Communications in Mathematical Analysis
Volume 5, Number 2, pp. 26–37 (2008) www.commun-math-anal.org
ISSN 1938-9787

A REMARK ON STABILITY OF A CLASS OF POSITIVE LINEAR DELAY SYSTEMS IN BANACH LATTICES

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(Communicated by Hitoshi Kitada)

Abstract

In this paper, we give an extension of the classical Perron-Frobenius theorem to a class of positive quasi-polynomial operators. Then the obtained results are applied to show that delay-independent exponential stability and exponential stability of positive linear delay systems are equivalent, and are characterized by a simple criterion. Next, we show that delay-independent exponential stability radii and exponential stability radii coincide and can be computed via simple formula. Finally, a simple example is given to illustrate the obtained results. The obtained results are extensions of the recent in [19].

AMS Subject Classification: 93D09; 34K06; 47H07.

Keywords: robustness, delay-independent, exponential stability, stability radius, positive system, delay system.

1 Introduction

In recent years, the stability condition problem of linear systems in finite dimensional spaces has been paid considerable attention. There are many approaches proposed to handle this

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problem: the Perron-Frobenius theorem [19], two-variable criteria [15], matrix pencil techniques [20], frequency-sweeping tests [8], infinite-dimensional LMI conditions derived by using some appropriate quadratic Lyapunov-Krasovskii functionals [6, 7] and characterization of the delay interference phenomenon [18]. For further discussions and references, see e.g. [9, 12, 21]. However, in our view, the principal tool for analysis of the stability of positive systems is the Perron-Frobenius theorem, see e.g. [2, 5, 17, 19].

In this paper, we study extension of the Perron-Frobenius theorem to the quasi-polynomial operator associated with the linear delay system of the following form

$$\dot{u}(t) = A_0 u(t) + A_1 u(t-h_1) + \ldots + A_N u(t-h_N), \;\; t \geq 0, \tag{1.1}$$

where A_k are operators on a Banach space X, and $h_i \in \mathbb{R}^+ := (0, +\infty)$, for all $i \in \underline{N} := 1, 2, \ldots, N$.

Recall that the quasi-polynomial operator associated with the delay system (1.1) is defined by, see e.g. [1],

$$P(\lambda) := A_0 + \sum_{i=1}^{N} e^{-\lambda h_i} A_i. \tag{1.2}$$

It is well known that, in the case X is a finite dimensional space, system (1.1) is exponentially stable if and only if characteristic roots of equation $\det P(\lambda) = 0$ lie in the open left half of complex plane, which means that the exponential stability of system (1.1) is controlled by the location of the spectrum of its quasi-polynomial matrix. And, in general, this is not the case if the finite dimension assumption is dropped. Thus, it is natural to investigate for which kind of systems this is true. In this work, we show that if system (1.1) is positive then we can get above conclusion. Then, the results obtained are used to derive necessary and sufficient conditions for exponential stability of positive systems (1.1).

The organization of this paper is as following. In next section, we summarize some results on Metzler operators and delay systems, and give an extension of the classical Perron-Frobenius theorem to a class of positive quasi-polynomial operators associated with systems (1.1). In section 3, we address some necessary and sufficient conditions for exponential stability of positive linear delay systems and show that under positivity assumption the concept of delay-independent exponential stability and exponential stability coincide. Moreover, we also show that delay-independent exponential stability of positive linear delay systems is robust and its radii can be computed via a simple formula. Finally, a simple example is given.

2 Preliminaries

2.1 Positive operator and Metzler operators

Let X be a complex Banach space. For a closed linear operator A, let $\sigma(A)$ denote the spectrum of A, $\rho(A) := \mathbb{C} \backslash \sigma(A)$ the resolvent set of A, and $R(\lambda, A) := (\lambda I - A)^{-1} \in \mathcal{L}(X)$ the resolvent of A defined on $\rho(A)$. The spectral radius $r(A)$ and the spectral bound $s(A)$ of A are defined by

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} \quad s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

Assume that X,Y are complex Banach lattices, let X^+, and Y^+ denote the positive cones of X and Y respectively; and $\mathcal{L}^{\mathbb{R}}(X,Y)$ and $\mathcal{L}^+(X,Y)$ denote the set of all the real and positive linear operators from X to Y respectively. Throughout the paper, we always assume that all spaces considered are complex Banach lattices.

Definition 2.1. [10] A closed operator A is said to be a *Metzler operator* if there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and $R(t,A)$ is positive for all $t \in (\omega,\infty))$.

Metzler operators are also called resolvent positive operator in the literature. For an introduction to these operators we refer to [3]. Now we recall some results related to Metzler operator which will used in sequel.

Theorem 2.2. [17] Suppose $T \in \mathcal{L}^+(X)$. Then

(i) $r(T) \in \sigma(T)$;

(ii) $R(\lambda,T) \geq 0$ if and only if $\lambda \in \mathbb{R}$ and $\lambda > r(T)$.

Theorem 2.3. [10] Let A be a Metzler operator on X. Then

(i) $s(A) \in \sigma(A)$ and $s(A) = t - [r(R(t,A))]^{-1}$, $t > s(A)$;

(ii) the function $R(\cdot,A)$ is positive and decreasing for $t > s(A)$

$$s(A) < t_1 \leq t_2 \Longrightarrow 0 \leq R(t_1,A) \leq R(t_2,A);$$

(iii) if A generates a positive C_0-semigroup, then we have $R(t,A)$ is positive if and only if $t > s(A)$.

2.2 Delay systems

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup generated by the operator $(A,D(A))$ on the Banach space X, we consider following quantities:

— the *spectral bound*

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\},$$

— the *abscissa of uniform boundedness*

$$s_0(A) := \inf\{\omega \in \mathbb{R} : \{\Re\lambda > \omega\} \subset \rho(A) \text{ and } \sup_{\Re\lambda > \omega} ||R(\lambda,A)|| < \infty\},$$

— the *growth bound*

$$\omega_1(A) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } ||S(t)x|| \leq Me^{\omega t}||x||_{D(A)},\ \forall t \geq 0, x \in D(A)\},$$

— the *uniform growth bound*

$$\omega_0(A) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } ||S(t)|| \leq Me^{\omega t},\ \forall t \geq 0\}.$$

We say that $(S(t))_{t\geq 0}$, or operator A, is *exponentially stable* if $\omega_1(A) < 0$. It is known that
$$s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A) < \infty.$$
The inequality $s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A)$ might be strict, that is, the exponential stability of a C_0-semigroup is, in general, not controlled by the location of the spectrum of its generator.

Remark 2.4. If A generates a uniformly continuous or eventually norm continuous C_0-semigroup, then $s(A) = \omega_0(A)$, see $[22, Theorem\, 2.3.2]$. And if A generates a positive C_0-semigroup, then $s(A) = \omega_1(A)$, see $[16, Proposition\, VI.1.14]$.

Given $p \in [1,\infty)$, non-negative real numbers $0 \leq h_1 < h_2 < \ldots < h_n =: h$, bounded linear operators $A_1,\ldots,A_N$ and A_0 generating a C_0-semigroup $(T(t))_{t\geq 0}$ on X, we rewrite the delay system (1.1) as
$$\begin{cases} \dot{u}(t) = A_0 u(t) + \sum\limits_{i=1}^{N} A_i u(t-h_i), \;\; t \geq 0, \\ u(0) = x, \\ u(t) = f(t), \;\; t \in [0,-h). \end{cases} \tag{2.1}$$
Here, $x \in X$ is the initial value and $f \in L^p([-h,0];X)$ is the 'history' function. A mild solution of (1.1) is the function $u(\cdot) \in L^p_{loc}([-h,\infty];X)$ satisfying
$$u(t) = \begin{cases} T(t)x + \int_0^t T(t-s) \sum\limits_{i=1}^{n} A_i u(s-h_i) ds, \; t \geq 0, \\ f(t), \;\; t \in [-h,0). \end{cases}$$
System (1.1) is called *exponentially stable* is there exist $M > 0$ and $\omega > 0$ such that the solution of (2.1) satisfies
$$||u(t)|| \leq Me^{-\omega t}(||x|| + ||f||_{L^p([-h,0];X)}), t \geq 0.$$

In order to study the asymptotic behavior of these solutions by semigroup method, we introduce the product space
$$\mathcal{X} := X \times L^p([-h,0];X),$$
endowed with the norm $||(x,f)|| := ||x|| + ||f||_{L^p([-h,0];X)}$); and the operator $\mathcal{A}$ on $\mathcal{X}$ defined by
$$\mathcal{A}(x,f) := (A_0 x + \sum_{i=1}^{N} A_i f(\cdot - h_i), f'),$$
with the domain
$$D(\mathcal{A}) := \{(x,f) \in \mathcal{X} : f \in W^{1,p}([-h,0];X), f(0) = x \in D(A_0)\},$$
where $W^{1,p}([-h,0];X)$ denotes the space of absolutely continuous X-valued functions f on $[-h,0]$ which are automatically differentiable a.e. with derivatives $f'(t) \in L^p([-h,0];X)$. Then, as it has been shown in [4, 11], $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ which is defined by
$$(\mathcal{T}(t))(x,f) := (u(t), u_t), t \geq 0,$$
where $u(t)$ is mild solution of (1.1) and $u_t(s) := u(t+s), s \in [-h,0]$. Moreover, system (1.1) is exponentially stable if and only if C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is exponentially stable, i.e. $\omega_1(\mathcal{A}) < 0$. And the resolvent and the spectrum of $\mathcal{A}$ are given by following.

Proposition 2.5. [11] We have $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(P(\lambda))$. In this case the resolvent of $\mathcal{A}$ is given by

$$R(\lambda, \mathcal{A}) = E_\lambda R(\lambda, P(\lambda)) H_\lambda F + T_\lambda,$$

where $E_\lambda \in \mathcal{L}(X, \mathcal{X}), H_\lambda \in \mathcal{L}(\mathcal{X}, X), F \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ and $T_\lambda, \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ are defined by

$$\begin{aligned} E_\lambda x &:= (x, e^{\lambda \cdot} x); \\ H_\lambda(x, f) &:= x + \int_{-h}^{0} e^{\lambda s} f(s) ds; \\ F(x, f) &:= (x, \sum_{j=1}^{n} \chi_{[-h_j, 0]}(\cdot) A_j f(-h_j - \cdot)); \\ T_\lambda(x, f) &:= (0, \int_{\cdot}^{0} e^{\lambda(\cdot - s)} f(s) ds). \end{aligned}$$

Recall that the spectral set, the resolvent set, and the spectral bound of quasi-polynomial operator $P(\cdot)$ are defined by

$$\sigma(P(\cdot)) := \{\lambda : \lambda \in \sigma(P(\lambda))\},$$

$$\rho(P(\cdot)) := \mathbb{C} \backslash \sigma(P(\cdot)),$$

$$s(P(\cdot)) := \sup\{\Re\lambda : \lambda \in \sigma(P(\cdot))\},$$

respectively.

Remark 2.6. From the above proposition, it is easy to see that $\rho(\mathcal{A}) = \rho(P(.))$, hence $s(\mathcal{A}) = s(P(\cdot))$. So if $\mathcal{A}$ generates a uniformly continuous or eventually norm continuous C_0-semigroup, or a positive C_0-semigroup, as in Remark 2.4, then system (1.1) is exponentially stable if and only if $s(P(\cdot)) < 0$.

The above remark shows that studying the exponential stability of system (1.1) turns out to be equivalent to studying the characteristics of quasi-polynomial operator (1.2) in some cases. In next, we present some results on an extension of Perron-Frobenius theorem to quasi-polynomial operator (1.2).

Definition 2.7. The quasi-polynomial operator (1.2) is called *positive* if A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, for all $i \in \underline{N}$.

It is important to note that if the quasi-polynomial operator (1.2) is positive then the delay system (1.1) is a positive system, that is, for any history function $f \in L^p([-h, 0); X^+)$ and initial value $x \in X^+$, the corresponding solution $u(t, x, f),\ t \geq 0$ satisfies $u(t, x, f) \in X^+$ for all $t \geq 0$.

By the representation of $R(\cdot, \mathcal{A})$ in Proposition 2.5, we obtain the following result.

Proposition 2.8. Let the quasi-polynomial operator (1.2) be positive. Then, for $\lambda_1, \lambda_2 \in \mathbb{R}$, the following statements are equivalent:

(i) $R(\lambda_1, P(\lambda_1)) \geq R(\lambda_2, P(\lambda_2)) \geq 0$;

(ii) $R(\lambda_1, \mathcal{A}) \geq R(\lambda_2, \mathcal{A}) \geq 0$.

Proof: Due to the fact that E_λ, H_λ, F, and T_λ are positive operators for $\lambda \in \mathbb{R}$, we get the $(i) \Rightarrow (ii)$. On the other hand, let $f = 0$, for all $x \in X$, we get

$$R(\lambda, \mathcal{A})(x, 0) = R(\lambda, P(\lambda))(x).$$

This implies $(ii) \Rightarrow (i)$. □

Note that $\mathcal{A}$ is a generator of a positive C_0-semigroup if A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, for all $i \in \underline{N}$, see [11]. Using Theorem 2.3 and Proposition 2.8, we get a Perron-Frobenius theorem for the positive quasi-polynomial operators (1.2).

Theorem 2.9. Let the quasi-polynomial operator (1.2) be positive. Then

(i) $s(P(\cdot)) \in \sigma(P(\cdot))$;

(ii) for $\lambda \in \mathbb{R}$, we have $R(\lambda, P(\lambda)) \in \mathcal{L}^+(X)$ if and only if $\lambda > s(P(\cdot))$;

(iii) $R(\lambda_1, P(\lambda_1)) \geq R(\lambda_2, P(\lambda_2))$ for $\lambda_2 \geq \lambda_1 > s(P(\cdot))$.

It is remarkable that Theorem 2.9 generalizes results in [19] to Perron-Frobenius theorem for positive quasi-polynomial operators in Banach spaces.

3 Main results

In this section, we apply the results obtained in the previous section to derive some necessary and sufficient conditions for the exponential stability of the positive delay system (1.1). We say that system (1.1) is *positive* if its associated quasi-polynomial operator is positive, i.e. A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$ for all $i \in \underline{N}$.

Theorem 3.1. Let system (1.1) be positive. Then the following statements are equivalent:

(i) System (1.1) is exponentially stable;

(ii) $s(A_0 + A_1 + \ldots + A_N) < 0$;

(iii) $s(A_0) < 0$ and $r(-A_0^{-1}(A_1 + \ldots + A_N)) < 1$;

(iv) $(-A_0 - A_1 - \ldots - A_N)^{-1} \geq 0$.

Proof. Assume that system (1.1) is exponentially stable. Then, since $\mathcal{A}$ generates a positive C_0-semigroup, this is equivalent to $s(\mathcal{A}) = s(P(\cdot)) < 0$. By Theorem 2.9, the previous statement happens if and only if the operator $(-A_0 - A_1 - \ldots - A_N)^{-1}$ exists and $(-A_0 - A_1 - \ldots - A_N)^{-1} \in \mathcal{L}^+(X)$. Thus, $(i) \Leftrightarrow (iv)$. Moreover, the operator $A_0 + A_1 + \ldots + A_N$ generates a positive C_0-semigroup, see $[16, Corollary\, VI.1.11]$. Thus, by Theorem 2.3, we get $(iv) \Leftrightarrow (ii)$.

Now, we show that $(ii) \Rightarrow (iii)$. Using Theorem 2.3 (i), we also get

$$\begin{aligned} s(A_0) &= t_0 - [r(R(t_0,A_0))]^{-1}, \forall t_0 > s(A_0), \\ s(A_0+\ldots+A_N) &= t_0 - [r(R(t_0,A_0+\ldots+A_N))]^{-1}, \forall t_0 > s(A_0+\ldots+A_N). \end{aligned} \tag{3.1}$$

Moreover, for t_0 large enough, we have

$$R(t_0,A_0+\ldots+A_N) = R(t_0,A_0)\sum_{n=0}^{\infty}[R(t_0,A_0)(A_1+\ldots+A_N)]^n.$$

So, we get $R(t_0,A_0+\ldots+A_N) \geq R(t_0,A_0) \geq 0$. This implies $r[R(t_0,A_0+\ldots+A_N)] \geq r[R(t_0,A_0)]$. Hence, from (3.1), we have $s(A_0) \leq s(A_0+A_1\ldots+A_N)$. Then (ii) implies $s(A_0) < 0$. And, by the same arguments, we also obtain $s(A_0+\frac{1}{t}(A_1\ldots+A_N)) < 0$ for all $t \geq 1$. Moreover,

$$tI - (-A_0)^{-1}(A_1+\ldots+A_N) = tA_0^{-1}[A_0+\frac{1}{t}(A_1+\ldots+A_N)],$$

which implies $[1;\infty) \subset \rho((-A_0)^{-1}(A_1+\ldots+A_N))$. By Theorem 2.2, $r[(-A_0)^{-1}(A_1+\ldots+A_N)] < 1$.

The remains $(iii) \Rightarrow (iv)$ of the proof is obvious by following equation

$$-A_0-A_1-\ldots-A_N = (-A_0)[I-(-A_0)^{-1}(A_1+\ldots+A_N)].$$

$\square$

From the above theorem, it is easy to see that the exponential stability of positive linear delay systems does not depend on delay parameters and that A_0 generates a exponentially stable C_0-semigroup is the necessary condition for exponential stability of the positive system (1.1). Next, we address some remarks about the relationship between delay-independent exponential stability and exponential stability of system (1.1).

Definition 3.2. System (1.1) is *delay-independently exponentially stable* if it is exponentially stable for each $(h_1,\ldots,h_N) \in (\mathbb{R}^+)^N$.

The concept of delay-independent exponential stability, has interested many researchers as in [6, 7, 8, 13, 18] and references therein. In fact, if the system is delay-independently exponentially stable then it is exponential stable, but the converse is not true. However, in the case of a positive system, both concepts are the same. This is stated in implications following corollary of Theorem 3.1.

Theorem 3.3. Let system (1.1) be positive. Then the following statements are equivalent:

(i) System (1.1) is exponentially stable;

(ii) System (1.1) is delay-independently exponentially stable.

Example 3.4. Consider the scalar equation

$$\dot{u}(t) = a_0u(t)+a_1u(t-h_1)+\ldots+a_Nu(t-h_N),$$

where $a_0 \in \mathbb{R}, a_i \in [0,\infty), i \in \underline{N}$. By Theorem 3.3, this system is exponentially stable as well as delay-independently exponentially stable if and only if $a_0+a_1+\ldots+a_N < 0$.

Suppose that all operators $A_i,\ i \in \underline{N}$, are subjected to perturbations of the form

$$A_i \to A_i + D_i \Delta_i E_i, i \in N, \tag{3.2}$$

where $D_i \in \mathcal{L}(U_i, X), E_i \in \mathcal{L}(X, Y_i), i \in \underline{N}$, are given operators determining structure of perturbations and $\Delta_i \in \mathcal{L}(Y_i, U_i), i \in \underline{N}$ are unknown operators. Then the perturbed system has a form

$$\begin{cases} \dot{u}(t) = (A_0 + D_0\Delta_0 E_0)u(t) + \sum_{i=1}^{N}(A_i + D_i\Delta_i E_i)u(t-h_i), t \geq 0, \\ u(0) = x, \\ u(t) = f(t), t \in [-h, 0). \end{cases} \tag{3.3}$$

Definition 3.5. Let system (1.1) be exponentially stable. The complex, real and positive exponential stability radii of (1.1) under perturbation of the form (3.2) are defined by

$$r_{\mathbb{C}} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}(Y_i, U_i), i = 0, 1, ..., N \text{ and } (3.3) \text{ is not exponentially stable}\},$$

$$r_{\mathbb{R}} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^{\mathbb{R}}(Y_i, U_i), i = 0, 1, ..., N \text{ and } (3.3) \text{ is not exponentially stable}\},$$

$$r_{+} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^{+}(Y_i, U_i), i = 0, 1, ..., N \text{ and } (3.3) \text{ is not exponentially stable}\},$$

respectively, where we set $\inf \emptyset = \infty$.

In [1], the robust exponential stability of system (1.1) is studied via a concept of complex exponential stability radii. Especially, for positive system, the complex, real and positive exponential stability radii coincide and can be computed via simple formulae.

Theorem 3.6. [1] Suppose that system (1.1) is positive and exponentially stable, and all operators D_i and E_i are positive, $i \in \underline{N}$. If $D_i = D_j$ or $E_i = E_j$ for all $i, j \in \underline{N}$, then

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_{+} = \frac{1}{\max_{i \in \{0,1,...,N\}} ||E_i(-A_0 - A_1 - ... - A_N)^{-1} D_i||}. \tag{3.4}$$

However, the delay-independent exponential stability property is not robust to perturbations of the parameters A_i. In particular, the set of delay-independently exponentially stable systems is not open as well as closed for the product topology, even in finite dimension, such as in following examples with $X = \mathbb{R}$.

Example 3.7. Consider following systems:

$$\dot{x}(t) = -x(t) + ix(t-\tau), \tag{3.5}$$

and

$$\dot{x}(t) = -(1 + \frac{1}{n})x(t) - x(t-\tau). \tag{3.6}$$

Using the Theorem 9.1 [14], we can check that system (3.5) is delay-independently exponentially stable, but the systems $\dot{x}(t) = -(1-\frac{1}{n})x(t) + ix(t-\tau)$ are not delay-independently exponentially stable for any $n \in \mathbb{N}$. This means the set of delay-independently exponentially stable systems is not open. And we also check that systems (3.6) are delay-independently exponentially stable for any $n \in \mathbb{N}$, but systems $\dot{x}(t) = -x(t) - x(t-\tau)$ is not delay-independently exponentially stable. This means the set of delay-independently exponentially stable systems is not closed.

In the rest of paper, we will prove that under the positivity assumption, the delay-independent exponential stability is robust even in the context of infinite dimension. We do this via the following quantities, which estimate the distance from a equation to the set of systems which are not delay-independently exponentially stable, shortly *d.i.e.s.*.

Definition 3.8. Suppose that system (1.1) is delay-independently exponentially stable. The complex, real and positive delay-independent exponential stability radii of (1.1) under perturbations of the form (3.2) is defined by

$$r_{\mathbb{C}}^{dies} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}(Y_i, U_i), i = 0, 1, ..., N \text{ and (3.3) is not d.i.e.s.}\},$$
$$r_{\mathbb{R}}^{dies} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^{\mathbb{R}}(Y_i, U_i), i = 0, 1, ..., N \text{ and (3.3) is not d.i.e.s.}\},$$
$$r_{+}^{dies} = \inf\{\sum_{i=0}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^{+}(Y_i, U_i), i = 0, 1, ..., N \text{ and (3.3) is not d.i.e.s.}\},$$

respectively.

From the definition, we get the inequality

$$r_{\mathbb{C}}^{dies} \leq r_{\mathbb{R}}^{dies} \leq r_{+}^{dies} \leq r_{+}. \tag{3.7}$$

Theorem 3.9. Suppose that (1.1) is positive and exponentially stable, and all operators D_i and E_i of the perturbation, $i \in \underline{N}$, are positive. If $D_i = D_j$ (or $E_i = E_j$) for all $i, j \in \underline{N}$, then

$$r_{\mathbb{C}}^{dies} = r_{\mathbb{R}}^{dies} = r_{+}^{dies} = \frac{1}{\max\limits_{i \in \{0,1,...,N\}} ||E_i(-A_0 - A_1 - ... - A_N)^{-1} D_i||} > 0.$$

Proof. By inequality (3.7), we need only to prove $r_{\mathbb{C}}^{dies} \geq r_+$. Indeed, let $\Delta_i' \in \mathcal{L}(Y_i, U_i),\ i \in \underline{N}$ such that $\sum_{i=0}^{N} ||\Delta_i'|| < r_+$ and $(h_1', ..., h_N') \in \mathbb{R}_+^N$, by Theorem (3.1) the equation with delay parameter $(h_1', ..., h_N')$ is also positive and exponentially stable. Hence, by Theorem 3.6, the perturbed equations with delay parameter $(h_1', ..., h_N')$ and perturbed operators $\Delta_i', i \in \underline{N}$, is also exponentially stable. And this completes the proof. □

The above theorem not only shows that delay-independent exponential stability radii and exponential stability radii coincide, but also proves the robustness of delay-independent exponential stability for the class of positive systems.

4 An example

Let $X = l_2(\mathbb{R})$, consider the following linear delay system

$$\dot{u}(t) = A_0u(t) + A_1u(t-1) + A_2u(t-2), \;\; t \geq 0, \tag{4.1}$$

in the form of system (1.1) with $N = 2$, $h_1 = 1$, $h_2 = 2$, and

$$A_0(x_1, x_2, x_3, \ldots) = (-3x_1, -4x_2, -5x_3, -6x_4, \ldots), \;\; A_1 = A_2 = Id.$$

Since A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, system (4.1) is positive. Applying Theorem 3.3 and Theorem 3.1, from $s(A_0 + A_1 + A_2) = -1 < 0$, we can get that system (4.1) is delay-independently exponentially stable.

Now suppose that all operators A_i, $i = 0, 1, 2$, are subjected to perturbations of the form (3.2) with $E_0 = E_1 = E_2 = E$ are the left shift operators defined by $E(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$, $D_0 = Id$, $D_1(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ is the right shift operator, and $D_2(x_1, x_2, x_3, \ldots) = (0, 0, x_1, x_2, x_3, \ldots)$ is the twice right shift operator. Then

$$E(-A_0 - A_1 - A_2)^{-1}D_0(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots),$$

$$E(-A_0 - A_1 - A_2)^{-1}D_1(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \ldots),$$

$$E(-A_0 - A_1 - A_2)^{-1}D_2(x_1, x_2, x_3, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{4}x_2, \frac{1}{5}x_3, \ldots).$$

So, by Theorem 3.9, we get

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = r_{\mathbb{C}}^{dies} = r_{\mathbb{R}}^{dies} = r_+^{dies} = 2.$$

Acknowledgments

The authors thank the referees for their careful reading of the manuscript and insightful comments.

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