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NONLOCAL CAUCHY PROBLEM FOR SOME FRACTIONAL ABSTRACT INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

In this paper we study the existence and uniqueness of solutions for fractional integrodifferential equations with nonlocal condition in a Banach space. The results are established by the application of the contraction mapping principle and the Krasnoleskii fixed point theorem.

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1 Introduction

Recently, G. M. N'Guérékata [10] proved the existence and uniqueness of solutions to the Cauchy problem for the fractional differential equations with nonlocal conditions of the

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form $D^q x(t) = f(t, x(t)),\ t \in [0,T],\ x(0) + g(x) = x_0$, where $0 < q < 1$. Let $(X, \|.\|)$ be a Banach space, and $I := [0,T], T > 0$, a compact interval in R. Denote by $\mathcal{C} = C([0,T],X)$ the Banach space of all continuous function $[0,T] \to X$ endowed with the topology of uniform convergence (the norm in this space will be denoted by $\|.\|_{C}$).

Several authors have studied the following Cauchy problem for semilinear differential equations with nonlocal conditions in a Banach space.

$$\begin{cases} x'(t) & = Ax(t) + f(t,x), \quad t \in [0,T] \\ x(0) + g(x) & = x_0. \end{cases}$$

As indicated in several recent papers (see for instance [1, 2, 3, 4, 5, 6, 10, 11, 13, 14, 15]), the nonlocal condition $x(0) + g(x) = x_0$ can be applied in physics with better effect than the classical Cauchy problem with initial condition $x(0) = x_0$. For instance the authors used

$$g(x) = \sum_{i=1}^{p} c_i x(t_i),$$

where $c_i = 1, 2, \ldots\ldots, p$ are given constants and $0 < t_1 < t_2 < \ldots\ldots, < t_p \leq T$. To describe the diffusion phenomenon of a small amount in a transparent tube. In this case, the Cauchy problem allows the additional measurements at $t_i, i = 1, 2, \ldots\ldots p$.

Recent studies of fractional differential equations are done by Lakshmikantham in his papers [7, 8, 9]. The reader may also consult [12]. In this work we consider the following Cauchy problem for the nonlocal conditions fractional integro differential equation

$$\begin{cases} D^q x(t) = \int_0^t k(t,s,x(s))ds, \quad t \in I, \\ x(0) + g(x) = x_0, \end{cases} \tag{1}$$

where $0 < q < 1$; $k : \Delta \times X \to X, g : C(C,I) \to X$ are given functions. Here Δ denotes the set $\{(t,s) : 0 \leq s \leq t \leq T\}$.

We investigate in our paper the Cauchy problem for the nonlinear fractional integrodifferential equation (1) with the following assumptions.

(H1). $k : \Delta \times X \to X$ is continuous and there exist a constant $K_1 > 0$ such that

$$\|k(t,s,x_1) - k(t,s,x_2)\| \leq K_1 \|x_1 - x_2\|, \quad x_1, x_2 \in X, (t,s) \in \Delta.$$

(H2). $g : C \to X$ is bounded, continuous, and $\|g(x) - g(y)\| \leq b\|x - y\| \quad \forall x, y \in \mathcal{C}$.
(H3). For any positive number r there exists $h_r \in L^1(I)$ such that

$$sup_{\|x\| \leq r} \|k(t,s,x)\| \leq h_r(t), \quad x \in X, (t,s) \in \Delta$$

2 Main Results

2.1 Existence and uniqueness result

Now we are ready to present our results.

Theorem 2.1. *Under assumptions (H1)-(H2), if $b < \frac{1}{2}$ and $K_1 \leq \frac{\Gamma(q+1)}{2T^q}$, then Eq. (1) has a unique solution.*

Proof. Define $F : \mathcal{C} \to C$ by

$$Fx(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds, \quad t \in [0,T].$$

Let $G = sup_{x\in C}\|g(x)\|$, and $K_2 = max\{\|k(t,s,0)\| : (t,s) \in \Delta\}$, and choose $r \geq 2(\|x_0\| + G + \frac{K_2 T^q}{\Gamma(q+1)})$ Then we can show that $FB_r \subset B_r$ where $B_r := \{x \in \mathcal{C} : \|x\| \leq r\}$. So let $x \in B_r$. Then we get

$$\begin{aligned}
\|Fx(t)\| &\leq \|x_0\| + G \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \|k(s,\tau,x(\tau))\| d\tau ds \\
&\leq \|x_0\| + G + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s (\|k(s,\tau,x(\tau) - k(s,\tau,0)\| + \|k(s,\tau,0)\|) d\tau ds \\
&\leq \|x_0\| + G + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s (K_1\|x(\tau)\| + K_2) d\tau ds \\
&\leq \|x_0\| + G + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (K_1\|x(s)\| + K_2) ds \\
&\leq \|x_0\| + G + (K_1 r + K_2) \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\
&\leq \|x_0\| + G + (K_1 r + K_2) \frac{T^q}{\Gamma(q+1)} \leq r
\end{aligned}$$

by the choice of K_1, K_2 and r. Now we take $x, y \in C$. Then we get

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\leq \|g(x) - g(y)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\| d\tau ds \\
&\leq \Omega_{b,K_1,T,q} \|x - y\|,
\end{aligned}$$

where $\Omega_{b,K_1,T,q} := (b + \frac{K_1 T^q}{\Gamma(q+1)})$ depends only on the parameters of the problem. And since $\Omega_{b,K_1,T,q} < 1$, the result follows in view of the contraction mapping principle. □

2.2 Existence result

In this subsection we prove the result based on the well-known theorem

Theorem 2.2. *(Krasnoselkii). Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*

1. *$Ax + By \in M$ whenever $x, y \in M$;*
2. *A is compact and continuous ;*
3. *B is a contraction mapping.*

Then there exists $z \in M$ *such that* $z = Az + Bz$.

Now we present our second result.

Theorem 2.3. *Assume (H1)-(H3) with* $b < 1$. *Then Eq.(1) has at least one solution on I.*

Proof. Choose $r \geq \|x_0\| + G + \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)}$ and consider $B_r : \{x \in C : \|x\| \leq r\}$. Now define on B_r the operators A, B by

$$(Ax)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds,$$

and

$$(Bx)(t) := x_0 - g(x).$$

Let's observe that if $x, y \in B_r$; then $Ax + By \in B_r$. Indeed it is easy to check the inequality

$$\|Ax + By\| \leq \|x_0\| + G + \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)} \leq r.$$

By$(H2)$, it is also clear that B is a contraction mapping for $b < 1$. Since x is continuous, then $(Ax)(t)$ is continuous in view of $(H1)$. Let's now note that A is uniformly bounded on B_r. This follows from the inequality

$$\|(Ax)(t)\| \leq \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)}.$$

Now let's prove that $(Ax)(t)$ is equicontinuous.

Let $t_1, t_2 \in I$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r$ (thus $\sup_{(t,s) \in I \times B_r} \|k(t,s,x(s))\| := c_0 < \infty$), we will get

$$\begin{aligned}
\|Ax(t_1) - Ax(t_2)\| &= \frac{1}{\Gamma(q)} \| \int_0^{t_1} (t_1 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \\
&\quad - \int_0^{t_2} (t_2 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \| \\
&= \frac{1}{\Gamma(q)} \| \int_{t_2}^{t_1} (t_1 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \\
&\quad - \int_0^{t_2} (t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \int_0^s k(s,\tau,x(\tau))d\tau ds \| \\
&\leq \frac{1}{\Gamma(q)} \left(\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))d\tau ds \| \right) \\
&\quad + \frac{1}{\Gamma(q)} \left(\| \int_0^{t_2} (t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \int_0^s k(s,\tau,x(\tau))d\tau ds \| \right) \\
&\leq \frac{c_0}{\Gamma(q+1)} |2(t_1 - t_2)^q + t_2^q - t_1^q| \\
&\leq \frac{2c_0}{\Gamma(q+1)} |t_1 - t_2|^q,
\end{aligned}$$

which does not depend on x. So $A(B_r)$ is relatively compact. By the Arzela-Ascoli Theorem, A is compact. We now conclude the proof of the theorem using the Krasnoselkii's theorem above. □

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