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# GROWTH AND OSCILLATION THEORY OF NON-HOMOGENEOUS COMPLEX DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

**BENHARRAT BELAÏDI***
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem
B. P 227 Mostaganem, Algeria

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**Abstract**

In this paper, we investigate the growth and the complex oscillation theory of the linear differential equation $f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f^{'}+A_0f=F$, where $A_0,A_1,...,$ $A_{k-1},F\not\equiv 0$ are entire functions. We also investigate the relation between the solutions of a pair non-homogeneous linear differential equations. We improve some results due to the author, S. Abbas and Z. X. Chen, S. A. Gao.

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## 1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [7]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f$, $\rho(f)$ to denote the order of growth of $f$.

For $k\geq 2$ we consider the linear differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f^{'}+A_0f=F, \tag{1.1}$$

where $A_0,A_1,...,A_{k-1},F\not\equiv 0$ are entire functions of finite order.

It is well-known that all solutions of equation (1.1) are entire functions and if at least one coefficient $A_s(z)$ is transcendental, then at least some of the solutions are of infinite order.

*E-mail address: belaidi@univ-mosta.dz

On the other hand, there exist equations of this form that possess one or more solutions of finite order. For example $f(z) = e^z$ satisfies $f''' - e^z f'' - e^{-z} f' + e^z f = e^z - 1$.

Recently the complex oscillation theory of the complex differential equations has been investigated actively [1, 2, 3, 4]. In [4], Chen and Gao have investigated the complex oscillation of (1.1) and have obtained the following results:

**Theorem 1.1.** ( [4], *p. 455*) *Suppose that* $k \geq 2$ *is a natural number and that* $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ *are finite order entire functions, where there exists an* $A_s$ $(0 \leq s \leq k-1)$ *such that for real constants* $\alpha > 0$, $\beta > 0$, *we have* $\rho(A_j) < \beta$ $(j \neq s)$ *and* $\rho(F) \geq \beta$. *Suppose that for any given* $\varepsilon > 0$, *there exist two finite collections of real numbers* $\{\varphi_m\}$ *and* $\{\theta_m\}$ *which satisfy*

$$\varphi_1 < \theta_1 < \varphi_2 < \theta_2 < ... < \varphi_n < \theta_n < \varphi_{n+1} = \varphi_1 + 2\pi \tag{1.2}$$

*and*

$$\sum_{m=1}^{n} (\varphi_{m+1} - \theta_m) < \varepsilon, \tag{1.3}$$

*such that*

$$|A_s(z)| \geq \exp\left\{(1 + o(1))\,\alpha\,|z|^{\beta}\right\} \tag{1.4}$$

*as* $z \to \infty$ *in* $\varphi_m \leq \arg(z) \leq \theta_m$ $(m = 1, ..., n)$. *Then all solutions of (1.1) satisfy*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty \tag{1.5}$$

*with some possible finite order solutions. All of the possible finite order solutions have the same order of growth* $\rho$ $(0 \leq \rho < \infty)$. *If there exist two finite order solutions* $f_0$, $f_1$ $(f_0 \not\equiv f_1)$ *of (1.1), then* $f_0 - f_1$ *is a polynomial with* $\deg(f_0 - f_1) \leq s - 1$, *and* $f_0$ *satisfies*

$$\rho = \rho(f_0) \leq \max\left\{\rho(A_s), \rho(F), \overline{\lambda}(f_0)\right\}. \tag{1.6}$$

*If* $\rho(A_s) \neq \rho(F)$, $\overline{\lambda}(f_0) < \rho$, *then* $\rho(f_0) = \max\{\rho(A_s), \rho(F)\}$. *Furthermore, if among* $A_{s-1}, ..., A_0$, *there exist and only exist* $A_{m_1}, ..., A_{m_d}$ $(s-1 \geq m_1 > m_2 > ... > m_d \geq 0)$ *being transcendental,* $\rho\left(A_{m_j}\right)$ $(j = 1, ..., d)$ *are unequal to each other or* $d = 1$, *and if* (1) $m_d = 0$ *or* (2) $m_d > 0$ *and polynomials* $A_{m_d-1}, A_{m_d-2}, ..., A_0$ *satisfy that* $\deg(A_j) - j$ $(j = m_d - 1, m_d - 2, ..., 0)$ *are unequal to each other, or* $m_d = 1$ *and* $A_0 \not\equiv 0$, *then all solutions of (1.1) satisfy (1.5) with at most one possible finite order solution.*

**Theorem 1.2.** ( [4], *p. 455*) *Suppose that* $A_0, ..., A_s, ..., A_{k-1}$, $\alpha$, $\beta$, $\{\varphi_m\}$, $\{\theta_m\}$ *and* $k$ *satisfy the hypotheses of Theorem 1.1, and that* $F \not\equiv 0$ *is an entire function with* $\rho(F) < \beta$. *Then all solutions of (1.1) satisfy (1.5) with some possible polynomial solutions of degree* $\leq s - 1$.

The purpose of this paper is to improve the above results by expressing the growth condition of the coefficient $A_s(z)$ more explicitly without making use of $o(1)$. We will prove the following theorems:

**Theorem 1.3.** *Suppose that $k \geq 2$ is a natural number and that $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ are finite order entire functions, where there exists an $A_s$ $(0 \leq s \leq k-1)$ such that for real constants $\alpha$, $\delta$ with $\alpha > 0$, $\delta > 0$, we have*

$$\max\left\{\rho(A_j) : j = 0, 1, ..., s-1, s+1, ..., k-1, \rho(F)\right\} < \delta \tag{1.7}$$

*and either*
$(i)$ $\max\left\{\rho(A_j) : j = 1, ..., s-1, \rho(F)\right\} < \rho(A_0)$ *or*
$(ii)$ $\max\{\rho(A_j) : j = 0, 1, ..., s-1\} < \rho(F)$.

*Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\varphi_m\}$ and $\{\theta_m\}$ which satisfy (1.2) and (1.3) such that*

$$|A_s(z)| \geq \exp\left\{\alpha |z|^{\delta}\right\} \tag{1.8}$$

*as $z \to \infty$ in $\varphi_m \leq \arg(z) \leq \theta_m$ $(m = 1, ..., n)$.*
$(a)$ *Then every solution $f$ of (1.1), satisfies*

$$\lambda(f) = \overline{\lambda}(f) = \rho(f) = \infty. \tag{1.9}$$

$(b)$ *If $\varphi$ is a finite order entire function, then*

$$\lambda(f-\varphi) = \overline{\lambda}(f-\varphi) = \rho(f) = \infty. \tag{1.10}$$

From Theorem 1.3, we obtain the following corollary:

**Corollary 1.4.** *Suppose that $A_0, ..., A_{k-1}, F \not\equiv 0, \alpha, \delta, \varepsilon, \{\phi_m\}, \{\theta_m\}$ and $k$ satisfy the hypotheses of Theorem 1.3. Then every solution $f$ of* (1.1) *has infinitely many fixed points and*

$$\lambda(f-z) = \overline{\lambda}(f-z) = \rho(f) = \infty. \tag{1.11}$$

**Theorem 1.5.** *Suppose that $k \geq 2$ is a natural number and that $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ are finite order entire functions, where there exists an $A_s$ $(0 \leq s \leq k-1)$ such that for real constants $\alpha$, $\delta$ with $\alpha > 0$, $\delta > 0$, we have*

$$\max\left\{\rho(A_j) : j = 0, 1, ..., s-1, s+1, ..., k-1\right\} < \delta \tag{1.12}$$

*and*

$$\max\left\{\rho(A_j) : j = 1, ..., s-1\right\} < \rho(A_0), \tag{1.13}$$

*let $F \not\equiv 0$ be a finite order entire function such that $\rho(F) \geq \delta$. Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\phi_m\}$ and $\{\theta_m\}$ that satisfy (1.2),(1.3) such that* (1.8) *holds as $z \to \infty$ in $\phi_m \leq \arg z \leq \theta_m$ $(m = 1, ..., n)$.*

(a) *Then all solutions of (1.1) satisfy (1.9), with at most one finite order solution $f_0$.*

(b) *If there exists a finite order solution $f_0$ in case $(a)$, then $f_0$ satisfies*

$$\rho(f_0) \leq \max\left\{\rho(F), \rho(A_s), \overline{\lambda}(f_0)\right\}. \tag{1.14}$$

*If $\overline{\lambda}(f_0) < \rho(f_0)$, $\rho(F) \neq \rho(A_s)$, then $\rho(f_0) = \max\{\rho(F), \rho(A_s)\}$.*

(c) *If $f$ is an infinite order solution of equation (1.1) and if $\varphi$ is a finite order entire function which is not solution of (1.1), then*

$$\lambda(f-\varphi)=\overline{\lambda}(f-\varphi)=\rho(f)=\infty. \tag{1.15}$$

From Theorem 1.5, we obtain the following corollary:

**Corollary 1.6.** *Suppose that $A_0,...,A_{k-1}$, $F\not\equiv 0$, $\alpha,\delta,\varepsilon,\{\phi_m\}$, $\{\theta_m\}$ and $k$ satisfy the hypotheses of Theorem 1.5. Then*

(i) *If there exists a finite order solution $f_0$ in case $(a)$, then for any infinite order solution $f$ of equation (1.1) and for any constant $C\neq 1$, we have*

$$\lambda(f-Cf_0)=\overline{\lambda}(f-Cf_0)=\rho(f)=\infty. \tag{1.16}$$

(ii) *If $f$ is an infinite order solution of (1.1), then $f$ has infinitely many fixed points and*

$$\lambda(f-z)=\overline{\lambda}(f-z)=\rho(f)=\infty. \tag{1.17}$$

In what follows, we investigate the relation between the solutions of a pair non-homogeneous linear differential equations and we obtain the following results :

**Theorem 1.7.** *Suppose that $A_0,...,A_{k-1}$, $\alpha,\delta,\varepsilon,\{\phi_m\}$, $\{\theta_m\}$ and $k$ satisfy the hypotheses of Theorem 1.5, and that $F_1\not\equiv 0$, $F_0\not\equiv 0$ are finite order entire functions such that $\rho(F_1)<\delta$, $\rho(F_0)\geq\delta$. If the equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f=F_1 \tag{1.18}$$

*has a solution $f_1$, then equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f=F_0 \tag{1.19}$$

*has at most one infinite order solution $f_0$ that satisfies*

$$\rho(f_1-Cf_0)\leq\max\left\{\rho(F_0),\rho(A_s),\overline{\lambda}(f_1-Cf_0)\right\} \tag{1.20}$$

*and all other solutions $f_2$ of $(1.19)$ satisfy*

$$\lambda(f_1-Cf_2)=\overline{\lambda}(f_1-Cf_2)=\rho(f_1-Cf_2)=\infty \tag{1.21}$$

*for any constant $C$.*

**Theorem 1.8.** *Suppose that $A_0,...,A_{k-1}$, $\alpha,\delta,\varepsilon,\{\phi_m\}$, $\{\theta_m\}$ and $k$ satisfy the hypotheses of Theorem 1.5, and that $F_1\not\equiv 0$, $F_0\not\equiv 0$ are finite order entire functions such that $F_1\not\equiv CF_0$ for any constant $C$. Suppose that*

$$\max\left\{\rho(A_j):j=0,1,...,s-1,s+1,...,k-1,\rho(F_0),\rho(F_1)\right\}<\delta \tag{1.22}$$

*and*

$$\max\left\{\rho(A_j):j=1,...,s-1,\rho(F_0),\rho(F_1)\right\}<\rho(A_0). \tag{1.23}$$

*Then all solutions of (1.18) and (1.19) satisfy (1.21).*

*Remark* 1.9. In Theorem 1.3, if we don't suppose the additional conditions

$$\max\left\{\rho(A_j) : j = 1,...,s-1, \rho(F)\right\} < \rho(A_0)$$

or

$$\max\left\{\rho(A_j) : j = 0,1,...,s-1\right\} < \rho(F),$$

then equation (1.1) may have polynomial solution. For example the equation

$$f^{(4)} + \sinh z^3 f^{(3)} + z^2 e^{z^2} f'' - z e^{z^2} f' + z f = z^3 \tag{1.24}$$

has a polynomial solution $f(z) = z^2$. The equation (1.24) satisfies the other additional hypotheses of Theorem 1.3 but does not satisfies

$$\max\{\rho(A_j) : j = 1,2, \rho(F)\} < \rho(A_0)$$

or

$$\max\{\rho(A_j) : j = 0,1,2\} < \rho(F).$$

## 2 Preliminary Lemmas

**Lemma 2.1.** ( [4]) *Suppose that $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ are entire functions with at least one $A_s$ $(0 \le s \le k-1)$ being transcendental. Let the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f' + A_0 f = 0 \tag{2.1}$$

*be the corresponding homogeneous differential equation of (1.1). Then both (1.1) and (2.1) must have infinite order solutions.*

**Lemma 2.2.** ***(Phragmen-Lindelöf Theorem,*** *see [9], p. 214). Let $f(z)$ be analytic in the region $D = \{z : \alpha_1 < \arg z < \beta_1,\ r_0 < |z| < \infty\}$ and continuous on $\overline{D} = D \cup \Gamma$, where $\Gamma$ is the boundary of $D$. If for any given small $\varepsilon > 0$, there exists $r_1(\varepsilon) > 0$ such that for $|z| \ge r_1(\varepsilon)$, $z \in D$, we have*

$$|f(z)| < \exp\left\{\varepsilon |z|^{\frac{\pi}{\beta_1 - \alpha_1}}\right\}, \tag{2.2}$$

*and for $z \in \Gamma$, we have $|f(z)| \le M$ ($M > 0$ is a constant), then $|f(z)| \le M$ for all $z \in D$. $|f(z)| = M$ if and only if $f$ is a constant.*

*Remark* 2.3. ( [4]) Now suppose that $g(z)$ is analytic in the region $D = \{z : \alpha_1 < \arg z < \beta_1,\ r_0 \le |z| < \infty\}$ and $|g(z)| \le \exp\left\{|z|^{\sigma}\right\}$ for some constant $0 \le \sigma < \infty$. If a subset $E \subset (\alpha_1, \beta_1)$ has linear measure zero and for any $\psi_0 \in (\alpha_1, \beta_1) - E$, $|g(z)|$ is bounded for all $z$ satisfying $\arg z = \psi_0$ and $|z| \ge r_0$, then for any given small $\varepsilon > 0$, there exists $r_1(\varepsilon) > r_0$ such that $|g(z)| \le \exp\left\{\varepsilon |z|^{\sigma+1}\right\}$ for $|z| = r > r_1$. We may choose points $\theta_j \in (\alpha_1, \beta_1) - E$ $(j = 1,...,n)$ such that $\theta_1 < \theta_2 < ... < \theta_n$ $(\alpha_1 < \theta_1 \le \alpha_1 + \varepsilon,\ \beta_1 - \varepsilon \le \theta_n < \beta_1)$ and $\max\{\theta_{j+1} - \theta_j : 1 \le j \le n-1\} < \frac{\pi}{\sigma+1}$. Now from Lemma 2.1, $|g(z)| \le M$ in the sectors $\{z : \theta_j \le \arg z \le \theta_{j+1}, |z| \ge r_0\}$ $(j = 1,...,n-1)$. Hence $|g(z)| \le M$ in the sector $\{z : \alpha_1 + \varepsilon \le \arg z \le \beta_1 - \varepsilon,\ |z| \ge r_0\}$.

**Lemma 2.4.** ( [4]) *Let $A_0$, $A_1, ..., A_{k-1}$, $F \not\equiv 0$ be finite order entire functions. If $f$ is a solution of* $(1.1)$ *with* $\rho(f) = \infty$, *then* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

**Lemma 2.5.** *[5] Let $f$ be a transcendental entire function of finite order $\rho$, let*

$$\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$$

*denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ $(i = 1, ..., m)$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all $z$ satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$, *and for all $(k, j) \in \Gamma$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.3}$$

**Lemma 2.6.** *([8], [6], Lemma 3) Let $f(z)$ be an entire function and suppose that $\left|f^{(k)}(z)\right|$ is unbounded on some ray* $\arg z = \theta$. *Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, ...)$, where $r_n \to +\infty$ such that $f^{(k)}(z_n) \to \infty$ and*

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \leq \frac{1}{(k-j)!}(1 + o(1)) |z_n|^{k-j} \quad (j = 0, ..., k-1). \tag{2.4}$$

**Lemma 2.7.** *Suppose that $k \geq 2$ is a natural number and that $A_0, A_1, ..., A_{k-1}$ are entire functions. For real constants $\alpha, \delta, \theta_1, \theta_2$ such that $\alpha > 0$, $\delta > 0$, $\theta_1 < \theta_2$, we have for some $A_s$ $(0 \leq s \leq k-1)$*

$$|A_s(z)| \geq \exp\left(\alpha |z|^{\delta}\right) \tag{2.5}$$

*as $z \to \infty$ in $S = \{z : \theta_1 \leq \arg z \leq \theta_2\}$ and*

$$\max\left\{\rho(A_j) : j = 0, 1, ..., s-1, s+1, ..., k-1\right\} < \delta. \tag{2.6}$$

*If $f \not\equiv 0$ is a solution of equation (2.1) with $\rho(f) = \rho < \infty$, then for any given small $\varepsilon > 0$, there is a constant $M > 0$ such that $|f(z)| \leq M |z|^s$ for all $z$ in $S(\varepsilon) = \{z : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\}$ with $|z| \geq r_0 > 0$.*

*Proof.* Set $\rho(f) = \rho < \infty$. Then by Lemma 2.5 there exists a set $E \subset [0, 2\pi)$ with linear measure zero such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all $j = s+1, ..., k$

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq |z|^{(j-s)(\rho-1+\varepsilon)} \leq |z|^{(k-s)\rho} \quad (0 < \varepsilon < 1) \tag{2.7}$$

as $|z| \geq R_0$ along $\arg z = \psi_0$.

Now suppose that $\left|f^{(s)}(z)\right|$ is unbounded on some ray $\arg z = \phi_0$, where $\phi_0 \in [\theta_1, \theta_2] - E$. Then by Lemma 2.6, there exists an infinite sequence of points $z_n = r_n e^{i\phi_0}$, where $r_n \to \infty$ such that $f^{(s)}(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right| \leq \frac{1}{(s-j)!}(1+o(1))|z_n|^{s-j} \leq 2|z_n|^s \quad (j=0,\ldots,s-1) \tag{2.8}$$

as $z_n \to \infty$. By (2.1) and (2.6)-(2.8), we have as $z_n \to \infty$

$$\begin{aligned} |A_s(z_n)| \leq & \left|\frac{f^{(k)}(z_n)}{f^{(s)}(z_n)}\right| + \left|\frac{f^{(k-1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{k-1}| + \ldots \\ & + \left|\frac{f^{(s+1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{s+1}| + \left|\frac{f^{(s-1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{s-1}| + \ldots \\ & + \left|\frac{f(z_n)}{f^{(s)}(z_n)}\right| |A_0| \leq M_0 \exp\left\{o(1)|z_n|^{\delta}\right\} |z_n|^{M_1}, \end{aligned} \tag{2.9}$$

where $M_0 > 0$, $M_1 > 0$ are some constants. Thus (2.9) contradicts (2.5). Therefore, $f^{(s)}(z)$ is bounded on any arbitrary ray $\arg z = \psi_0 \in [\theta_1, \theta_2] - E$. From Remark 2.3, it is easy to see that

$$\left|f^{(s)}(z)\right| \leq M_2 \tag{2.10}$$

in $S(\varepsilon) = \{z : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\}$ for some constant $M_2 > 0$. By $s$-fold iterated integration along the line segment $[0, z]$, we obtain

$$\begin{aligned} f(z) = & f(0) + f'(0)\frac{z}{1!} + \ldots + \frac{1}{(s-1)!} f^{(s-1)}(0) z^{s-1} \\ & + \int_0^z \ldots \int_0^z \int_0^z f^{(s)}(t)\, dt \ldots dt. \end{aligned} \tag{2.11}$$

Therefore, by an elementary triangle inequality and (2.10), we obtain from (2.11) for an arbitrary point $z$ in $S(\varepsilon)$ with $|z| \geq r_0 > 0$

$$|f(z)| \leq |f(0)| + \left|f'(0)\right| \frac{|z|}{1!} + \left|f''(0)\right| \frac{|z|^2}{2!} + \ldots + M_2 \frac{|z|^s}{s!} \leq M|z|^s, \tag{2.12}$$

where $M > 0$ is some constant.

**Lemma 2.8.** *Suppose that $A_0, \ldots, A_{k-1}, \alpha, \delta, \varepsilon, \theta_1, \theta_2, S, S(\varepsilon)$ and $k$ satisfy the hypotheses of Lemma 2.7 and $F \not\equiv 0$ is an entire function with $\rho(F) < \delta$. Then every solution $f$ of (1.1) with $\rho(f) = \rho < \infty$, satisfies $|f(z)| \leq M|z|^s$ for all $z$ in $S(\varepsilon)$ with $|z| \geq r_0 > 0$ for some constant $M > 0$.*

*Proof.* Let $\rho(f) = \rho < \infty$. Using the same reasoning as in Lemma 2.7, it is easy to see that (2.7) holds for $f(z)$. If we assume $\left|f^{(s)}(z)\right|$ is unbounded on a some ray $\arg z = \phi_0$ where $\phi_0 \in [\theta_1, \theta_2] - E$, then using the same reasoning as in Lemma 2.7, it is easy to see that (2.8) holds. Since $\left|f^{(s)}(z_n)\right| \to \infty$ $(n \to \infty)$, we may assume that $\left|f^{(s)}(z_n)\right| \geq 1$ for all $n$. Hence

$$\left|\frac{F(z_n)}{f^{(s)}(z_n)}\right| \leq |F(z_n)| \leq \exp\left\{o(1)|z_n|^{\delta}\right\}. \tag{2.13}$$

From (1.1), (2.6), (2.7), (2.8) and (2.13) we get

$$|A_s(z_n)| \leq \left|\frac{f^{(k)}(z_n)}{f^{(s)}(z_n)}\right| + \left|\frac{f^{(k-1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{k-1}| + \dots$$

$$+ \left|\frac{f^{(s+1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{s+1}| + \left|\frac{f^{(s-1)}(z_n)}{f^{(s)}(z_n)}\right| |A_{s-1}| + \dots$$

$$+ \left|\frac{f(z_n)}{f^{(s)}(z_n)}\right| |A_0| + \left|\frac{F(z_n)}{f^{(s)}(z_n)}\right| \leq M_2 \exp\left\{o(1)|z_n|^{\delta}\right\} |z_n|^{M_1}, \tag{2.14}$$

where $M_2 > 0$, $M_1 > 0$ are some constants. This contradicts (2.5). Therefore $\left|f^{(s)}(z)\right|$ is bounded on any arbitrary ray $\arg z = \psi_0 \in [\theta_1, \theta_2] - E$. And also using the same reasoning as in Lemma 2.7, we have $|f(z)| \leq M|z|^s$ for all $z$ in $S(\varepsilon)$ with $|z| \geq r_0 > 0$ for some constant $M > 0$.

**Lemma 2.9.** *Suppose that $k \geq 2$ is a natural number and $A_0, A_1, \dots, A_{k-1}$, are finite order entire functions. Suppose that there exists an $A_s$ $(0 \leq s \leq k-1)$ such that for real constants $\alpha$, $\delta$, with $\alpha > 0$, $\delta > 0$, we have*

$$\max\left\{\rho(A_j) : j = 0, 1, \dots, s-1, s+1, \dots, k-1\right\} < \delta \tag{2.15}$$

*and*

$$\max\left\{\rho(A_j) : j = 1, \dots, s-1\right\} < \rho(A_0). \tag{2.16}$$

*Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\phi_m\}$ and $\{\theta_m\}$ that satisfy (1.2), (1.3). In $D_m = \{z : \phi_m \leq \arg z \leq \theta_m\}$ $(m = 1, \dots, n)$ as $z \to \infty$, we have*

$$|A_s(z)| \geq \exp\left\{\alpha |z|^{\delta}\right\}. \tag{2.17}$$

*Then every solution $f$ of (2.1) satisfies $\rho(f) = \infty$.*

*Proof.* Let $f \not\equiv 0$ be a solution of (2.1) with $\rho(f) = \rho < \infty$. By the hypotheses, for any given $\varepsilon \left(0 < \varepsilon < \frac{\pi}{3\left(\rho + \frac{1}{4}\right)}\right)$, there exist two finite collections of real numbers $\{\phi_m\}$ and $\{\theta_m\}$ that satisfy (1.2), (1.3). Suppose $A_s$ satisfies (2.17) in $D_m$ $(m = 1, \dots, n)$ and $A_j$ $(j = 0, 1, \dots, s-1, s+1, \dots, k-1)$ satisfy

$$\left|A_j(z)\right| \leq \exp\left\{o(1)|z|^{\delta}\right\}. \tag{2.18}$$

Then by Lemma 2.7, we have

$$|f(z)| \leq M|z|^s \ \ (M > 0 \text{ is some constant}) \tag{2.19}$$

in $D_m^*(\varepsilon) = \{z : \phi_m + \varepsilon \leq \arg z \leq \theta_m - \varepsilon\}$ $(m = 1, \dots, n)$ with $|z| \geq r_1$.

On the other hand, in the sector $\theta_m - \varepsilon \leq \arg z \leq \phi_{m+1} + \varepsilon$ $(m = 1, \dots, n)$, from $\rho(f) = \rho < \infty$, we have $|f(z)| \leq \exp\left\{r^{\rho + \frac{1}{4}}\right\}$ holds for $|z| = r > r_0 > 0$. By (1.3) and $0 < \varepsilon <$

$\frac{\pi}{3(\rho+\frac{1}{4})}$, we have $(\phi_{m+1}+\varepsilon)-(\theta_m-\varepsilon)<3\varepsilon<\frac{\pi}{\rho+\frac{1}{4}}$. Hence there exists $R>1+r_0$ such that for $r>R$, we have $r^{\rho+\frac{1}{4}}<\varepsilon r^{\frac{\pi}{[(\phi_{m+1}+\varepsilon)-(\theta_m-\varepsilon)]}}$. Therefore, in $\theta_m-\varepsilon\leq\arg z\leq\phi_{m+1}+\varepsilon$ $(m=1,...,n)$

$$\left|\frac{f(z)}{z^s}\right|\leq|f(z)|\leq\exp\left\{|z|^{\rho+\frac{1}{4}}\right\}<\exp\left\{\varepsilon|z|^{\frac{\pi}{[(\phi_{m+1}+\varepsilon)-(\theta_m-\varepsilon)]}}\right\} \tag{2.20}$$

holds for $|z|=r>R$. And on the rays $\arg z=\theta_m-\varepsilon$, $\arg z=\varphi_{m+1}+\varepsilon$ with $|z|\geq r_1>0$, we have $\left|\frac{f(z)}{z^s}\right|\leq M$ holds from (2.19). Hence by Lemma 2.2, $\left|\frac{f(z)}{z^s}\right|\leq M$ holds in $\theta_m-\varepsilon\leq\arg z\leq\varphi_{m+1}+\varepsilon$ $(m=1,...,n)$ with $|z|>R$. So $|f(z)|\leq M|z|^s$ holds in the whole plane from (2.19). Therefore, $f(z)$ is a polynomial.

If $f$ is a polynomial of $\deg f\geq s$, then $\rho\left(f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f\right)=\rho(A_s)\geq\delta>0$ and this contradicts (2.1). If $f$ is a non zero polynomial of $\deg f\leq s-1$, then $\rho\left(A_{s-1}f^{(s-1)}+...+A_0f\right)=\rho(A_0)>0$. This contradicts (2.1). Therefore, every solution $f\not\equiv 0$ of (2.1) satisfies $\rho(f)=\infty$.

**Lemma 2.10.** *Suppose that* $A_0,...,A_{k-1},\alpha,\delta,\varepsilon,\{\phi_m\},\{\theta_m\}$, *k and* $D_m$ *satisfy the hypotheses of Lemma 2.9, and that* $F\not\equiv 0$ *is a finite order entire function such that* $\rho(F)<\delta$ *and that either:*

*(i)* $\max\left\{\rho(A_j):j=1,...,s-1,\rho(F)\right\}<\rho(A_0)$ *or that*

*(ii)* $\max\left\{\rho(A_j):j=0,1,...,s-1\right\}<\rho(F)$.

*Then every solution* $f$ *of (1.1) satisfies* $\rho(f)=\infty$.

*Proof.* Let $f$ be a solution of (1.1) with $\rho(f)=\rho<\infty$. Using the same reasoning as in Lemma 2.9 and together with Lemma 2.8, it easy to see that $|f(z)|\leq M|z|^s$ for some constant $M>0$ in

$$D_m^*(\varepsilon)=\{z:\phi_m+\varepsilon\leq\arg z\leq\theta_m-\varepsilon\}\ \ (m=1,...,n)$$

with $|z|\geq r_1$. Similarly in $\theta_m-\varepsilon\leq\arg z\leq\phi_{m+1}+\varepsilon$ $(m=1,...,n)$, also using the same reasoning as in Lemma 2.9, we see that $|f(z)|\leq M|z|^s$ in $D_m^{**}(\varepsilon)=\{z:\theta_m-\varepsilon\leq\arg z\leq\phi_{m+1}+\varepsilon\}$ $(m=1,...,n)$ with $|z|>R$. Hence $f$ is a polynomial.

If $f$ is a polynomial with $\deg f\geq s$, then

$$\rho\left(f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f\right)=\rho(A_s)\geq\delta>\rho(F).$$

This contradicts (1.1). If $f$ is a non zero polynomial with $\deg f\leq s-1$, then if

$$\max\left\{\rho(A_j):j=1,...,s-1,\rho(F)\right\}<\rho(A_0),$$

we have $\rho(A_{s-1}f^{(s-1)}+...+A_0f)=\rho(A_0)>\rho(F)$. This contradicts (1.1). If $\max\{\rho(A_j):j=0,1,...,s-1\}<\rho(F)$, then

$$\rho\left(f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f\right)\leq\max\left\{\rho(A_j):j=0,1,...,s-1\right\}<\rho(F).$$

This contradicts (1.1). Therefore, every solution $f$ of (1.1) has infinite order.

# 3 Proof of Theorem 1.3.

(a) From $f(z)$ is a solution of (1.1) and Lemma 2.10, we know that $\rho(f) = +\infty$. Together with Lemma 2.4, we know that $f(z)$ satisfies (1.9).

(b) Set $g = f - \varphi$. Since $f$ is a solution of $(1.1)$, then we have $\rho(g) = \rho(f) = \infty$. Substituting $f = g + \varphi$ into equation (1.1), we get

$$g^{(k)} + A_{k-1}g^{(k-1)} + \ldots + A_0 g = F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \ldots + A_0\varphi\right). \quad (3.1)$$

By $\rho(\varphi) < +\infty$, then $\varphi$ is not a solution of (1.1). Thus, we have

$$F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \ldots + A_0\varphi\right) \not\equiv 0 \quad (3.2)$$

and by Lemma 2.4, we know that

$$\lambda(f - \varphi) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty. \quad (3.3)$$

# 4 Proof of Theorem 1.5.

(a) Assume that $f_0$ is a solution of (1.1) with $\rho(f_0) = \rho < \infty$. If $f_1$ is another finite order solution of (1.1), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation (2.1) of (1.1), but $\rho(f_1 - f_0) = \infty$ from Lemma 2.9. This is a contradiction. Hence (1.1) has at most one finite order solution $f_0$ and all other solutions $f_1$ of (1.1) satisfy (1.9) by Lemma 2.4.

(b) Assume that $f_0$ is a solution of (1.1) with $\rho(f_0) < \infty$. By (1.1) we can write

$$\frac{1}{f_0} = \frac{1}{F}\left(\frac{f_0^{(k)}}{f_0} + A_{k-1}\frac{f_0^{(k-1)}}{f_0} + \ldots + A_1\frac{f_0'}{f_0} + A_0\right). \quad (4.1)$$

It follows that if $f_0$ has a zero at $z_0$ of order $d > k$, then $F$ must have a zero at $z_0$ of order $d - k$. Hence

$$n\left(r, \frac{1}{f_0}\right) \le k\overline{n}\left(r, \frac{1}{f_0}\right) + n\left(r, \frac{1}{F}\right) \quad (4.2)$$

and

$$N\left(r, \frac{1}{f_0}\right) \le k\overline{N}\left(r, \frac{1}{f_0}\right) + N\left(r, \frac{1}{F}\right). \quad (4.3)$$

Applying the Lemma of the logarithmic derivative [7]

$$m\left(r, \frac{f_0^{(j)}}{f_0}\right) = O(\ln r) \quad (j = 1, \ldots, k) \quad (\rho(f_0) < +\infty). \quad (4.4)$$

From (4.1) we have

$$m\left(r, \frac{1}{f_0}\right) \le m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\left(r, \frac{f_0^{(j)}}{f_0}\right) + O(1). \quad (4.5)$$

Then we get from (4.3), (4.4) and (4.5)

$$T(r, f_0) = T\left(r, \frac{1}{f_0}\right) + O(1)$$

$$\leq k\overline{N}\left(r, \frac{1}{f_0}\right) + T(r,F) + \sum_{j=0}^{k-1} T(r,A_j) + O(\log r). \tag{4.6}$$

Now set $\alpha = \max\{\rho(A_s), \rho(F)\}$. Then for given small $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r,F) < r^{\alpha+\varepsilon},\ T(r,A_j) < r^{\alpha+\varepsilon}\ (j = 0, \ldots, k-1). \tag{4.7}$$

So, by (4.6), (4.7)

$$T(r, f_0) \leq k\,\overline{N}\left(r, \frac{1}{f_0}\right) + (k+1)\,r^{\alpha+\varepsilon} + O(\log r) \tag{4.8}$$

holds for a sufficiently large $r$. Therefore,

$$\rho(f_0) \leq \max\left\{\alpha,\ \overline{\lambda}(f_0)\right\} = \max\left\{\rho(A_s),\ \rho(F),\ \overline{\lambda}(f_0)\right\}. \tag{4.9}$$

If $\overline{\lambda}(f_0) < \rho(f_0)$, $\rho(A_s) \neq \rho(F)$, then by (4.9) we have

$$\rho(f_0) \leq \max\{\rho(A_s),\ \rho(F)\} \tag{4.10}$$

and by $(1.1)$, we get

$$\rho(f_0) \geq \max\{\rho(A_s),\ \rho(F)\}. \tag{4.11}$$

Therefore,

$$\rho(f_0) = \max\{\rho(A_s),\ \rho(F)\}. \tag{4.12}$$

(c) Set $g = f - \varphi$. If $f$ is a solution of $(1.1)$ with $\rho(f) = \infty$, then we have $\rho(g) = \rho(f) = \infty$. Substituting $f = g + \varphi$ into equation (1.1) we get

$$g^{(k)} + A_{k-1}g^{(k-1)} + \ldots + A_0 g = F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \ldots + A_0\varphi\right). \tag{4.13}$$

Since $\varphi$ is not a solution of (1.1), then we have

$$F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \ldots + A_0\varphi\right) \not\equiv 0 \tag{4.14}$$

and by Lemma 2.4, we know that

$$\lambda(f - \varphi) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty. \tag{4.15}$$

## 5 Proof of Theorem 1.7

Suppose that $f_1$ is a solution of equation (1.18) and $f_2$ is a solution of equation (1.19). Set $g = f_1 - Cf_2$. Then $g$ is a solution of equation $g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = F_1 - CF_0$. By $\rho(F_1 - CF_0) = \rho(F_0) \geq \delta$ and Theorem 1.5 the equation $g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = F_1 - CF_0$ has at most one finite order solution $g_0 = f_1 - Cf_0$ that satisfies

$$\rho(g_0) \leq \max\left\{\rho(F_0), \rho(A_s), \overline{\lambda}(g_0)\right\} \tag{5.1}$$

and all other solutions $g = f_1 - Cf_2$ of $g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = F_1 - CF_0$ satisfy

$$\lambda(f_1 - Cf_2) = \overline{\lambda}(f_1 - Cf_2) = \rho(f_1 - Cf_2) = \infty. \tag{5.2}$$

## 6 Proof of Theorem 1.8

Suppose that $f_1$ is a solution of equation (1.18) and $f_2$ is a solution of equation (1.19). Set $g = f_1 - Cf_2$. Then $g$ is a solution of equation $g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = F_1 - CF_0$. By $F_1 - CF_0 \not\equiv 0$, $\rho(F_1 - CF_0) < \delta$ and Theorem 1.3, we have $\rho(g) = \infty$. Thus, by Lemma 2.4, we know that

$$\lambda(f_1 - Cf_2) = \overline{\lambda}(f_1 - Cf_2) = \rho(f_1 - Cf_2) = \infty. \tag{6.1}$$

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