n + [\Lambda Z]_n \to L'l + \chi \ (n \to \infty) \text{ for some } \chi \in \mathbb{C}.$$

Now we are led to deal with the cases $L' = 0$ and $L' \neq 0$.

1- Assume $L' = 0$. Here $x_n \to L'l + \chi = \chi$ and

$$\frac{x_1 + \ldots + x_n}{n} \to \chi.$$

Since $n/\lambda_n \to \infty$ and

$$\frac{x_1 + \ldots + x_n}{\lambda_n} = \frac{x_1 + \ldots + x_n}{n} \frac{n}{\lambda_n} \to l \ (n \to \infty)$$

we conclude that $\chi = 0$ and $x_n \to 0 = L'l$ $(n \to \infty)$.

2- Case $L' \neq 0$. Here since $x_n \to L'l + \chi$ $(n \to \infty)$ we also have

$$\frac{x_1 + \ldots + x_n}{n} \to L'l + \chi \ (n \to \infty).$$

Then

$$\frac{x_1 + \ldots + x_n}{\lambda_n} = \frac{x_1 + \ldots + x_n}{n} \frac{n}{\lambda_n} \to \frac{L'l + \chi}{L'} = l \ (n \to \infty)$$

So $\chi = 0$ and $x_n \to L'l$ $(n \to \infty)$. This concludes the proof. □

We deduce the following

Corollary 2.9. *Let $X = (x_n)_{n \geq 1}$ be a sequence with*

$$(x_1 + \ldots + x_n)/n \to l \text{ and } n(x_n - x_{n-1}) \to l' \ (n \to \infty)$$

for some $l, l' \in \mathbb{C}$. Then

(i) $x_n \to l$ $(n \to \infty)$,

(ii) $l' = 0$.

Proof. Condition (2.9) is trivially satisfied and $L = L' = 1$. Since $\Lambda/L = \Lambda$ defined in the proof of Proposition 2.8 is regular we have here $l' = \chi$. As we have just seen we successively get $x_n \to l + \chi$ $(n \to \infty)$, $l + \chi = l + l' = l$ and $l' = 0$. □

Remark 2.10. It can be seen that Proposition 2.8 is an extension of Hardy's Tauberian theorem. For this show there is $\mu \in U^+$ with $(n/\mu_n)_n \in U^+ \backslash l_\infty$ such that (2.9) holds. Take for instance $\mu_n = 2^i/i$ when $n = 2^i$, $i \geq 1$ and $\mu_n = n^3$ otherwise. Let n be a given integer and put $I_n = \{2^i : 2^i \leq n\}$. Using the notation $\overline{I'_n} = \overline{I_n} \bigcap [1,n]$ we successively get

$$\begin{aligned} s_n &= \frac{1}{n} \sum_{m=1}^{n} \frac{m}{\mu_m} = \frac{1}{n} \sum_{m \in I_n} \frac{m}{\mu_m} + \frac{1}{n} \sum_{m \in \overline{I'_n}} \frac{m}{\mu_m} \\ &\leq \frac{1}{n} \sum_{k \in \{i : 2^i \leq n\}} k + \frac{1}{n} \sum_{m \in \overline{I'_n}} \frac{1}{m^2}. \end{aligned}$$

Putting $N = \max\left\{i : 2^i \leq n\right\}$ and $S = \sum_{m=1}^{\infty} 1/m^2$ we deduce

$$s_n \leq \frac{1}{n}\frac{N(N+1)}{2} + \frac{S}{n}.$$

Since $2^N \leq n$ we get

$$s_n \leq \frac{1}{2^{N+1}} N(N+1) + \frac{S}{n}.$$

Finally from the definition of N and the inequality $N \leq \ln n/\ln 2$ we obtain $N = E(\ln n/\ln 2)$. So N tends to infinity as n tends to infinity and s_n tends to zero.

Since we have

$$n(x_n - x_{n-1}) = \frac{n}{\mu_n}\mu_n(x_n - x_{n-1})$$

and $(n/\mu_n)_{n\geq 1} \notin l_\infty$ the condition $\mu_n(x_n - x_{n-1}) \to l'$ $(n \to \infty)$ does not imply $n(x_n - x_{n-1}) = O(1)$ $(n \to \infty)$ and we have shown that Proposition 2.8 is an extension of Hardy's Tauberian theorem.

More precisely we can state another result when $C(\lambda)X \in c_0$. For this we need to recall the next well-known result

Lemma 2.11. *$A \in (c_0, c_0)$ if and only if $A \in S_1$ and $\lim_{n\to\infty} a_{nm} = 0$ for all m.*

We can state the following.

Proposition 2.12. *Let λ, $\mu \in U^+$and assume*

$$\sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{m}{\mu_m}\right) < \infty \tag{2.13}$$

and

$$\sup_n \frac{\lambda_n}{n} < \infty. \tag{2.14}$$

For any given sequence $(x_n)_{n\geq 1}$ the conditions $(x_1 + \ldots + x_n)/\lambda_n \to 0$ and $\mu_n(x_n - x_{n-1}) \to 0$ imply $x_n \to 0$ $(n \to \infty)$.

Proof. We will use the same notations that in the proof of Proposition 2.8. Here from (2.13) we have $\Lambda \in S_1$ and trivially $\lim_{n\to\infty}\Lambda_{nm} = 0$ for all $m \geq 1$. Then $\Lambda \in (c_0, c_0)$ and

$$\frac{1}{n+1}\sum_{m=1}^{n}\frac{m}{\mu_m} z_m \to 0 \ \ (n \to \infty) \text{ for all } Z \in c_0.$$

By (2.14) we also have

$$\frac{\lambda_n}{n+1} y_n = \frac{\lambda_n}{n}\frac{n}{n+1} y_n = O(1)\,o(1) = o(1) \ \ (n \to \infty)$$

and from the identity (2.12) given in the previous proof we conclude $x_n \to 0$ $(n \to \infty)$. □

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