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Simple MCD-Graphs on 28 Vertices

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Abstract Let S_n be the set of simple graphs on n vertices in which no two cycles have the same length. A graph G is called a simple MCD-graph if there exists no graph G' in S_n with $|E(G')|>|E(G)|$. A simple MCD-graph G is called a 2-connected simple MCD-graph if G is a 2-connected graph. In this paper we prove that there does not exist a 2-connected simple MCD-graph on 28 vertices containing a subgraph homeomorphic to K_4. Consequently the following theorem stated in Discrete Math. 126(1994) 325~338 is flawless: "There exists a 2-connected simple MCD-graph on n vertices containing a subgraph homeomorphic to K_4 if and only if $n\in\{10, 11, 14, 15, 16, 21, 22\}$."

Keywords cycle distributed graph; MCD-graph; simple MCD-graph

1 Introduction

Let S_n be the set of simple graphs on n vertices in which no two cycles have the same length. A graph G in S_n is called a simple MCD-graph if there exists no graph G' in S_n with $|E(G')|>|E(G)|$. A simple MCD-graph G is called a 2-connected simple MCD-graph if G is a 2-connected graph.

In this paper we assume that G is a 2-connected simple graph containing exactly one longest cycle denoted by C, and that all vertices in V(G)-V(C) and all edges in E(G)-E(C) are drawn inside the bounded region of C. The paths with two and only two end vertices lying on C are called bridges.

Two bridges b_1 and b_2 are said to be skew if they have no common vertex and they cross each other.

Two bridges are said to be parallel if they are internally disjoint and not skew.

Two bridges b_1 and b_2 are said to be independent if there exists a bridge b such that b_1, b and b_2 are parallel, and b_1 and b_2 are separated by b (though they may have end vertices in common).

A family of parallel bridges is said to be dependent if no two bridges of it are independent.

A family of parallel bridges (at least three bridges) is said to be independent if no three

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bridges of it are dependent.

A bridge b is said to be a skew bridge if there exists a bridge b' such that b and b' are skew.

A cycle C' is said to be a skew cycle if C' contains two bridges which are skew.

If C' is not a skew cycle of G, then the plane is partitioned into three disjoint sets called the interior, exterior and itself of C'. We shall express the interior and exterior of C' respectively, by intC' and extC'. If a bridge b lies in intC', then we write $b \in \text{int}C'$.

For any two vertices u and v of C, let $C[u, v]$ denote the (u, v)-path which follows the clockwise orientation of C; similarly we shall use the symbol $C(u, v)$ to denote the path $C[u, v] - \{u, v\}$. Let $b = (v_1, v_2)$ be a bridge of G and let $C' = b \cup C[v_1, v_2]$, $C'' = b \cup C[v_2, v_1]$, $k = \min\{|V(C')|, |V(C'')|\}$. If $|V(C')| \neq |V(C'')|$, then the k-cycle in $\{C', C''\}$ is said to be a side cycle of G. If C_1 is a cycle in $\{C', C''\}$, then the other cycle in $\{C', C''\}$ is denoted by $\overline{C}_1$.

Let $m(G)$ denote the number of cycles in a graph G. For all other notations and terminologies not explained here we refer to [1~4].

The main result of this paper is

Main Theorem If G is a 2-connected simple MCD-graph on 28 vertices, then G does not contain a subgraph homeomorphic to K_4.

2 Some lemmas

In the coming discussion, we always denote by $|C'|$ the length of C' (where C' is a cycle or a path).

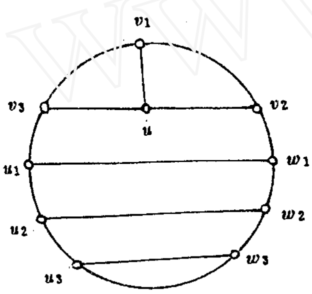

Fig. 1

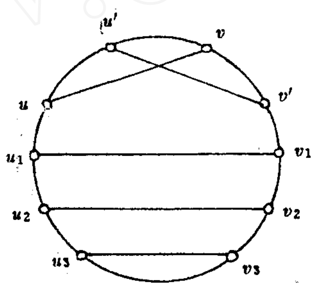

Fig. 2

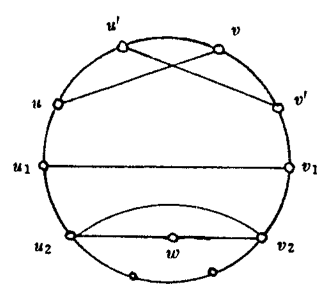

Fig. 3

Lemma 1 Let G be a 2-connected simple graph on 28 vertices such that (1) $|C| = 27$; (2) there exists exactly one vertex $u \in \text{int}C$ such that $N(u) = \{v_1, v_2, v_3\}$; (3) G-u has exactly three bridges u_1w_1, u_2w_2 and u_3w_3 which are contained in intC' (where $C' = v_3uv_2 \cup C[v_2, v_3]$); (4) the four bridges v_3uv_2, u_1w_1, u_2w_2 and u_3w_3 are independent (though they may have end vertices in common) (See Fig. 1.). Then G contains two cycles having the same length.

Proof Clearly G has a subgraph homeomorphic to K_4 and $|E(G)| - |V(G)| = 5$. By Proposition 3.1 in [3], $m(G) \geqslant 25$. Suppose that G contains no two cycles having the same length, then by the first condition of the lemma, G must contain exactly one k-cycle for each integer $3 \leqslant k \leqslant 27$. Let $p = 27$.

Let C_1, C_2, C_3, C_4, C_5 and C_6 be the cycles $v_1uv_3 \cup C[v_3, v_1]$, $v_2uv_1 \cup C[v_1, v_2]$, $v_2uv_3 \cup C$

$[v_3, v_2]$, $u_3w_3 \cup C[w_3, u_3]$, $u_2w_2 \cup C[w_2, u_2]$ and $u_1w_1 \cup C[w_1, u_1]$, respectively (See Fig. 1). Let $x_1 = |C(v_3, v_1)|$ and $x_2 = |C(v_1, v_2)|$. Clearly $x_1 \geqslant 2$ and $x_2 \geqslant 2$. Since $|C_1| \neq |C_2|$, we may assume that $x_1 < x_2$. Clearly $x_1 \leqslant 3$. For otherwise, $|\bar{C}_i| \leqslant p-3$ for $i \in \{1, 2\}$, and then we can find $|C_4| = 3$ and $|C_5| = 4$, respectively, by considering the (p−1)-cycle and the (p−2)-cycle of G, and hence G has two 3-cycles (in this case $u_2 = u_3$ or $w_2 = w_3$), a contradiction. There are two possible cases only.

Case 1 $x_1 = 2$. Clearly $|\bar{C}_1| = p-1$. If $x_2 \geqslant 6$, then $|\bar{C}_2| \leqslant p-5$. In this case we can find $|C_4| = 4$ and $|C_5| = 6$, respectively, by considering the $(p-2)$-cycle and the $(p-4)$-cycle of G, and hence G has two 4-cycles, a contradiction. If $x_2 = 3$, then $\bar{C}_2 = p-2$. In this case we can find $|C_4| = 5$ by considering the $(p-3)$-cycle of G, and hence G has two $(p-5)$-cycles, a contradiction. If $x_2 = 4$, then $|\bar{C}_2| = p-3$. In this case we can find $|C_4| = 4$ by considering the $(p-2)$-cycle of G, and hence G has two $(p-3)$-cycles, a contradiction. Therefore $x_2 = 5$. In this case $|\bar{C}_2| = p-4$. We can find $|C_4| = 4$ by considering $(p\text{-}2)$-cycle of G. It is easily seen that $G - \{u_1w_1, u_2w_2\}$ has one $(p-i)$-cycle for $i = 1, 2, 3, 4, 6, 7, 9$ but no $(p-5)$-cycle.

Furthermore we can find $|C_5| = 7$ by considering the $(p-5)$-cycle, and hence G has two 5-cycles, a contradiction.

Case 2 $x_1 = 3$. Clearly $|\bar{C}_1| = p-2$. If $x_2 \geqslant 6$, then $|\bar{C}_2| \leqslant p-5$. In this case we can find $|C_4| = 3$ and $|C_5| = 6$, respectively, by considering the $(p-1)$-cycle and the $(p-4)$-cycle, and hence G has two 6-cycles, a contradiction. If $x_2 = 4$, then $|\bar{C}_2| = p-3$. In this case we can find $|C_4| = 3$ by considering the $(p-1)$-cycle, and hence G has two $(p-3)$-cycles, a contradiction. Therefore $x_2 = 5$. In this case $|\bar{C}_2| = p-4$. We can find $|C_4| = 3$ by considering the $(p-1)$-cycle. Clearly $G - \{u_1w_1, u_2w_2\}$ has one $(p-i)$-cycle for $i = 1, 2, 3, 4, 5, 8, 9$ but no $(p-6)$-cycle. Furtheremore we can find $|C_5| = 8$ by considering the $(p-6)$-cycle, and hence G has two 8-cycles, again a contradiction. □

In the coming discussion, we assume that B is the set of bridges of G, X is the set of skew bridges of G. Let $B' \subseteq B$. If C' is an k-cycle containing at least one bridge such that every bridge contained in C' is an element of B', then we say that C' is covered by B' and we write $k \in C(B')$.

Lemma 2 Let G be a 2-connected simple graph on 28 vertices such that (1) $|C| = 27$; (2) G has exactly one pair skew bridges (u, v) and (u', v') such that C is separated by the four vertices u, u', v, v' into four paths $C[u, u']$, $C[u', v]$, $C[v, v']$ and $C[v', u]$; (3) G has exactly five bridges; (4) the end vertices of the three bridges (u_1, v_1), (u_2, v_2), (u_3, v_3) lie on $C[v', u]$ and there exist four bridges in G such that they are independent (though they may have end vertices in common) (See Fig. 2). Then G contains two cycles having the same length.

Proof Clearly G has a subgraph homeomorphic to K_4 and $j = |E(G)| - |V(G)| = 5$, by Proposition 3. 1 in [3], $m(G) \geqslant 25$. Suppose that G contains no two cycles having the same length, then by the first condition of the lemma, G must contain exactly one k-cycle for each integer $3 \leqslant k \leqslant 27$. Let $p = 27$.

Let C_3 be the 3-cycle of G. Suppose that C_3 is not a side cycle. There are only two cases by considering the $(p-1)$-cycle of G.

Case 1 G has one side cycle of length 5 containing the unique bridge b^* of length 2. Suppose that C_3 contains b^* (See Fig. 3). Then another bridge contained in C_3 and b^* have the same end vertices. Let $b^*=u_2wv_2$ and $b'=u_2v_2$. Clearly b^* is not a skew bridge. Let $C_4=b'\cup C[v_2, u_2]$, $C_5=b^*\cup C[v_2, u_2]$. Clearly C_i is a i-cycle for $i=4, 5$, and $\overline{C}_4$ is the $(p-2)$-cycle, $\overline{C}_5$ is the $(p-1)$-cycle. It follows that the $(p-3)$-cycle must be a skew cycle containing two skew bridges and containing no other bridges. Clearly another skew cycle containing two skew bridges and containing no other bridges is the 7-cycle. Let C_6 be the 6-cycle, then C_6 is not a side cycle. Let $P=C[u_2, u_1]\cup u_1v_1\cup C[v_1, v_2]$, then C_6 does not contain the path P. Let $C'=C[u_1, u']\cup u'v'\cup C[v', v_1]\cup v_1u_1$ and $C''=C[u_1, u]\cup uv\cup C[v, v_1]\cup v_1u_1$, then $C_6=C'$ or $C_6=C''$. We may assume that $C_6=C'$. Then $C[u, u']$ must be a path of length 1. It follows that C'' is the 9-cycle. Let C_8 is the 8-cycle. Clearly C_8 does not contain the path P. Thus C_8 must be the side cycle $C[u, v]\cup vu$. In this case, $C[u', v']\cup v'u'$ is the 11-cycle and $u'v'\cup C[v', v_1]\cup v_1u_1\cup C[u_1, u]\cup uv\cup C[u', v]$ is the 12-cycle. It follows that the 10-cycle must contain the path P, and hence G has two cycles having the same length, a contradiction.

Therefore C_3 does not contain b^*. In this case there exists a subgraph $G^*\subseteq G$ such that G^* has one side cycle of length 3 and one side cycle of length 5. It follows that G^* has two (p^*-1)-cycles (where P^* is the length of the longest cycle in G^*), a contradiction.

Case 2 G has one skew cycle of length $p-1$. In this case there exists a subgraph $G^*\subseteq G$ such that G^* has one side cycle of length 3 and one pair skew bridges. It follows that G^* has two (P^*-1)-cycles, again a contradiction.

Similarly, we can find that the 4-cycle denoted by C_4 is a side cycle by considering the $(p-2)$-cycle.

Let $b_3(b_4)$ denote the bridge contained in $C_3(C_4)$. Clearly $b_4\notin X$. Let $b_4=(u_3, v_3)$, then $b_3\in X$. We assume that $b_3=(u, v)$.

Let $B_2=\{b_3, b_4\}$, then $C(B_2)=\{3, 4, p-1, p-2, p-3\}$. Two cases now arise, depending on whether the $(p-4)$-cycle C_{p-4} is a skew cycle or not.

Case 1 C_{p-4} is a skew cycle. We express by b_7 the other skew bridge. Let $B_3=\{b_3, b_4, b_7\}$. There are two possible subcases only.

Case 1.1 b_7 is the bridge of length 2. In this case $C(B_3)=\{3, 4, 9, 10, p-1, p-2, p-3, p-4, p-5, p-6, p-7\}$. Clearly the $(p-8)$-cycle is not a skew cycle, and hence G has one 10-cycle which is a side cycle. a contradiction.

Case 1.2 b_7 is a bridge of length 1. In this case $C(B_3)=\{3, 4, 7, 8, p-1, p-2, p-3, p-4, p-5, p-6, p-7\}$. Clearly the $(p-8)$-cycle denoted by C_{p-8} is not a skew cycle. Let b_{10} be a bridge contained in C_{p-8}, then $\overline{C}_{p-8}$ is clearly a side cycle. Let $B=\{b_3, b_4, b_7, b_{10}\}$.

Case 1.2.1 $b_4\in \mathrm{int}\overline{C}_{p-8}$. In this case b_{10} is a bridge of length 2. Thus $C(B_4)=C(B_3)\cup\{10, 12, p-8, p-9, p-12, p-13\}$. By considering the $(p-10)$-cycle, it follows that G has one 12-cycle which contains exactly one bridge of length 1, a contradiction.

Case 1.2.2 $b_4\notin \mathrm{int}\overline{C}_{p-8}$. In this case b_{10} is a bridge of length 1. Thus $C(B_4)=C(B_3)\cup$

$\{5, 6, 9, 10, p-8, p-10\}$. Clearly the $(p-9)$-cycle denoted by C_{p-9} is not a skew cycle, and hence $\overline{C}_{p-9}$ is a side cycle. Let b_{11} be a bridge contained in C_{p-9}. If $b_3 \in \mathrm{int} C_{p-9}$, then G has two $(p-10)$-cycles, a contradiction. Therefore $b_3 \in \mathrm{int}\overline{C}_{p-9}$. By considering the length of b_{11} is 1 or 2, it follows that G has two 7-cycles or two 9-cycles, a contradiction.

Case 2 C_{p-4} is not a skew cycle. Clearly $\overline{C}_{p-4}$ is a side cycle. Let b_6 be a bridge contained in $\overline{C}_{p-4}$. Clearly the length of any skew cycle is at least 9. Therefore $b_4 \in \mathrm{int}\overline{C}_{p-4}$, and hence b_6 is a bridge of length 2. Let $B_3 = \{b_3, b_4, b_6\}$. Then $C(B_3) = \{3, 4, 6, 8, p-1, p-2, p-3, p-4, p-5\}$. It is easy to see that the 5-cycle denoted by C_5 is neither a skew cycle nor a side cycle. Thus C_5 must be a cycle containing two bridges which are parallel. If C_5 contains a skew bridge, then G clearly has two 6-cycles, a contradiction. Therefore C_5 must contain b_6. Let b^* be a bridge contained in C_5 and $b^* \neq b_6$. Let $B_4 = \{b_3, b_4, b_6, b^*\}$. Then $C(B_4) = C(B_3) \cup \{5, 7, 9, p-7, p-8\}$. Clearly the $(p-6)$-cycle is not a skew cycle. It follows that G has one side cycle of length 8, which contains exactly one bridge of length 1, a contradiction. □

Replacing by "$|C| = 28$" the equality "$|C| = 27$" relative to Lemma 2 results in the following.

Lemma 3 (Omission)

Proof By contradiction. Suppose that G contains no two cycles having the same length. Then for each integer $k \in \{3, 4, \cdots, 28\}$, except one, G has exactly one k-cycle. In this proof, the k-cycle of G is denoted by C_k. If C_k is a side cycle, then the bridge contained in C_k is denoted by b_k. Let $p = 28$.

We first prove the following two facts.

Fact A Suppose $B' \subseteq B$ is such that for some $k(1 \leqslant k \leqslant p/2-1)$, $p-i \in C(B')$ for each $i \in \{1, 2, \cdots, k-1\}$ and $p-k \notin C(B')$, and G has C_{p-k} and C_{k+2}. Then either $B' = B$ or G has b_{k+2}.

Proof If $B' \neq B$, then C_{p-k} contains a bridge $b \notin B'$.

Suppose that C_{p-k} is a skew cycle. Consider two cases.

Case 1 C_{p-k} contains at least three bridges. Let C' be the cycle containing only the bridge b such that $X \subseteq \mathrm{int} C'$. Then $|C'| > |C_{p-k}|$ and thus G has two $|C'|$-cycles, which is false.

Case 2 C_{p-k} contains only two skew bridges. In this case, there exists a subgraph $G^* \subseteq G$ such that G^* contains one side cycle of length $k+2$ and two skew bridges. Thus G has two $(|V(G^*)|-k)$-cycles, which is false.

Suppose that C_{p-k} is not a skew cycle and that C_{p-k} also contains a bridge $b' \neq b$. Now by replacing the bridge b' by the arc between the end vertices of b', we obtain a cycle containing b and having length longer than $p-k$, and hence G has two cycles having the same length, which is also false.

Therefore C_{p-k} contains only one bridge b, and hence $b = b_{k+2}$.

Fact B Suppose that $B' \subseteq B$, G has no C_{p-t} and for some $k(1 \leqslant t < k < p/2-1)$, $p-i \in C(B')$ for each $i \in \{1, 2, \cdots, k-1\} - \{t\}$ and $p-k \notin C(B')$. Then either $B' = B$ or G has b_{k+2}.

Proof This is similar to the proof of Fact A.

Now we continue to prove this lemma.

Clearly G has C_3. For otherwise, G is a 4-UPC[1]-graph, this contradicts Theorem 4 in [4].

Suppose that G has no C_{p-1}, then G must have C_{p-2}. By Fact B, G has b_4. Note that $b_4 \notin X$. Let $B_1=\{b_4\}$, then $C(B_1)=\{4, p-2\}$. By Fact B, G has b_5. Let $B_2=\{b_4, b_5\}$. Clearly $b_5 \notin X$. For otherwise, $C(B_2)=\{4, 5, p-2, p-3, p-5\}$, by Fact B, G has b_6; it is easily seen that the length of any skew cycle is at least 10, and hence $b_4 \in \mathrm{int}C_6$; consequently G has two 4-cycles, which is false. Thus $b_4 \in \mathrm{int}C_5$ and $C(B_2)=\{3, 4, 5, p-2, p-3\}$. By Fact B, G has b_6. Clearly $b_5 \notin \mathrm{int}C_6$. Thus b_6 must be a skew bridge. Let $B_3=\{b_4, b_5, b_6\}$. Then $C(B_3)=\{3, 4, 5, 6, p-2, p-3, p-4, p-6, p-7\}$. By Fact B, G has b_7. Clearly b_7 is not a skew bridge. Thus $b_5 \in \mathrm{int}C_7$, and hence G has two 4-cycles, a contradiction. Therefore G must have C_{p-1}.

By Fact A, G has b_3. Let $B_1=\{b_3\}$, then $C(B_1)=\{3, p-1\}$. There are two possible cases only.

Case 1 $b_3 \in X$

Suppose that G has no C_4. Then G must have C_{p-2} and C_{p-2} must be a skew cycle. It follows that G has $b_5 \in X$. Let $B_2=\{b_3, b_5\}$. Then $C(B_2)=\{3, 5, 6, p-1, p-2, p-3\}$, where C_6 is a skew cycle. Since G has C_{p-4}, by Fact A, G has b_6 and C_6 is a side cycle, a contradiction. Therefore G has C_4.

Suppose that G has no C_{p-2}. By Fact B, G has b_5. Clearly $b_5 \notin X$. Let $B_2=\{b_3, b_5\}$, then $C(B_2)=\{3, 5, p-1, p-3, p-4\}$. By Fact B, G has b_7. Clearly $b_7 \notin X$. Also, the length of any skew cycle is at least 9. Hence $b_5 \in \mathrm{int}C_7$. Let $B_3=\{b_3, b_5, b_7\}$. Then $C(B_3)=\{3, 4, 5, 7, p-1, p-3, p-4, p-5, p-6\}$. By Fact B, G has b_9. Clearly $b_9 \notin X$. Thus $b_7 \in \mathrm{int}C_9$, and hence G has two 4-cycles, a contradiction. Therefore G has C_{p-2}.

By Fact A, G has b_4. Note that $b_4 \notin X$. Let $B_2=\{b_3, b_4\}$. Then $C(B_2)=\{3, 4, p-1, p-2, p-3\}$.

Suppose that G has no C_{p-4}. By Fact B, G has b_7. Clearly $b_7 \notin X$, and hence $b_4 \in \mathrm{int}C_7$. Let $B_3=\{b_3, b_4, b_7\}$. Then $C(B_3)=\{3, 4, 5, 7, p-1, p-2, p-3, p-5, p-6\}$. By Fact B, G has b_9. Clearly $b_9 \notin X$, and hence $b_7 \in \mathrm{int}C_9$. Thus G has two 4-cycles, a contradiction. Therefore G has C_{p-4}.

Suppose that G has no C_6. Then C_{p-4} must be a skew cycle. It follows that G has $b_7 \in X$. Let $B_3=\{b_3, b_4, b_7\}$. Then $C(B_3)=\{3, 4, 7, 8, p-1, p-2, p-3, p-4, p-5, p-6, p-7\}$. Since G has C_5, there exists a subgraph $G^* \subseteq G$ such that G^* has a side cycle of length 5 and contains two skew bridges. In this case G^* has two cycles of length $|V(G^*)|-4$, a contradiction. Therefore G has C_6.

By Fact A, G has b_6. Clearly $b_6 \notin X$, and hence $b_4 \in \mathrm{int}C_6$. Thus G has two 4-cycles, again a contradiction.

Case 2 $b_3 \notin X$

Suppose that G has no C_4. Then G must have C_{p-2} and C_6, and C_{p-2} is the maximum skew cycle and C_6 is the minimum skew cycle. Let $B_3 = X \cup \{b_3\}$. Then $p-i \in C(B_3)$ for $i=1, 2, 3$ and $p-4 \notin C(B_3)$. Since G has C_{p-4} and C_6, by Fact A, C_6 is a side cycle. Thus G has two 6-cycles, a contradiction. Therefore G has C_4.

Suppose that G has no C_{p-2}. By Fact B, G has b_5. Let $B_2 = \{b_3, b_5\}$. Consider two subcases.

Case 2. 1 $b_5 \in X$. In this case $C(B_2) = \{3, 5, p-1, p-3, p-4\}$. By Fact B, G has b_7. Suppose $b_7 \in X$. Then G has two $(p-6)$-cycles, a contradiction. Therefore $b_7 \notin X$. Thus $b_3 \in \mathrm{int}C_7$. Let $B_3 = \{b_3, b_5, b_7\}$. Then $C(B_3) = \{3, 5, 6, 7, p-1, p-3, p-4, p-5, p-8\}$. By Fact B, G has b_8. Clearly $b_8 \notin X$. Thus $b_7 \in \mathrm{int}C_8$, and hence G has two 3-cycles, a contradiction.

Case 2. 2 $b_5 \notin X$. In this case $C(B_2) = \{3, 4, 5, p-1, p-3\}$. By Fact B, G has b_6. Suppose that $b_6 \in X$. Let $B_3 = \{b_3, b_5, b_6\}$. Then $C(B_3) = \{3, 4, 5, 6, p-1, p-3, p-4, p-5, p-7\}$. By Fact B, G has b_8. Clearly $b_8 \notin X$. Thus $b_5 \in \mathrm{int}C_8$, and hence G has two 5-cycles, a contradiction. Therefore $b_6 \notin X$. Thus $b_5 \in \mathrm{int}C_6$, and hence G has two 3-cycles, a contradiction. Therefore G has C_{p-2}. By Fact A, G has b_4. *Clearly* $b_3 \in \mathit{int}C_4$. Consequently G has two 3-cycles, again a contradiction. □

3 The proof of main theorem

By Theorem 3. 1 in [2], G has at least 33 edges, i. e., $j = |E(G)| - |V(G)| \geqslant 5$. Suppose G has a subgraph homeomorphic to K_4. Then, by Proposition 3. 1 in [3], $m(G) \geqslant 25$, and hence $|C| \geqslant 27$. Since $|V(G)| = 28$, we must have $j = 5$, $m(G) \leqslant 26$, and $|C| = 27$ or 28. Let $p = |C|$.

Case 1 $p = 27$

In this case there exists exactly one vertex (say u) in $V(G)$-$V(C)$ and there are exactly 6 edges in $E(G)$-$E(C)$. Let $N(u)$ be the set of vertices of G adjacent to u, and $d(u) = |N(u)|$. Let T be the set of bridges of G-u. Clearly $|T| = 6$-$d(u)$. There are three possible subcases only.

Case 1. 1 $d(u) \geqslant 4$

Let $w \in N(u)$ and $G^* = G - uw$. Clearly G^* contains a subgraph homeomorphic to K_4 and $|E(G^*)| - |V(G^*)| = 4$, by Proposition 3. 1 in [3], $m(G^*) \geqslant 18$. Now we count the number of cycles in G containing uw : For each $v \in N(u) - \{w\}$, G has exactly two cycles containing $\{uv, uw\}$ but no other edges in $E(G) - E(C)$; for each $v \in N(u) - \{w\}$ and each $b \in T$, G has at least one cycle containing $\{uv, uw, b\}$ but no other edges in $E(G) - E(C)$. Thus it follows from $|T| \leqslant 2$ that the number of cycles in G containing uw is at least

$$\begin{aligned} & 2(d(u) - 1) + (d(u) - 1)|T| \\ &= (2 + |T|)(5 - |T|) \\ &= 10 + |T|(3 - |T|) \geqslant 10. \end{aligned}$$

Consequently, $m(G) \geqslant 28$, a contradiction.

Case 1. 2 $d(u) = 3$

In this case G must be a plane graph. For otherwise there exists $v \in N(u)$ such that $G^* = G$

$-uv$ contains a subgraph homeomorphic to K_4, and hence $m(G^*)\geqslant 18$. It is easily seen that the number of cycles in G containing uv is at least

$$2(d(u)-1)+|T|(d(u)-1)=10,$$

and hence $m(G)\geqslant 28$, again a contradiction.

Let $N(u)=\{v_1, v_2, v_3\}$. Then the three edges uv_1, uv_2 and uv_3 divide int$C$ into three regions R_1, R_2 and R_3. Let $T=\{b_1, b_2, b_3\}$. If b_1, b_2 and b_3 do not lie in the same region, then we may assume that b_1 lies in R_1, and b_2, b_3 do not lie in R_1. Since $G^*=G-b_1$ contains a subgraph homeomorphic to K_4, $m(G^*)\geqslant 18$. Now we count the number of cycles in G containing b_1: Clearly G has exactly two cycles containing b_1 but no other edges in $E(G)-E(C)$; for each pair of distinct elements $i, j\in\{1, 2, 3\}$, G has exactly one cycle containing $\{b_1, v_iuv_j\}$ but no other edges in $E(G)-E(C)$; for each $i\in\{2, 3\}$, G has exactly one cycle containing $\{b_1, b_i\}$ but no other edges in $E(G)-E(C)$; for any $b\in\{b_2, b_3\}$, there exists one pair of distinct elements $i, j\in\{1, 2, 3\}$ such that G has exactly one cycle containing $\{b, b_1, v_iuv_j\}$ but no other edges in $E(G)-E(C)$. Thus the number of cycles containing b_1 is at least $2+3+2+2=9$, consequently $m(G)\geqslant 27$, a contradiction. Therefore b_1, b_2 and b_3 must lie in the same region, say R_1. Let $C'=v_3uv_2\cup C[v_2, v_3]$ be the boundary of R_1, then the four bridges b_1, b_2, b_3 and v_2uv_3 are independent (See Fig. 1). For otherwise there exist three bridges (say b_i, b_j and v_3uv_2 for any two distinct elements $i, j\in\{1, 2, 3\}$) are dependent, and then it is easy to verify that G has at least 10 cycles containing b_1, and hence $m(G)\geqslant 18+10=28$, a contradiction.

Clearly G is such that it satisfies the conditions in Lemma 1. By Lemma 1, G has two cycles having the same length, a contradiction.

Case 1.3 $d(u)=2$

Let B be the set of bridges in G. Clearly $|B|=5$.

Suppose that G has two pairs of skew bridges. Then there exists $b_1, b_2\in B$ such that b_1 skews to b_2 and $G-b_1$ has one pair of skew bridges. Let $G^*=G-b_1$. Then G^* has a subgraph homeomorphic to K_4, and hence $m(G^*)\geqslant 18$. Since G is a simple graph, there exists at most one bridge such that it and b_1 have the same end vertices. Let $B_1\subseteq B$, the number of cycles containing all bridges in B_1 but no bridges in B-B_1 is denoted by $c(B_1)$. If $B_1=\{b_1, \cdots, b_k\}$, then $c(B_1)$ is denoted by $c(b_1, \cdots, b_k)$. Clearly $c(b_1)=2$; $c(b_1, b_2)=2$; for each $b\in B-\{b_1, b_2\}$, $c(b_1, b)=1$; there exist two bridges $b_3, b_4\in B-\{b_1, b_2\}$ such that $c(b_1, b_2, b_i)=1$ for $i=3, 4$. Therefore the number of cycles containing b_1 is at least $2+2+3+2=9$. Thus $m(G)\geqslant 27$, a contradiction. Therefore G has exactly one pair of skew bridges. Let $b_1=(u, v)$ and $b_2=(u', v')$ be one pair of skew bridges. Then the four vertices u, u', v and v' divide C into four paths $C[u, u']$, $C[u', v]$, $C[v, v']$ and $C[v', u]$ which are denoted by P_1, P_2, P_3 and P_4, respectively. Let $B-\{b_1, b_2\}=\{b_3, b_4, b_5\}$. Then all end vertices of b_3, b_4 and b_5 lie on a common path in $\{P_1, P_2, P_3, P_4\}$. For otherwise, we may assume that the end vertices of b_3 lie on P_3 and the end vertices of b_4 and b_5 do not lie on P_3. In this case let $G^*=G-b_3$. Then $m(G^*)\geqslant 18$. Also $c(b_3)=2$; for each $b\in B-\{b_3\}$, $c(b_3, b)=1$; clearly G has at least three cycles, each of which contains b_3 and contains at least three bridges. Thus the number of cycles containing b_3 is at least $2+4+3=9$, and hence $m(G) \geqslant 27$, a contradiction.

In fact, the four bridges b_1, b_3, b_4 and b_5 are independent (See Fig. 2), since otherwise we may assume that b_3, b_4 and b_1 are dependent. In this case $c(b_3)=2$; for each $b \in B-\{b_3\}$, $c(b, b_3)=1$; also G has at least three cycles, each of which contains b_3 and contains at least three bridges. Thus $m(G) \geqslant 18+2+4+3=27$, a contradiction.

Clearly G is such that the conditions in Lemma 2 are satisfied. By Lemma 2, G has two cycles having the same length, a contradiction.

Case 2 $p=28$

A similar discussion to that of Case 1.3 yields that G is such that the conditions in Lemma 3 are satisfied. By Lemma 3, G has two cycles having the same length, again a contradiction.

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28 个顶点的简单 MCD 图

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提　要　设 S_n 是 n 个顶点的没有等长圈的简单图的集合．若 $G \in S_n$ 且 S_n 中不存在图 G' 使 $|E(G')|>|E(G)|$，则称图 G 是简单 MCD 图．若简单 MCD 图 G 是 2 连通的，则称 G 是 2 连通简单 MCD 图．本文证明了不存在具有 28 个顶点的含有同胚于 K_4 的子图的 2 连通简单 MCD 图．于是结合 Discrete Math. 126 (1994)，我们完全证明了下述定理：存在 n 个顶点的含有同胚于 K_4 的子图的 2 连通简单 MCD 图当且仅当 $n \in \{10, 11, 14, 15, 16, 21, 22\}$.

关键词　圈分布图；MCD 图；简单 MCD 图

中图法分类号　O157.5