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NONLINEAR IMPULSIVE DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper we investigate the existence of solutions for nonlinear boundary value problems for second order impulsive differential inclusions with integral boundary conditions. The both case of convex as well as nonconvex valued right hand side are considered.

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1 Introduction

This paper is concerned with the existence of solutions of second order differential inclusions with integral boundary conditions. More precisely we consider the following boundary value problem (BVP for short)

$$y''(t) \in F(t, y(t)), \text{ for a.e. } t \in J := [0,1],\ t \neq t_i,\ i = 1, \ldots, m, \tag{1.1}$$

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$$\Delta y|_{t=t_i} = I_i(y(t_i^-)),\ i = 1,\dots,m, \tag{1.2}$$

$$\Delta y'|_{t=t_i} = \overline{I}_i(y(t_i^-)),\ i = 1,\dots,m, \tag{1.3}$$

$$y(0) - k_1 y'(0) = \int_0^1 h_1(s,y(s))ds, \tag{1.4}$$

$$y(1) + k_2 y'(1) = \int_0^1 h_2(s,y(s))ds, \tag{1.5}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $h_1,\ h_2 : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $I_i,\ \bar{I}_i : \mathbb{R} \to \mathbb{R}$, are given functions, $t_i \in J,\ 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1,\ k_1,\ k_2$ are nonnegative constants, $\Delta y|_{t=t_i} = y(t_i^+) - y(t_i^-)$, $\Delta y'|_{t=t_i} = y'(t_i^+) - y'(t_i^-)$, $y(t_i^+) = \lim_{h\to 0^+} y(t_i + h)$ and $y(t_i^-) = \lim_{h\to 0^+} y(t_i - h)$ are the right and left hand limits of $y(t)$ at $t = t_i$, respectively.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; for example we recommend the books by Benchohra *et al* [9], Lakshmikantham *et al* [27] and Samoilenko and Perestyuk [30] and the references therein.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Belarbi and Benchohra [4], Belarbi *et al* [5] Benchohra *et al* [7, 8, 10], Brykalov [12], Denche and Marhoune [16], Gallardo [18], Jankowskii [22], Karakostas and Tsamatos [23], Krall [26], Lomtatidze and Malaguti [29] and the references therein. Recently Ahmad *et al* [1] and Khan [24] have applied the generalized method of quasilinearization to a class of second order boundary value problem with integral boundary conditions. Benchohra *et al* considered in [6] multiple solutions for impulsive differential equations with integral conditions. Graef and Ouahab [19] studied a class of impulsive differential equations with integral boundary conditions on times scales,

We shall provide sufficient conditions ensuring some existence results for problem (1.1)-(1.5). In the convex case we present an existence result based on the nonlinear alternative of Leray Schauder type. In the nonconvex case two results will be given. The first one relies of the fixed point theorem due to Covitz and Nadler for contraction multivalued maps. The second one is based on a selection theorem for lower semicontinuous maps with closed decomposable values. Our results extend the previously cited results considered in the absence of impulses, and to the multivalued case those considered in [6].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le 1\},$$

and we let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^1 |y(t)|dt.$$

We consider the space

$$PC = \{y \; : \; [0,1] \to \mathbb{R} : y_i \in C(J_i,\mathbb{R}),\ i = 1,\dots,m, \\ y(t_i^-) \text{ and } y(t_i^+) \text{ exist } i = 1,\dots,m, \text{ and } y(t_i^-) = y(t_i)\}.$$

PC is a Banach space with the norm

$$\|y\|_{PC} = \max \|y_i\|_{J_i} : i = 0,\dots,m,$$

where y_i is the restriction of y to $J_i = [t_i,t_{i+1}] \subset [0,1]$, $i = 0,\dots,m$, and

$$\|y\|_{J_i} = \max_{t\in J_i} |y_i(t)|.$$

$AC^1((0,1),\mathbb{R})$ is the space of differentiable functions $y : (0,1) \longrightarrow \mathbb{R}$, whose first derivative y' is absolutely continuous. For a normed space $(X,|\cdot|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \mathbb{U}_{x\in B}G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y,G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable.

Let A be a subset of $[0,1]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u,v \in A$ and $\mathcal{J} \subset [0,1]$ measurable, $u\chi_{\mathcal{J}} + v\chi_{[0,1]-\mathcal{J}} \in A$, where χ stands for the characteristic function.

Let $G : X \to \mathcal{P}(X)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x)\cap B \neq \emptyset\}$ is open for any open set B in X.

For more details on multivalued maps see the books of Aubin and Cellina [2], Aubin and Frankowska [3], Deimling [15] and Hu and Papageorgiou [21].

Definition 2.1. A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t\longmapsto F(t,u)$ is measurable for each $u\in\mathbb{R}$;

(ii) $u\longmapsto F(t,u)$ is upper semicontinuous for almost all $t\in[0,1]$;

(iii) for each $q>0$, there exists $\varphi_q\in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,u)\|=\sup\{|v|:v\in F(t,u)\}\leq\varphi_q(t)\ \text{ for all } |u|\leq q \text{ and } \ a.e.\ t\in[0,1].$$

For each $y\in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y}=\{v\in L^1([0,1],\mathbb{R}):v(t)\in F(t,y(t))\ a.e.\ t\in[0,1]\}.$$

Definition 2.2. Let X be a separable metric space and let $N:X\to\mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has property (BC) if

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F}:C([0,1],\mathbb{R})\to\mathcal{P}(L^1([0,1],\mathbb{R}))$$

by letting

$$\mathcal{F}(y)=\{w\in L^1([0,1],\mathbb{R}):w(t)\in F(t,y(t)) \text{ for } a.e.\ t\in[0,1]\}.$$

The operator $\mathcal{F}$ is called the Nymetzki operator associated with F.

Definition 2.3. Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\longrightarrow\mathbb{R}_+\cup\{\infty\}$ given by

$$H_d(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\right\},$$

where $d(A,b)=\inf_{a\in A}d(a,b)$, $d(a,B)=\inf_{b\in B}d(a,b)$. Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space (see [25]).

Definition 2.4. A multivalued operator $N:X\to P_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\leq\gamma d(x,y),\quad \text{for each } x,\ y\in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.5. *[28]. Let X be a Banach space. Let $F : [0,1] \times X \longrightarrow P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$, then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,1],X) &\longrightarrow P_{cp,c}(C([0,1],X)), \\ y &\longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{aligned}$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Lemma 2.6. *(Nonlinear Alternative of Leray-Schauder Type, [20]). Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in X$. Suppose that $N : \overline{U} \to C$ is a continuous, compact map. Then either,*

$(C1)$ *N has a fixed point in $\overline{U}$; or*

$(C2)$ *There exists $\lambda \in (0,1)$ and $x \in \partial U$ (the boundary of U in C) with $x = \lambda N(x)$.*

Lemma 2.7. *[11]. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.8. *[14] Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Main Results

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.5) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

Definition 3.1. A function $y \in PC \cap \cup_{i=0}^{m} AC^1((t_i,t_{i+1}),\mathbb{R})$ is said to be a solution of (1.1)-(1.5) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,y(t))$, for a.e. $t \in [0,1]$, $t \neq t_i$, $i = 1,\dots,m$, and for each $i = 1,\dots,m, \Delta y|_{t=t_i} = I_i(y(t_i^-))$, $\Delta y'|_{t=t_i} = \overline{I}_i(y(t_i^-))$ and the boundary conditions (1.4)-(1.5) are satisfied.

We need the following auxiliary result. Its proof can be found in [6]. For completeness we present here the proof.

Lemma 3.2. *Let σ, ρ_1, $\rho_2 : J \to \mathbb{R}$ be integrable functions. If y is solution of the equation*

$$y(t) = p(t) + \int_0^1 G(t,s)\sigma(s)ds + \sum_{i=1}^{m} W_i(t), \tag{3.1}$$

where

$$G(t,s) = \alpha \begin{cases} (k_1+t)(1-s+k_2), & 0 \le t \le s, \\ (k_1+s)(1-t+k_2), & s \le t \le 1, \end{cases} \tag{3.2}$$

$$p(t) = -\alpha\left\{(1-t+k_2)\int_0^1 \rho_1(s)ds + (k_1+t)\int_0^1 \rho_2(s)ds\right\},$$

$$W_i(t) = -\alpha\begin{cases} (k_1+t)[-I_i(y(t_i)) - (1-t_i+k_2)\overline{I}_i(y(t_i))], & 0 \le t \le t_i, \\ (1-t+k_2)[I_i(y(t_i)) - (t_i+k_1)\overline{I}_i(y(t_i))], & t_i \le t \le 1, \end{cases} \quad (3.3)$$

and $\alpha = \frac{-1}{(1+k_1+k_2)}$, *then* y *is solution of the boundary value problem*

$$y''(t) = \sigma(t), \text{ a.e. } t \in J,\ t \ne t_i,\ i = 1,\dots,m, \quad (3.4)$$

$$\Delta y|_{t=t_i} = I_i(y(t_i^-)),\ i = 1,\dots,m, \quad (3.5)$$

$$\Delta y'|_{t=t_i} = \overline{I}_i(y(t_i^-)),\ i = 1,\dots,m, \quad (3.6)$$

$$y(0) - k_1 y'(0) = \int_0^1 \rho_1(s)ds, \quad (3.7)$$

$$y(1) + k_2 y'(1) = \int_0^1 \rho_2(s)ds. \quad (3.8)$$

Proof. Let y satisfy the integral equation (3.1) and $t \in J \setminus \{t_1,\dots,t_m\}$. Then we have

$$\begin{aligned} y'(t) &= [p(t) + \textstyle\int_0^1 G(t,s)\sigma(s)ds + \sum_{i=1}^m W_i(t)]' \\ &= p'(t) + \alpha\left[\textstyle\int_0^t (1-t+k_2)(k_1+s)\sigma(s)ds\right]' \\ &+ \alpha\left[\textstyle\int_t^1 (1-s+k_2)(k_1+t)\sigma(s)ds\right]' + \sum_{i=1}^m W_i'(t) \\ &= p'(t) - \alpha\textstyle\int_0^t (k_1+s)(s)ds - \alpha t(k_1+t)\sigma(t) \\ &+ \alpha(k_1+t)(1+k_2)\sigma(t) + \alpha\textstyle\int_t^1 (1-s+k_2)\sigma(s)ds \\ &- \alpha t(1-t+k_2)\sigma(t) - \alpha k_1(1-t+k_2)\sigma(t) \\ &+ \textstyle\sum_{i=1}^m W_i'(t) \\ &= p'(t) - \alpha\textstyle\int_0^t (k_1+s)\sigma(s)ds + \alpha\int_t^1 (1-s+k_2)\sigma(s)ds \\ &+ \textstyle\sum_{i=1}^m W_i'(t), \end{aligned}$$

where

$$p'(t) = \alpha\left\{\int_0^1 \rho_1(s)ds - \int_0^1 \rho_2(s)ds\right\},\ p''(t) = 0, \text{ for a.e. } t \in J$$

$$W_i'(t) = -\alpha\begin{cases} -I_i(y(t_i)) - (1-t_i+k_2)\overline{I}_i(y(t_i)), & 0 \le t \le t_i, \\ -[I_i(y(t_i)) - (t_i+k_2)\overline{I}_i(y(t_i))], & t_i \le t \le 1, \end{cases}$$

and

$$W_i''(t) = 0,\ \ i = 1,...,m, \text{ for each } t \in J.$$

Thus

$$\begin{aligned} y''(t) &= p''(t) - \alpha\left[\textstyle\int_0^t (k_1+s)\sigma(s)ds\right]' + \alpha\left[\int_t^1 (1-s+k_2)\sigma(s)ds\right]' \\ &= -\alpha(1+k_1+k_2)\sigma(t) = \sigma(t), \end{aligned}$$

then (3.4) is satisfied. Now, we show the conditions (3.5)-(3.8) hold. From (3.1), we have

$$\begin{aligned} y(0) &= p(0) + \textstyle\int_0^1 G(0,s)\sigma(s)ds + \sum_{i=1}^m W_i(0) \\ &= -\alpha(1+k_2)\textstyle\int_0^1 \rho_1(s)ds - \alpha k_1\int_0^1 \rho_2(s)ds \\ &+ \alpha k_1\textstyle\int_0^1 (1-s+k_2)\sigma(s)ds \\ &- \alpha\textstyle\sum_{i=1}^m k_1[-I_i(y(t_i)) - (1-t_i+k_2)\overline{I}_i(y(t_i))], \end{aligned}$$

and

$$\begin{array}{rcl} y'(0) & = & p'(0)+\alpha\int_0^1(1-s+k_2)\sigma(s)ds \\ & + & \alpha\sum_{i=1}^m[I_i(y(t_i))+(1-t_i+k_2)\overline{I}_i(y(t_i))] \\ & = & -\alpha\int_0^1\rho_2(s)ds+\alpha\int_0^1\rho_1(s)ds+\alpha\int_0^1(1-s+k_2)\sigma(s)ds \\ & + & \alpha\sum_{i=1}^m[I_i(y(t_i))+(1-t_i+k_2)\overline{I}_i(y(t_i))]. \end{array}$$

Then

$$\begin{array}{rcl} y(0)-k_1y'(0) & = & -\alpha(1+k_2)\int_0^1\rho_1(s)ds-\alpha k_1\int_0^1\rho_2(s)ds \\ & + & k_1\alpha\int_0^1\rho_2(s)ds-\alpha k_1\int_0^1\rho_1(s)ds \\ & = & \int_0^1\rho_1(s)ds, \end{array}$$

so (3.7) holds. Similarly, we will have

$$y(1)+k_2y'(1)=\int_0^1\rho_2(s)ds.$$

To show that (3.5) holds, we see for $i=1,\dots,m$,

$$\begin{array}{rcl} y(t_i) & = & p(t_i)+\int_0^1G(t_i,s)\sigma(s)ds+\sum_{i=1}^mW_i(t_i) \\ & = & p(t_i)+\int_0^1G(t_i,s)\sigma(s)ds \\ & - & \alpha\sum_{i=1}^m(k_1+t_i)[-I_i(y(t_i))-(1-t_i+k_2)\overline{I}_i(y(t_i))], \end{array}$$

and

$$\begin{array}{rcl} y(t_i^+) & = & p(t_i^+)+\int_0^1G(t_i^+,s)\sigma(s)ds+\sum_{i=1}^mW_i(t_i^+) \\ & = & p(t_i)+\int_0^1G(t_i,s)\sigma(s)ds \\ & - & \alpha\sum_{i=1}^m(1-t_i+k_2)[-I_i(y(t_i))-(t_i+k_1)\overline{I}_i(y(t_i))]. \end{array}$$

Then

$$\begin{array}{rcl} y(t_i^+)-y(t_i^-) & = & -\alpha\sum_{i=1}^m(k_2-t_i+1)[I_i(y(t_i))-(t_i+k_1)\overline{I}_i(y(t_i)) \\ & + & \alpha\sum_{i=1}^m(k_1+t_i)[-I_i(y(t_i))-(1-t_i+k_2)\overline{I}_i(y(t_i))] \\ & = & -\alpha(1+k_1+k_2)I_i(y(t_i)) \\ & = & I_i(y(t_i)). \end{array}$$

Similarly, we find

$$y'(t_i^+)-y'(t_i^-)=\overline{I}_i(y(t_i)),\ \text{for each } i=1,\dots,m,$$

so, the lemma is proved. □

Let us introduce the following hypotheses which are assumed hereafter:

(H1) The function $F:[0,1]\times\mathbb{R}\to P_{cp,c}(\mathbb{R})$ is L^1-Carathéodory.

(H2) There exist constants $c_1,\ c_2>0$ with $\overline{M}:=1-\frac{(c_1+c_2)(2+k_1+k_2)}{1+k_1+k_2}>0$ such that

$$|I_i(u)|\le c_1|u|,\ \text{and } |\overline{I}_i(u)|\le c_2|u|,\ \text{for each } u\in\mathbb{R},\ i=1,\dots,m.$$

(H3) There exist functions $g_1,\ g_2:[0,\infty)\to[0,\infty)$ continuous, nondecreasing and $q_1,\ q_2\in L^1(J,\mathbb{R}^+)$ such that

$$|h_1(t,u)|\le q_1(t)g_1(|u|) \text{ for each } u\in\mathbb{R} \text{ and for } t\in J,$$

$$|h_2(t,u)|\le q_2(t)g_2(|u|) \text{ for each } u\in\mathbb{R} \text{ and for } t\in J.$$

(H4) There exist a continuous non-decreasing function $g : [0,\infty) \longrightarrow (0,\infty)$ and a function $q \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,u)\| \le q(t)g(|u|) \text{ for each } (t,u) \in [0,1]\times\mathbb{R}.$$

(H5) There exists a number $M > 0$ such that

$$\frac{\overline{M}M}{g_1(M)\int_0^1 q_1(s)ds + g_2(M)\int_0^1 q_2(s)ds + G^* g(M)\int_0^1 q(s)ds} > 1, \tag{3.9}$$

where

$$G^* = \sup_{(t,s)\in J^2} |G(t,s)|.$$

Theorem 3.3. *Suppose that the hypotheses (H1)-(H5) are satisfied, then the problem (1.1)-(1.5) has at least one solution.*

Proof. Transform the problem (1.1)-(1.5) into a fixed point one. Consider the operator

$$N(y) = \{ f \in PC : \; f(t) = P(t) + \int_0^1 G(t,s)v(s)ds + \sum_{i=1}^m W_i(t), \; v \in S_{F,y}\},$$

where

$$P(t) = -\alpha\left\{(1-t+k_2)\int_0^1 h_1(s,y(s))ds + (k_1+t)\int_0^1 h_1(s,y(s))ds\right\}, \tag{3.10}$$

$G(t,s)$ and $W_i(t), \; i = 1,\ldots,m$ are the functions defined by (3.2) and (3.3), respectively.

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)–(1.5). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in PC$.*

Indeed, if $f_1, \; f_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in [0,1]$ we have

$$f_j(t) = P(t) + \int_0^1 G(t,s)v_j(s)ds + \sum_{i=1}^m W_i(t), \; j = 1,2.$$

Let $0 \le d \le 1$. Then, for each $t \in [0,1]$, we have

$$(d f_1 + (1-d) f_2)(t) = P(t) + \int_0^1 G(t,s)[dv_1(s) + (1-d)v_2(s)]ds + \sum_{i=1}^m W_i(t).$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$d f_1 + (1-d) f_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in PC.*

Let $B_r=\{y\in PC:\|y\|_\infty\le r\}$ be a bounded set in PC and $y\in B_r$. Then for each $f\in N(y)$, there exists $v\in S_{F,y}$ such that

$$f(t)=P(t)+\int_0^1 G(t,s)v(s)ds+\sum_{i=1}^m W_i(t).$$

From (H2)-(H4) we have

$$\begin{array}{rcl} |f(t)| & \le & |P(t)|+\int_0^1|G(t,s)||v(s)|ds+\sum_{0\le t\le t_i}|W_i(t)|+\sum_{t_i\le t\le 1}|W_i(t)| \\ & \le & p_r+G^*g(r)\int_0^1 q(s)ds+\frac{k_1+1}{1+k_1+k_2}c_1|y(t_i)|+\frac{k_2+1}{1+k_1+k_2}c_2|y(t_i)| \\ & + & \frac{1+k_2}{1+k_1+k_2}c_1|y(t_i)|+\frac{1+k_1}{1+k_1+k_2}c_2|y(t_i)| \\ & \le & p_r+G^*g(r)\int_0^1 q(s)ds+\frac{c_1r(2+k_1+k_2)}{1+k_1+k_2}+\frac{c_2r(2+k_1+k_2)}{1+k_1+k_2}=:l, \end{array}$$

where

$$p_r=\frac{1}{1+k_1+k_2}\left\{(1+k_2)g_1(r)\int_0^1 q_1(s)ds+(1+k_1)g_2(r)\int_0^1 q_2(s)ds\right\}.$$

Step 3: N maps bounded sets into equicontinuous sets of PC.

Let $\tau_1,\tau_2\in[0,1]$, $\tau_1<\tau_2$, B_r be a bounded set of PC as in Step 2 and $y\in B_r$. For each $f\in N(y)$:

$$\begin{array}{rcl} |f(\tau_2)-f(\tau_1)| & \le & |P(\tau_2)-P(\tau_1)| \\ & + & \int_0^1|G(\tau_2,s)-G(\tau_1,s)||v(s)|ds \\ & + & \sum_{i=1}^m|W_i(\tau_2)-W_i(\tau_1)| \\ & \le & -\alpha(\tau_1-\tau_2)\left[\int_0^1|h_1(s,y(s))|ds+\int_0^1|h_2(s,y(s))|ds\right] \\ & + & g(r)\int_0^1 q(s)|G(\tau_2,s)-G(\tau_1,s)|ds \\ & + & \sum_{i=1}^m|W_i(\tau_2)-W_i(\tau_1)|. \end{array}$$

The right-hand side of the above inequality tends to zero as $\tau_2-\tau_1\to 0$.
As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N:PC\longrightarrow PC$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n\to y_*$, $h_n\in N(y_n)$ and $f_n\to f_*$. We need to show that $f_*\in N(y_*)$.
$f_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in[0,1]$,

$$f_n(t)=P_n(t)+\int_0^1 G(t,s)v_n(s)ds+\sum_{i=1}^m W_{n_i}(t),$$

where

$$p_n(t)=-\alpha\left\{(1-t+k_2)\int_0^1 h_1(s,y_n(s))ds+(k_1+t)\int_0^1 h_2(s,y_n(s))ds\right\},$$

and

$$W_{n_i}(t)=-\alpha\begin{cases}(k_1+t)[-I_i(y_n(t_i))-(1-t_i+k_2)\overline{I}_i(y_n(t_i))], & 0\le t\le t_i,\\ (1-t+k_2)[I_i(y_n(t_i))-(t_i+k_1)\overline{I}_i(y_n(t_i))], & t_i\le t\le 1.\end{cases}$$

We must show that there exists $f_* \in S_{F,y_*}$ such that, for each $t \in [0,1]$,

$$f_*(t) = P_*(t) + \int_0^1 G(t,s)v_*(s)ds + \sum_{i=1}^{m} W_i(t, y_*(t_i)).$$

Clearly we have

$$\|(f_n - P_n - W_{n_i}) - (f_* - P_* - W_{*_i}\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

Consider the continuous linear operator

$$\Gamma : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^1 G(t,s)v(s)ds.$$

From Lemma 2.5, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$\Big(f_n(t) - P_n(t)\Big) \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.5 that

$$f_*(t) = P_*(t) + \int_0^1 G(t,s)v_*(s)ds + \sum_{i=1}^{m} W_i(t, y_*(t_i))$$

for some $v_* \in S_{F,y_*}$.

Step 5: *A priori bounds on solutions.*

Let y be such that $y \in \lambda N(y)$ for $\lambda \in (0,1)$. Then there exists $v \in S_{F,y}$ such that, for each $t \in [0,1]$,

$$y(t) = \lambda \left[p(t) + \int_0^1 G(t,s)v(s)ds + \sum_{i=1}^{m} W_i(t) \right],$$

so

$$|y(t)| \le |p(t)| + G^* \int_0^1 q(s)g(|y(s)|)ds + \sum_{i=1}^{m} |W_i(t)|.$$

(H3) implies that

$$\begin{array}{rcl} |p(t)| & \le & -\alpha \left\{ (1+k_2)\int_0^1 q_1(s)g_1(|y(s)|)ds + (k_1+1)\int_0^1 q_2(s)g_2(|y(s)|)ds \right\} \\ & \le & -\alpha \left\{ (1+k_2)g_1(\|y\|_{PC})\int_0^1 q_1(s)ds + (k_1+1)g_2(\|y\|_{PC})\int_0^1 q_2(s)ds \right\} \\ & \le & g_1(\|y\|_{PC})\int_0^1 q_1(s)ds + g_2(\|y\|_{PC})\int_0^1 q_2(s)ds. \end{array}$$

Using hypothesis (H2), we have

$$\begin{array}{rcl} \sum_{i=1}^{m} |W_i(t, y(t_i))| & \le & \sum_{0 \le t \le t_i} |W_i(t, y(t_i))| + \sum_{t_i \le t \le 1} |W_i(t, y(t_i))| \\ & \le & \frac{k_1+1}{1+k_1+k_2} c_1 |y(t_i)| + \frac{k_2+1}{1+k_1+k_2} c_2 |y(t_i)| \\ & + & \frac{1+k_2}{1+k_1+k_2} c_1 |y(t_i)| + \frac{1+k_1}{1+k_1+k_2} c_2 |y(t_i)| \\ & \le & \|y\|_{PC} \left[\frac{c_1(2+k_1+k_2)}{1+k_1+k_2} + \frac{c_2(2+k_1+k_2)}{1+k_1+k_2} \right]. \end{array}$$

Then

$$\begin{array}{rcl}\|y\|_{PC} & \leq & g_1(\|y\|_{PC})\int_0^1 q_1(s)ds + g_2(\|y\|_{PC})\int_0^1 q_2(s)ds \\ & + & G^* g(\|y\|_{PC})\int_0^1 q(s)ds + \|y\|_{PC}\left[\frac{(c_1+c_2)(2+k_1+k_2)}{1+k_1+k_2}\right],\end{array}$$

this implies that

$$\frac{\overline{M}\|y\|_{PC}}{g_1(\|y\|_{PC})\int_0^1 q_1(s)ds + g_2(\|y\|_{PC})\int_0^1 q_2(s)ds + G^* g(\|y\|_{PC})\int_0^1 q(s)ds} \leq 1,$$

then by (H5), we can affirm that there a constant $M > 0$ such that

$$\|y\|_{PC} \neq M.$$

Set

$$U = \{y \in PC : \|y\|_{PC} < M\}.$$

$N : \overline{U} \to PC$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [20], we deduce that N has a fixed point in U which is a solution to problem (1.1)-(1.5). □

We present now a result for the problem (1.1)-(1.5) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for multivalued map given by Covitz and Nadler [14]. We need the following hypotheses:

(H6) $F : [0,1] \times \mathbb{R} \longrightarrow P_{cp}(\mathbb{R})$ has the property that $F(\cdot,u) : [0,1] \to P_{cp}(\mathbb{R})$ is measurable for each $u \in PC$.

(H7) There exists $l \in L^1([0,1],\mathbb{R}^+)$ such that

$$H_d(F(t,u),F(t,\overline{u})) \leq l(t)|u-\overline{u}| \text{ for a.e. } t \in [0,1] \text{ and } u,\ \overline{u} \in \mathbb{R}$$

and

$$d(0,F(t,0)) \leq l(t) \text{ for a.e. } t \in [0,1].$$

(H8) There exist constants c, $\overline{c} > 0$ such that

$$|h_1(t,u) - h_1(t,v)| \leq c|u-v|, \text{ for each } u,\ v \in \mathbb{R}, \text{ and each } t \in J,$$

$$|h_2(t,u) - h_2(t,v)| \leq \overline{c}|u-v|, \text{ for each } u,\ v \in \mathbb{R} \text{ and each } t \in J.$$

(H9) There exist constants d, $\overline{d} > 0$ such that

$$|I_i(u) - I_i(v)| \leq d|u-v| \text{ for each } u,\ v \in \mathbb{R},$$

$$|\overline{I}_i(u) - \overline{I}_i(v)| \leq \overline{d}|u-v| \text{ for each } u,\ v \in \mathbb{R}.$$

Theorem 3.4. *Assume that (H6)-(H9) are satisfied. If*

$$\left[\left(\frac{c(1+k_2)+\overline{c}(1+k_1)}{1+k_1+k_2}\right)+G^*\int_0^1 l(s)ds+\left(\frac{(d+\overline{d})(2+k_1+k_2)}{1+k_1+k_2}\right)\right]<1, \tag{3.11}$$

then the BVP (1.1)-(1.5) has at least one solution.

Proof. For each $y\in PC$, the set $S_{F,y}$ is nonempty since by (H6), F has a measurable selection (see [13], Theorem III.6). We shall show that N satisfies the assumptions of Lemma 2.8. The proof will be given in two steps.

Step 1: $N(y)\in P_{cl}(PC)$ *for each* $y\in PC$.

Indeed, let $(y_n)_{n\geq 0}\in N(y)$ such that $y_n\longrightarrow\tilde{y}$ in PC. Then, $\tilde{y}\in PC$ and there exists $v_n\in S_{F,y}$ such that, for each $t\in[0,1]$,

$$y_n(t)=P(t)+\int_0^1 G(t,s)v_n(s)ds+\sum_{i=1}^m W_i(t).$$

Using the fact that F has compact values and from (H7), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1_w([0,1],\mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem ([31]) implies that v_n converges strongly to v in $L^1([0,1],\mathbb{R})$ and hence $v\in S_{F,y}$. Then, for each $t\in[0,1]$,

$$y_n(t)\longrightarrow\tilde{y}(t)=p(t)+\int_0^1 G(t,s)v(s)ds+\sum_{i=1}^m W_i(t).$$

So, $\tilde{y}\in N(y)$.

Step 2: *There exists* $\gamma<1$ *such that*

$$H_d(N(y),N(\overline{y}))\leq\gamma\|y-\overline{y}\|_{PC}\ \textit{for each}\ y,\overline{y}\in PC.$$

Let $y,\overline{y}\in PC$ and $f_1\in N(y)$. Then, there exists $v_1(t)\in F(t,y(t))$ such that for each $t\in[0,1]$

$$f_1(t)=P(t)+\int_0^1 G(t,s)v_1(s)ds+\sum_{i=1}^m W_i(t).$$

From (H7) it follows that

$$H_d(F(t,y(t)),F(t,\overline{y}(t)))\leq l(t)|y(t)-\overline{y}(t)|.$$

Hence, there exists $w\in F(t,\overline{y}(t))$ such that

$$|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|,\ t\in[0,1].$$

Consider $U:[0,1]\to\mathcal{P}(\mathbb{R})$ given by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable (see Proposition III.4 in [13]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t, \bar{y}(t))$, and for each $t \in [0,1]$,

$$|v_1(t) - v_2(t)| \leq l(t\|y(t) - \bar{y}(t)|.$$

Let us define for each $t \in [0,1]$

$$f_2(t) = \bar{P}(t) + \int_0^1 G(t,s)v_2(s)ds + \sum_{i=1}^m \bar{W}_i(t),$$

where

$$\bar{p}(t) = -\alpha\left\{(1-t+k_2)\int_0^1 h_1(s,\bar{y}(s))ds + (k_1+t)\int_0^1 h_2(s,\bar{y}(s))ds\right\},$$

and

$$\bar{W}_i(t) = -\alpha\begin{cases} (k_1+t)[-I_i(\bar{y}(t_i)) - (1-t_i+k_2)\bar{I}_i(\bar{y}(t_i))], & 0 \leq t \leq t_i, \\ (1-t+k_2)[I_i(\bar{y}(t_i)) - (t_i+k_1)\bar{I}_i(\bar{y}(t_i))], & t_i \leq t \leq 1. \end{cases}$$

Then we have

$$\begin{aligned} |f_1(t) - f_2(t) &\leq |p(t) - \bar{p}(t)| + \textstyle\int_0^1 |G(t,s)||v_1(s) - v_2(s)|ds \\ &+ |\textstyle\sum_{i=1}^m W_i(t) - \sum_{i=1}^m \bar{W}_i(t)|. \end{aligned}$$

By (H8) we have

$$|p(t) - \bar{p}(t)| \leq \left(\frac{c(1+k_2) + \bar{c}(1+k_1)}{1+k_1+k_2}\right)\|y - \bar{y}\|_{PC},$$

and from (H9)

$$|\sum_{i=1}^m W_i(t) - \sum_{i=1}^m \bar{W}_i(t)| \leq \left(\frac{(d+\bar{d})(2+k_1+k_2)}{1+k_1+k_2}\right)\|y - \bar{y}\|_{PC}.$$

It follows then

$$\begin{aligned} |f_1(t) - f_2(t)| &\leq \left(\tfrac{c(1+k_2)+\bar{c}(1+k_1)}{1+k_1+k_2}\right)\|y-\bar{y}\|_{PC} \\ &+ \|y-\bar{y}\|_{PC} G^* \textstyle\int_0^1 l(s)ds \\ &+ \left(\tfrac{(d+\bar{d})(2+k_1+k_2)}{1+k_1+k_2}\right)\|y-\bar{y}\|_{PC} \\ &\leq \left[\left(\tfrac{c(1+k_2)+\bar{c}(1+k_1)}{1+k_1+k_2}\right) + G^* \textstyle\int_0^1 l(s)ds \right. \\ &+ \left.\left(\tfrac{(d+\bar{d})(2+k_1+k_2)}{1+k_1+k_2}\right)\right]\|y-\bar{y}\|_{PC}. \end{aligned}$$

By an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$\begin{aligned} H_d(N(y) - N(\bar{y})) &\leq \left[\left(\tfrac{c(1+k_2)+\bar{c}(1+k_1)}{1+k_1+k_2}\right) + G^* \textstyle\int_0^1 l(s)ds \right. \\ &+ \left.\left(\tfrac{(d+\bar{d})(2+k_1+k_2)}{1+k_1+k_2}\right)\right]\|y-\bar{y}\|_{PC}. \end{aligned}$$

So, by (3.11), N is a contraction and thus, by Lemma 2.8, N has a fixed point y which is solution to (1.1)–(1.5). The proof is complete. □

In this part, by using the nonlinear alternative of Leray Schauder type combined with the selection theorem of Bresssan and Colombo for semi-continuous maps with decomposable values, we shall establish an existence result for the problem (1.1)-(1.5). We need the following hypothesis:

(H10) $F:[0,1]\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that:
a) $(t,u)\mapsto F(t,u)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable;
b) $u\mapsto F(t,u)$ is lower semi-continuous for each $t\in[0,1]$.

The following lemma is of great importance in the proof of our next result.

Lemma 3.5. *[17] Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assume $(H4)$ and $(H10)$ hold. Then F is of l.s.c. type.*

Theorem 3.6. *Assume that (H2)-(H5) and (H10) hold. Then the BVP (1.1)–(1.5) has at least one solution.*

Proof. Note that (H4), (H10) and Lemma 3.5 imply that F is of l.s.c. type. Then from Lemma 2.7, there exists a continuous function $f:PC\to L^1([0,1],\mathbb{R})$ such that $f(y)\in\mathcal{F}(y)$ for all $y\in PC$. Consider the problem

$$y''(t)=f(y)(t),\ \text{for a.e. } t\in J:=[0,1],\ t\neq t_i,\ i=1,\ldots,m, \tag{3.12}$$

$$y(t_i^+)-y(t_i)=I_i(y(t_i^-)),\ \ i=1,\ldots,m, \tag{3.13}$$

$$y'(t_i^+)-y'(t_i)=\overline{I}_i(y(t_i^-)),\ \ i=1,\ldots,m, \tag{3.14}$$

$$y(0)-k_1y'(0)=\int_0^1 h_1(s,y(s))ds, \tag{3.15}$$

$$y(1)-k_2y'(1)=\int_0^1 h_2(s,y(s))ds. \tag{3.16}$$

It is clear that if $y\in PC\cap AC^1((0,1),\mathbb{R})$ is a solution of (3.12)–(3.16), then y is a solution to the problem (1.1)–(1.5). Consider the operator $\widetilde{N}:PC\to PC$ defined by:

$$\widetilde{N}(y)(t)=P(t)+\int_0^1 G(t,s)f(y)(s)ds+\sum_{i=1}^m W_i(t).$$

We can easily show that $\widetilde{N}$ is continuous and completely continuous. The remainder of the proof is similar to that of Theorem 3.1. □

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