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SUPERPOSITION OPERATORS BETWEEN VARIOUS ALMOST PERIODIC FUNCTION SPACES AND APPLICATIONS

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Abstract

We study the superposition operators (also called Nemytskii operators) between spaces of almost periodic functions with values in a complete metric space or in a Banach space. We establish new results on their continuity and their differentiability by using several different methods. We give applications to evolution equations and to differential equations in Banach spaces. We also establish new results on the superposition operators between spaces of asymptotically almost periodic functions and spaces of almost automorphic functions.

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1 Introduction

From a mapping $f : X \times \mathbb{R} \longrightarrow Y$, where X and Y are complete metric spaces, or Banach spaces when differentiability is studied, we consider a superposition operator in the following form

$$[t \mapsto u(t)] \longmapsto [t \mapsto f(u(t),t)]$$

where $u : \mathbb{R} \to X$ is an almost periodic function, or an asymptotically almost periodic function, or an almost automorphic function.
Such operators are useful for the functional-analytic methods in the study of oscillations in various kinds of differential equations. First studies of this kind of operators are presumably due to Nemytskii (see the preface of [40]); that is why such operators are sometime called Nemytskii operators.

Our aim is to obtain the continuity and differentiability of such operators under assumptions which are weaker than the generally used ones.

Superposition operators are currently used in numerous fields of Mathematical Analysis, including the study of almost periodic functions; for instance the book [5] is totally devoted to these operators and contains a lot of examples.

Let us describe the content of this paper.

In Section 2 we fix our notations about the notions used in the sequel.

In Section 3 we study the continuity of the superposition operators between spaces of almost periodic functions in the sense of Harald Bohr. We improve results of Yoshizawa [41] by deleting an assumption of separability. The first main result is Theorem 3.5 where we do not consider the usual Lipschitz condition as in [23] for instance. We give three different proofs of this theorem since each of these proofs contains arguments which are interesting for themselves. The first proof is the english translation of the one given by Cieutat in his Thesis Dissertation [20] (in French) and which is unpublished in a scientific journal. The second proof is based on a generalization to the almost periodicity of a method that we have only encountered in the well-known book "Cours d'Analyse" by Laurent Schwartz. This method essentially uses a variation of the classical theorem of Heine on uniform continuity. In another setting (this one of the spaces of bounded sequences), this method has been used by Blot and Crettez in [12]. The third proof is based on the Bohr compactification from which one can transform almost periodic functions into continuous functions defined on a compact group. Finally we establish the converse of Theorem 3.5. This new result shows that, among the various notions of almost periodic functions with parameters, the notion used by Yoshizawa [41] is the appropriate notion that yields to the continuity of the superposition operators. In Section 4, by using the new result of section 3, we improve results on the semilinear evolution equations which are given in Hino, Naito, Minh and Shin's book [29].

In Section 5 we establish results on the differentiability of superposition operators between spaces of almost periodic functions.

In Section 6, by using results of Section 5, we establish a result on the perturbation of almost periodic solutions of differential equations in Banach spaces. In Section 7 we establish new results on the continuity and the differentiability of the superposition operators in the setting of the n-times differentiable almost periodic functions. Such functions are specially studied in works of N'Guérékata and several co-authors [34], [19], [7].

The methods used in this section are adaptations of these ones of Section 3 and of Section 5.

We also adapt the methods of Section 3 and Section 5 to establish new results on on the continuity and the differentiability of the superposition operators in the setting of the asymptotically almost periodic functions in Section 8 (where we also use important results due to Zaidman) and in the setting of almost automorphic functions in Section 9.

Note that we can find some results on the superposition operators on spaces of Besicovitch-almost periodic functions in [10], [20], [13], and on spaces of almost periodic sequences in [14]. Pankov's book [36] contains some particular results on superposition operators in Stepanov and Besicovich spaces. We do not study these settings in the present paper.

2 Notation

X and Y are complete metric spaces.

When A is a topological space, $C^0(A,X)$ denotes the space of all continuous mappings from A into X. When moreover A is compact, $C^0(A,X)$ is endowed with the supremum distance $d_\infty(\varphi,\psi) := \sup_{a\in A} d(\varphi(a),\psi(a))$.

When X and Y are Banach spaces, $\mathcal{L}(X,Y)$ is the Banach space of all linear continuous mappings from X into Y. And when $n \in \mathbb{N}_* := \mathbb{N}\setminus\{0\}$, $\mathcal{L}_n(X^n,Y)$ is the Banach space of all continuous n-linear mappings from the product X^n into Y.

When X and Y are Banach spaces and when $n \in \mathbb{N}_*$, $C^n(X,Y)$ denotes the space of all n-times continuously Fréchet-differentiable mappings from X into Y.

$BC^0(X,Y)$ denotes the space of the bounded continuous mappings from X into Y. Endowed with the distance of the supremum, defined by $d_\infty(\varphi,\psi) := \sup_{x\in X} d(\varphi(x),\psi(x))$, $BC^0(X,Y)$ is a complete metric space, [38], Corollaire 2, p. 196. When Y is a Banach space, endowed with the sup norm $\|\varphi\|_\infty := \sup_{x\in X}\|\varphi(x)\|$, $BC^0(X,Y)$ is a Banach space, [38], Corollaire 3, p. 196. When X and Y are Banach spaces and when $n \in \mathbb{N}_*$, $BC^n(X,Y)$ is the space of the mappings which belong to $C^n(X,Y)\cap BC^0(X,Y)$ such that all their differentials, up to order n, are bounded on X. Endowed with the norm $\|\varphi\|_{C^n} := \|\varphi\|_\infty + \sum_{1\le k\le n}\|D^k\varphi\|_\infty$, it is a Banach space.

When X is a Banach space, $C^0_0(\mathbb{R}_+,X)$ denotes the space of all functions $u \in C^0(\mathbb{R}_+,X)$ such that $\lim_{t\to\infty} u(t) = 0$. It is a Banach subspace of $BC^0(\mathbb{R}_+,X)$.

$AP^0(\mathbb{R},X)$ stands for the space of all almost periodic functions in the sense of Harald Bohr [31], [4], [17], [33], [21], [41], [24]. Endowed with the distance d_∞, it is a complete metric space. When X is a Banach space and $n \in \mathbb{N}_*$, $AP^n(\mathbb{R},X)$ denotes the space of the functions of $BC^n(\mathbb{R},X)\cap AP^0(\mathbb{R},X)$ such that all their derivatives, up to order n, belong to $AP^0(\mathbb{R},X)$ [34], [19], [7].

We denote by $\mathcal{P}_c(X)$ the set of all compact subsets of X. We define $APU(X \times \mathbb{R}, Y)$ as the set of the mappings $f : X \times \mathbb{R} \to Y$ which are continuous on $X \times \mathbb{R}$ and satisfy the following property:

for all $K \in \mathcal{P}_c(X)$, for all $\varepsilon > 0$, there exists $\ell = \ell(K, \varepsilon) > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\ell]$ satisfying $\| f(x,t+\tau) - f(x,t) \| \leq \varepsilon$ for all $(x,t) \in K \times \mathbb{R}$; see [41], Definition 2.1, p. 5-6.

In the finite-dimensional setting this notion is used in [11], [13] and [6], and under an equivalent form it is also used in [30] and in [43], Section 3.4 in Chapter 3, p. 175. Such mappings are called *almost periodic in t uniformly in x*.

Remark 2.1. If $\phi \in C^0(X,Y)$, and if we set $f(x,t) := \phi(x)$ for all $(x,t) \in X \times \mathbb{R}$, then we have $f \in APU(X \times \mathbb{R}, Y)$.

When X is a Banach space, $AAP(\mathbb{R}_+, X)$ denotes the space of all functions from $\mathbb{R}_+$ in to X which are asymptotically almost periodic in the sense of Fréchet [42]. Recall that $u \in AAP(\mathbb{R}_+, X)$ means that $u = u_1 + u_2$ with $u_1 \in AP^0(\mathbb{R}_+, X)$ and $u_2 \in C_0^0(\mathbb{R}_+, X)$. $AAP(\mathbb{R}_+, X)$ is a Banach subset of $BC^0(\mathbb{R}_+, X)$. Following Zaidman [28], a mapping $f : X \times \mathbb{R}_+ \to Y$, $(x,t) \mapsto f(x,t)$, is called *asymptotically almost periodic in t uniformly in x* when f is continuous and when f satisfies the following condition: for all $K \in \mathcal{P}_c(X)$, for all $\varepsilon > 0$, there exist $T = T(K, \varepsilon) \geq 0$ and $\ell = \ell(K, \varepsilon) > 0$ such that, for all $r \in \mathbb{R}_+$, there exists $\tau \in [r, r+\ell]$ satisfying $\| f(x,t+\tau) - f(x,t) \| \leq \varepsilon$ for all $x \in K$ and for all $t \geq T$. We denote by $AAPU(X \times \mathbb{R}_+, Y)$ the set of all such mappings. One can see a recent use of these notions in evolution equations e.g. in [28].

We denote by $AA(\mathbb{R}, X)$ the space of all functions from $\mathbb{R}$ into X which are almost automorphic in the sense of Bochner [15], [16], [34]. Recall that $u \in AA(\mathbb{R}, X)$ means that $u \in C^0(\mathbb{R}, X)$ and that u satisfies the following property: for all real sequence $(s'_n)_n$ there exists a subsequence $(s_n)_n$ of $(s'_n)_n$ such that $\lim_{m\to\infty} u(t - s_m)$ exists in X for all $t \in \mathbb{R}$, and $\lim_{n\to\infty}(\lim_{m\to\infty} u(t - s_m + s_n)) = u(t)$ for all $t \in \mathbb{R}$.

Now we introduce a new notion. We say that a mapping $f : X \times \mathbb{R} \to Y$, $(x,t) \mapsto f(x,t)$, is *almost automorphic in t uniformly in x* when it satisfies the two following conditions:

(1) For all $x \in X$, $f(x,.) \in AA(\mathbb{R}, Y)$.

(2) For all $K \in \mathcal{P}_c(X)$, for all $\varepsilon > 0$, there exists $\delta = \delta(K, \varepsilon) > 0$ such that, for all $x, z \in K$, if $d(x,z) \leq \delta$ then we have $d(f(x,t), f(z,t)) \leq \varepsilon$ for all $t \in \mathbb{R}$.

We denote by $AAU(X \times \mathbb{R}, Y)$ the set of all such mappings.

Remark 2.2. Conditions (1) and (2) together are equivalent to the following one.

(3) $\Phi \in C^0(X, AA(\mathbb{R}, Y))$ where $\Phi(x) := [t \mapsto f(x,t)]$.

Indeed, let us assume that (1) and (2) fulfilled. In view of (1), the mapping $\Phi : X \to AA(\mathbb{R}, Y)$ is well-defined. Since X and $AA(\mathbb{R}, Y)$ are metric spaces, Φ is continuous on X if and only if the restriction of Φ to each compact subset of X is continuous. This last condition is ensured by (2), and consequently (3) is satisfied. Conversely assume that (3) fulfilled. Then, since $\Phi(x) \in AA(\mathbb{R}, Y)$ for all $x \in X$, (1) is satisfied. And since Φ is continuous on X, Φ is uniformly continuous on each $K \in \mathcal{P}_c(X)$, that is exactly (2).

Remark 2.3. In the above definition, note that conditions (1) and (2) imply that $f \in C^0(X \times \mathbb{R}, Y)$.

Remark 2.4. If $\phi \in C^0(X,Y)$, and if we set $f(x,t) := \phi(x)$ for all $(x,t) \in X \times \mathbb{R}$, then $f \in AAU(X \times \mathbb{R}, Y)$. If $f \in APU(X \times \mathbb{R}, Y)$ then $f \in AAU(X \times \mathbb{R}, Y)$. This notion is different from the one used in [34], Theorem 2.18, p. 60; it also differs from the one used in [32] where the authors use bounded subsets of X instead of compact subsets of X.

3 Continuity and Almost Periodicity

In this section we study the continuity of the superposition operators between spaces of almost periodic functions. X and Y are complete metric spaces.

First, let's present a series of lemmas.

Lemma 3.1. *Let $f \in APU(X \times \mathbb{R}, Y)$ and $K \in \mathcal{P}_c(X)$. Then the restriction of f to $K \times \mathbb{R}$ is uniformly continuous.*

In the particular case where X and Y are both finite-dimensinal normed vector spaces,Lemma 3.1is established in Yoshizawa's book [41] (see Theorem 2.1, p. 7). The proof therein can be easily generalized to complete metric spaces.

Lemma 3.2. *Let $\phi \in C^0(X,Y)$ and $v \in AP^0(\mathbb{R}, X)$. Then we have $\phi \circ v \in AP^0(\mathbb{R}, Y)$.*

Lemma 3.2 is Property 4, p. 3, in [31]. In the Banach spaces setting, a proof of this result is also given in [4], p. 6, and in [34], Proof of Theorem 1.34, p. 14.

Lemma 3.3. *Let $f \in C^0(X \times \mathbb{R}, Y)$ and $K \in \mathcal{P}_c(X)$. We define the mapping $f^K : \mathbb{R} \to C^0(K,Y)$ by setting $f^K(t) := [x \mapsto f(x,t)]$ for all $t \in \mathbb{R}$. Then the two following assertions are equivalent.*

(i) $f \in APU(X \times \mathbb{R}, Y)$

(ii) *For all $K \in \mathcal{P}_c(X)$, $f^K \in AP^0(\mathbb{R}, C^0(K,Y))$.*

Lemma 3.3 comes from [20] inside the proof of Théorème in Section 2 of Chapter 1, p. 7.

Proof. **(i $\Longrightarrow$ ii)**. Continuity of f^K is a straightforward consequence of Lemma 3.1. By using the definition of $APU(X \times \mathbb{R}, Y)$, we know that, for all $\varepsilon > 0$, there exists $\ell > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\ell]$ satisfying $d(f(x,t+\tau), f(x,t)) \leq \varepsilon$ for all $x \in K$ and all $t \in \mathbb{R}$. By taking the supremum over $x \in K$, we obtain $d_\infty(f^K(t+\tau), f^K(t)) \leq \varepsilon$, for all $t \in \mathbb{R}$.
(ii $\Longrightarrow$ i). Let $(x_n,t_n)_n$ be a sequence of elements of $X \times \mathbb{R}$ which converges toward $(x_*,t_*) \in X \times \mathbb{R}$. We set $K_* := \{x_n : n \in \mathbb{N}\} \cup \{x_*\} \in \mathcal{P}_c(X)$. Note that we have

$$\begin{aligned}\|f(x_n,t_n) - f(x_*,t_*)\| &\leq \|f(x_n,t_n) - f(x_n,t_*)\| + \|f(x_n,t_*) - f(x_*,t_*)\| \\ &\leq \|f^{K_*}(t_n) - f^{K_*}(t_*)\|_\infty + \|f^{K_*}(t_*)(x_n) - f^{K_*}(t_*)(x_*)\|.\end{aligned}$$

Since $\lim_{n\to\infty} t_n = t_*$, f^{K_*} is continuous on $\mathbb{R}$, and $f^{K_*}(t_*)$ is continuous on K_*, we deduce from the previous inequalities that $\lim_{n\to\infty} f(x_n,t_n) = f(x_*,t_*)$. And so we have proven that $f \in C^0(X\times\mathbb{R},Y)$. Now we arbitrarily fix $K \in \mathcal{P}_c(X)$ and $\varepsilon > 0$. Since $f^K \in AP^0(\mathbb{R},C^0(K,Y))$ we know that there exists $\ell > 0$ such that for all $r \in \mathbb{R}$, there exists $\tau \in [r,r+\ell]$ satisfying $\| f^K(t+\tau) - f^K(t)\|_\infty \le \varepsilon$ for all $t \in \mathbb{R}$, that implies: $\| f^K(t+\tau)(x) - f^K(t)(x)\| \le \varepsilon$ for all $x \in K$ and for all $t \in \mathbb{R}$. Noting that $f^K(t+\tau)(x) = f(x,t+\tau)$ and that $f^K(t)(x) = f(x,t)$, we obtain that $f \in APU(X\times\mathbb{R},Y)$. This completes the proof. □

Lemma 3.4. *Let $f \in APU(X\times\mathbb{R},Y)$ and $u \in AP^0(\mathbb{R},X)$. Then we have $[t \mapsto f(u(t),t)] \in AP^0(\mathbb{R},Y)$.*

Proof. We set $K := \overline{u(\mathbb{R})}$, the closure of $u(\mathbb{R})$, which is compact [31], Property 1, p. 2. We define the mapping $\phi : C^0(K,Y)\times K \to Y$ by $\phi(g,x) := g(x)$, and the function $v : \mathbb{R} \to C^0(K,Y)\times K$ by $v(t) := (f^K(t),u(t))$, where f^K is provided by Lemma 3.3. When $g,h \in C^0(K,Y)$ and $x,z \in K$, we have $d(\phi(g,x),\phi(h,z)) \le d(g(x),h(x)) + d(h(x),h(z)) \le d_\infty(g,h) + d(h(x),h(z))$. From these inequalities we can deduce that ϕ is continuous. In view of Lemma 3.3 we deduce that f^K is almost periodic, and by the Bochner criterion, [31], p. 4, we see that $v \in AP^0(\mathbb{R},C^0(K,Y)\times K)$. Since $f(u(t),t) = \phi\circ v(t)$ for all $t \in \mathbb{R}$, we conclude using Lemma 3.2. □

The proof of Lemma 3.4 is contained in [20] Chapter 1, p.7. Lemma 3.4 is established in [41] (Theorem 2.7, p. 16) when X and Y are finite-dimensional spaces. Theorem 2.7 may not be extened to general complete metric spaces since Yoshizawa uses an argument on the separability of X.

(4) $N^1_f : AP^0(\mathbb{R},X) \to AP^0(\mathbb{R},Y)$, $N^1_f(u) := [t \mapsto f(u(t),t)]$

where $f \in APU(X\times\mathbb{R},Y)$.

Following Remark 2.1, when $\phi \in C^0(X,Y)$, we can define the superposition operator $u \mapsto \phi\circ u$ from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$. The first main result of this section is the following theorem.

Theorem 3.5. *Let $f \in APU(X\times\mathbb{R},Y)$. Then the superposition N^1_f, defined in (4), is continuous from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$.*

We will give three proofs of Theorem 3.5. But first, we need the following lemma.

Lemma 3.6. *If $\mathcal{K}$ is a compact subset of $AP^0(\mathbb{R},X)$, then $S := \{u(t) : t \in \mathbb{R}, u \in \mathcal{K}\}$ is a relatively compact subset of X.*

Proof. We fix $\varepsilon > 0$. Since $\mathcal{K}$ is compact, it is also precompact, thus there exists $\{u_1,...,u_p\}$ a finite subset of $AP^0(\mathbb{R},X)$ such that $\mathcal{K} \subset \bigcup_{1\le j\le p}\{v \in AP^0(\mathbb{R},X) : d_\infty(v,u_j) \le \frac{\varepsilon}{2}\}$. Since $u_j(\mathbb{R})$ is relatively compact in X for all $j = 1,...,p$, their union $\bigcup_{1\le j\le p} u_j(\mathbb{R})$ also is relatively compact too, and consequently there exists $\{t_1,...,t_q\}$ a finite subset of $\mathbb{R}$ such that $\bigcup_{1\le j\le p} u_j(\mathbb{R}) \subset \bigcup_{1\le j\le p}\bigcup_{1\le k\le q}\{x \in X : d(x,u_j(t_k)) \le \frac{\varepsilon}{2}\}$. If $x \in S$ there exist $v \in \mathcal{K}$ and $t \in \mathbb{R}$ such that $x = v(t)$, and then there exists $j \in \{1,...,p\}$ such that $d_\infty(v,u_j) \le \frac{\varepsilon}{2}$, and consequently we have $d(x,u_j(t)) \le \frac{\varepsilon}{2}$. By using the previous inclusion, we obtain that there exist $i \in \{1,...,p\}$ and $k \in \{1,...,q\}$ such that $d(u_j(t),u_i(t_k)) \leq \frac{\varepsilon}{2}$. And so we have $d(x,u_j(t_k)) \leq d(x,u_j(t)) + d(u_j(t),u_i(t_k)) \leq 2\frac{\varepsilon}{2} = \varepsilon$. This reasoning proves the following inclusion: $S \subset \bigcup_{1\leq j\leq p} \bigcup_{1\leq k\leq q} \{x \in X : d(x,u_j(t_k)) \leq \varepsilon\}$ that means S is precompact, and since X is complete, we obtain that S is relatively compact, [22], Théorème 3.17.5, p. 63. □

Lemma 3.6 and its proof are included in [20], Lemme 3 in Section 3 of Chapter 2, p.15.
First Proof of Theorem 3.5.

Proof. Let $\mathcal{K}$ be a compact subset of $AP^0(\mathbb{R},X)$, let $u \in \mathcal{K}$ and let $\varepsilon > 0$. We set $S := \{u(t) : t \in \mathbb{R}, u \in \mathcal{K}\}$. By using Lemma 3.6 we know that the closure $\overline{S}$ is compact. Since $f \in APU(X \times \mathbb{R},Y)$ there exists $\ell > 0$ such that, for all $r \in \mathbb{R}$, there exists $-\tau \in [r,r+\ell]$ satisfying the following condition:

(5) $d(f(x,s+\tau), f(x,s)) \leq \frac{\varepsilon}{3}$ for all $(x,s) \in S \times \mathbb{R}$.

Since $\overline{S} \times [0,\ell]$ is compact as a product of two compacts, f is uniformly continuous on it, and consequently there exists $\delta = \delta(\overline{S} \times [0,\ell],\varepsilon) > 0$ such that, for all $s_1,s_2 \in [0,\ell]$ and for all $x_1,x_2 \in \overline{S}$ we have: $(d(x_1,x_2) \leq \delta, |s_1 - s_2| \leq \delta) \Longrightarrow d(f(x_1,s_1), f(x_2,s_2)) \leq \frac{\varepsilon}{3}$. This implies the following assertion:

(6) $d(x_1,x_2) \leq \delta \Longrightarrow d(f(x_1,s), f(x_2,s)) \leq \frac{\varepsilon}{3}$, for all $s \in [0,\ell]$.

If $v \in \mathcal{K}$ satisfies $d_\infty(v,u) \leq \delta$, by using (5) and (6), we obtain, for all $t \in \mathbb{R}$,

$$d(f(u(t),t), f(v(t),t)) \leq d(f(u(t),t), f(u(t),t-\tau)) +$$

$$d(f(u(t),t-\tau), f(v(t),t-\tau)) + d(f(v(t),t-\tau), f(v(t),t)) \leq 3\frac{\varepsilon}{3}\varepsilon.$$

And so, by taking the supremum on the $t \in \mathbb{R}$, we obtain $d_\infty(N_f^1(u), N_f^1(v)) \leq \varepsilon$. This proves that the restriction of N_f^1 to $\mathcal{K}$ is continuous for all compact subset $\mathcal{K}$ of $AP^0(\mathbb{R},X)$; and since $AP^0(\mathbb{R},X)$ and $AP^0(\mathbb{R},Y)$ are metric spaces, this proves the continuity of N_f^1 on $AP^0(\mathbb{R},X)$. □

This first proof of Theorem 3.5 is due to Cieutat [20], Proposition 1 in Section 3 of Chapter 2, p. 15-17. In order to give a second proof of Theorem 3.5 we need the following new result.

Lemma 3.7. *Let $f \in APU(X \times \mathbb{R},Y)$. Then the following assertion holds: for all $K \in \mathcal{P}_c(X)$ and for all $\varepsilon > 0$, there exists $\delta = \delta(K,\varepsilon) > 0$ such that, for all $x \in K$ and for all $z \in X$, if $d(x,z) \leq \delta$ then $d(f(x,t), f(z,t)) \leq \varepsilon$, for all $t \in \mathbb{R}$.*

Before presenting the proof of this lemma, let's first give the following remark.

Remark 3.8. This statement is not simply the uniform continuity of f on $K \times \mathbb{R}$ (Lemma 3.1). Note that we use $z \in X$ and not $z \in K$ in the conclusion. When X is locally compact, for instance when X is a finite-dimensional Banach space, we can find a relatively compact subset Ω of X which contains the compact K, [22], Result 3.18.2, p. 65, and then the uniform continuity of f on $\overline{\Omega} \times \mathbb{R}$ implies the conclusion of Lemma 3.7. But when X is not locally compact, for instance when X is an infinite-dimensional Banach space, such a reasoning becomes impossible.

Proof. We proceed by contradiction: let's assume that there exist $K \in \mathcal{P}_c(X)$ and $\varepsilon > 0$ such that, for all $\delta > 0$, there exist $x^\delta \in K$, $z^\delta \in X$ and $t^\delta \in \mathbb{R}$ satisfying $d(x^\delta, z^\delta) \leq \delta$ and $d(f(x^\delta, t^\delta), f(z^\delta, t^\delta) > \varepsilon$. For all $m \in \mathbb{N}_*$, we set $x_m := x^{1/m}$, $z_m := z^{1/m}$ and $t_m := t^{1/m}$, and so the following assertion holds:

(7) For all $m \in \mathbb{N}_*$ there exist $x_m \in K$, $z_m \in X$ and $t_m \in \mathbb{R}$ such that $d(x_m, z_m) \leq \frac{1}{m}$ and $d(f(x_m, t_m), f(z_m, t_m)) > \varepsilon$.

Since K is compact, by using the Weierstrass-Bolzano theorem, there exists a subsequence $(x_{\alpha(m)})_m$ of $(x_m)_m$ (where $\alpha : \mathbb{N}_* \to \mathbb{N}_*$ is strictly increasing) such that $\lim_{m\to\infty} x_{\alpha(m)} = \hat{x}$ for some $\hat{x} \in K$. Note that we have $d(x_{\alpha(m)}, z_{\alpha(m)}) \leq \frac{1}{\alpha(m)} \leq \frac{1}{m}$ for all $m \in \mathbb{N}_*$, that implies that we also have $\lim_{m\to\infty} z_{\alpha(m)} = \hat{x}$. We set $K_1 := \{z_{\alpha(m)} : m \in \mathbb{N}_*\} \cup \{\hat{x}\}$. By using the Heine-Borel-Lebesgue theorem we can see that K_1 is compact. And so $K_2 := K \cup K_1$ is also compact as a union of two compact subsets. Then by using Lemma 3.1, we obtain the following assertion:

(8) For all $\sigma > 0$ there exists $\beta_\sigma > 0$ such that, for all $x, z \in K_2$, if $d(x,z) \leq \beta_\sigma$ then $d(f(x,t), f(z,t)) \leq \sigma$ for all $t \in \mathbb{R}$.

By taking $\sigma_0 := \frac{\varepsilon}{2}$, $\beta := \beta_{\sigma_0}$ and choosing $m_\beta \in \mathbb{N}_*$ such that $\frac{1}{m_\beta} \leq \beta$, we obtain

$$d(f(x_{\alpha(m)}, t), f(z_{\alpha(m)}, t)) \leq \frac{\varepsilon}{2}$$

for all $t \in \mathbb{R}$ and for all $m \geq m_\beta$, that implies $d(f(x_{\alpha(m)}, t_{\alpha(m)}), f(z_{\alpha(m)}, t_{\alpha(m)})) \leq \frac{\varepsilon}{2}$ when $m \geq m_\beta$, which contradicts with (7). □

Second Proof of Theorem 3.5.

Proof. Let's fix $u \in AP^0(\mathbb{R}, X)$ and $\varepsilon > 0$. Since $K := \overline{u(\mathbb{R})} \in \mathcal{P}_c(X)$, we can consider $\delta = \delta(K, \varepsilon) > 0$ provided by Lemma 3.7. Then when $v \in AP^0(\mathbb{R}, X)$ satisfies $d_\infty(v, u) \leq \delta$, we have $d(v(t), u(t)) \leq \delta$ for all $t \in \mathbb{R}$, with $u(t) \in K$. Consequently Lemma 3.7 implies that we have $d(f(v(t), t), f(u(t), t)) \leq \varepsilon$ for all $t \in \mathbb{R}$, that means that $d_\infty(N_f^1(v), N_f^1(u)) \leq \varepsilon$. □

We need some additional lemmas in order to give a third proof of Theorem 3.5.

Lemma 3.9. *If $\phi \in C^0(X,Y)$ then the following assertion holds: for all $K \in \mathcal{P}_c(X)$ and all $\varepsilon > 0$ there exists $\delta = \delta(K, \varepsilon) > 0$ such that, for all $x \in K$ and all $z \in X$, if $d(x,z) \leq \delta$ then $f(\phi(x), \phi(z)) \leq \varepsilon$.*

Following Remark 2.1, Lemma 3.9 is a straightforward consequence of Lemma 3.7. Lemma 3.9 is established in [37], footnote (**), p. 355, in [38], Théorème (T.2, IX, 5; 1), p.109, and in [39], Théorème 2.7.20, p. 229. In [12], Lemma A1.3 in Appendix 1, p. 22, the authors call Lemma 3.9 as the lemma of Heine-Schwartz since its direct proof (given in Laurent Schwartz's book) is identical to the proof of the classical Heine's theorem on the uniform continuity (continuity on a compact space implies the uniform continuity).

Lemma 3.10. *Let A be a compact metric space and $F : X \times A \to Y$ be a mapping. Then the following assertions are equivalent.*

(i) *F is continuous on $X \times A$.*

(ii) *The superposition operator $N_F : C^0(A,X) \to C^0(A,Y)$ defined by $N_F(\varphi) := [a \mapsto F(\varphi(a),a)]$ is well-defined and continuous.*

Proof. **($i \Longrightarrow ii$)** We fix $\varphi \in C^0(A,X)$ and $\varepsilon > 0$. We set $K := \{(\varphi(a),a) : a \in A\}$ which is compact as the range of the compact A by the continuous mapping $a \mapsto (\varphi(a),a)$. We consider $\delta > 0$ provided by Lemma 3.9 where X is replaced by $X \times A$ and ϕ is replaced by F. When $\psi \in C^0(A,X)$ satisfies $d_\infty(\psi,\varphi) \le \delta$ then we have $d((\varphi(a),a),(\psi(a),a)) = d(\varphi(a),\psi(a)) + d(a,a) \le d_\infty(\psi,\varphi) \le \delta$ with $(\varphi(a),a) \in K$ for all $a \in A$. Then Lemma 3.9 implies $d(F(\varphi(a),a),F(\psi(a),a)) \le \varepsilon$ for all $a \in A$, that means: $d_\infty(N_F(\varphi),N_F(\psi)) \le \varepsilon$.

($ii \Longrightarrow i$) When $x \in X$ we consider the constant function $\varphi_x(a) := x$, and so $\varphi_x \in C^0(A,X)$ and the mapping $x \mapsto \varphi_x$ is continuous from X into $C^0(A,X)$. We fix $(x,a) \in X \times A$, and consider $((x_n,a_n))_n$ a sequence of points of $X \times A$ which converges towards (x,a). Then, for all $n \in \mathbb{N}$, the following inequalities hold:

$$\begin{aligned} d(F(x_n,a_n),F(x,a)) &\le d(F(x_n,a_n),F(x,a_n)) + d(F(x,a_n),F(x,a)) \\ &= d(N_F(\varphi_{x_n})(a_n),N_F(\varphi_x)(a_n)) + d(N_F(\varphi_x)(a_n),N_F(\varphi_x)(a)) \\ &\le d_\infty(N_F(\varphi_{x_n}),N_F(\varphi_x)) + d(N_F(\varphi_x)(a_n),N_F(\varphi_x)(a)). \end{aligned}$$

Note that $\lim_{n\to\infty} d_\infty(N_F(\varphi_{x_n}),N_F(\varphi_x)) = 0$ since N_F and $x \mapsto \varphi_x$ are continuous, and that $\lim_{n\to\infty} d(N_F(\varphi_x)(a_n),N_F(\varphi_x)(a)) = 0$ since $N_F(\varphi_x) \in C^0(A,Y)$. And so the last inequalities imply $\lim_{n\to\infty} d(F(x_n,a_n),F(x,a)) = 0$.
And since we work in metric spaces, we conclude that F is continuous. □

In the following lemma we use the Bohr compactification $\mathbb{R}_B$ of $\mathbb{R}$, [36], Subsection 1.1 in Section 1 in Chapter 1 p. 5. We denote by $i_B : \mathbb{R} \to \mathbb{R}_B$ the associated injective homomorphism of topological groups. Recall that $\mathbb{R}_B$ is a compact group, that $i_B(\mathbb{R})$ is dense in $\mathbb{R}_B$, and that $u \in AP^0(\mathbb{R},X)$ then there exists $u_B \in C^0(\mathbb{R}_B,X)$ such that $u_B(i_B(t)) = u(t)$ for all $t \in \mathbb{R}$.

Lemma 3.11. *Let $f \in APU(X \times \mathbb{R},Y)$. Then there exists $f_B \in C^0(X \times \mathbb{R}_B,Y)$ such that $f_B(x,i_B(t)) = f(x,t)$ for all $(x,t) \in X \times \mathbb{R}$.*

Proof. We consider the operator $I_Y : AP^0(\mathbb{R},Y) \to C^0(\mathbb{R}_B,Y)$ defined by $I_Y(v) := v_B$. It is easy to check that I_Y is a bijective isometry. We define $f^\sharp : X \to AP^0(\mathbb{R},Y)$ by setting $f^\sharp(x) := [t \mapsto f(x,t)]$. By using Lemma 3.7 we see that $f^\sharp$ is continuous. Denote by π_1 and π_2 the two projections on $X \times \mathbb{R}_B$. We define the mapping $E : C^0(\mathbb{R}_B,Y) \times \mathbb{R}_B \to Y$ by $E(\varphi,\xi) := \varphi(\xi)$. We can verify that E is continuous. Since a composition of continuous mappings is continuous, we obtain that $f_B := E \circ (I_Y \circ f^\sharp \circ \pi_1, \pi_2)$ is continuous from $X \times \mathbb{R}_B$ into Y. And when $(x,t) \in X \times \mathbb{R}$, we have $f_B(x,i_B(t)) = E(I_Y \circ f^\sharp(x), i_B(t)) = I_Y(f^\sharp(x))(i_B(t)) = f^\sharp(x)(t) = f(x,t)$. □

Third Proof of Theorem 3.5.

Proof. We consider $f_B \in C^0(X \times \mathbb{R}_B,Y)$ provided by Lemma 3.11. In view of Lemma 3.10 it follows that the superposition operator $N_{f_B} : C^0(\mathbb{R}_B,X) \to C^0(\mathbb{R}_B,Y)$, defined by $N_{f_B}(\varphi) = [\xi \mapsto f_B(\varphi(\xi),\xi)]$, is continuous, i.e. the following assertion holds:

(9) For all $\varphi \in C^0(\mathbb{R}_B, X)$ and all $\varepsilon > 0$ there exists $\eta = \eta(\varphi, \varepsilon) > 0$ such that, for all $\psi \in C^0(\mathbb{R}_B, X)$, if $d_\infty(\psi, \varphi) \leq \eta$, then we have $d_\infty(N_{f_B}(\psi), N_{f_B}(\varphi)) \leq \varepsilon$.

Fix $u \in AP^0(\mathbb{R}, X)$ and $\varepsilon > 0$. Let $\eta = \eta(u_B, \varepsilon) > 0$ be provided by (9). If $v \in AP^0(\mathbb{R}, X)$ satisfies $d_\infty(v, u) \leq \eta$ then we also have $d_\infty(v_B, u_B) \leq \eta$ and consequently, by using (9), we obtain, for all $t \in \mathbb{R}$, the following inequalities:

$$d(f(v(t),t), f(u(t),t)) = d(f_B(v(t), i_B(t)), f_B(u(t), i_B(t)))$$

$$= d(f_B(v_B(i_B(t)), i_B(t)), f_B(u_B(i_B(t)), i_B(t))) \leq d_\infty(N_{f_B}(v_B), N_{f_B}(u_B)) \leq \varepsilon,$$

that implies $d_\infty(N_f^1(v), N_f^1(u)) \leq \varepsilon$. □

In the finite-dimensional Banach spaces setting, Theorem 3.5 is stated in [30], Lemma 1.1, p. 5, with no proof. The authors rather refer to 1955 Amerio' works (in Italian), 1960 Perov's results (in Russian), 1962 Zubov's works (in Russian), and 1961 Corduneanu's results (in Romanian); see [30] for precise references. When X is a finite-dimensional space, a proof of Theorem 3.5 is also given in [11], Lemma 5, p. 706.

Now we establish the converse of Theorem 3.5.

Theorem 3.12. *Let $f : X \times \mathbb{R} \to Y$ be a mapping such that, for all $u \in AP^0(\mathbb{R}, X)$, we have $[t \mapsto f(u(t),t)] \in AP^0(\mathbb{R}, Y)$, and such that the superposition operator $N_f^1 : AP^0(\mathbb{R}, X) \to AP^0(\mathbb{R}, Y)$ is continuous. Then we have $f \in APU(X \times \mathbb{R}, Y)$.*

Proof. For all $x \in X$ we consider the constant function $u_x : \mathbb{R} \to X$ defined by $u_x(t) := x$ for all $t \in \mathbb{R}$. Then we have $u_x \in AP^0(\mathbb{R}, X)$, and by using the first assumption we obtain the following assertion:

(10) For all $x \in X$, $f(x,.) = N_f^1(u_x) \in AP^0(\mathbb{R}, Y)$.

It is clear that the mapping $U : X \to AP^0(\mathbb{R}, X)$, defined by $U(x) := u_x$, is continuous and since $N_f^1 : AP^0(\mathbb{R}, X) \to AP^0(\mathbb{R}, Y)$ is continuous, the composition $N_f^1 \circ U$ is continuous on X, and by using the Heine theorem, we obtain that, for all $K \in \mathcal{P}_c(X)$, the mapping $x \mapsto f(x,.)$ is uniformly continuous on K, i.e. the following assertion holds:

(11) For all $K \in \mathcal{P}_c(X)$ and all $\varepsilon > 0$ there exists $\delta = \delta(K, \varepsilon) > 0$ such that, for all $x, z \in K$, if $d(x,z) \leq \delta$ then $d(f(x,t), f(z,t)) \leq \varepsilon$ for all $t \in \mathbb{R}$.

This assertion means that, for all $K \in \mathcal{P}_c(X)$, the family $(f(.,t))_{t \in \mathbb{R}}$ is uniformly equicontinuous on K.

Now we fix $K \in \mathcal{P}_c(X)$ and $\varepsilon > 0$. Then there exists a finite subset $\{x_1, ..., x_n\}$ in K such that $K \subset \bigcup_{1 \leq i \leq n} \{x \in X : d(x, x_i) \leq \delta(K, \frac{\varepsilon}{3})\}$ where $\delta(K, \frac{\varepsilon}{3})$ is provided by (11). By using (10) we know that $f(x_i,.) \in AP^0(\mathbb{R}, Y)$ for all $i = 1, ..., n$, and by using the Bochner criterion, we see that the function $t \mapsto (f(x_1,.), ..., f(x_n,.))$ is almost periodic, from which we obtain the following assertion:

(12) For all $\sigma > 0$ there exists $\ell = \ell_\sigma > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r + \ell]$ satisfying $d(f(x_i, t + \tau), f(x_i, t)) \leq \sigma$ for all $i = 1, ..., n$ and all $t \in \mathbb{R}$.

We use (12) with $\sigma = \frac{\varepsilon}{3}$. Let $r \in \mathbb{R}$ and $\tau \in [r, r+\ell]$. Then we have $d(f(x_i, t+\tau), f(x_i, t)) \leq \frac{\varepsilon}{3}$ for all $i = 1, \dots, n$ and all $t \in \mathbb{R}$. When $x \in K$ there exists $i \in \{1, \dots, n\}$ such that $d(x, x_i) \leq \delta(K, \frac{\varepsilon}{3})$, and then, by using (11), the following inequalities hold: $d(f(x,t), f(x_i,t)) \leq \frac{\varepsilon}{3}$ and $d(f(x, t+\tau), f(x_i, t+\tau)) \leq \frac{\varepsilon}{3}$ for all $t \in \mathbb{R}$. Therefore we obtain

$$d(f(x,t+\tau), f(x,t)) \leq d(f(x,t+\tau), f(x_i,t+\tau)) + d(f(x_i,t+\tau), f(x_i,t))$$

$$+ d(f(x_i,t), f(x,t)) \leq 3\frac{\varepsilon}{3} = \varepsilon,$$

for all $t \in \mathbb{R}$. And so we have proven that for all $K \in \mathcal{P}_c(X)$ and for all $\varepsilon > 0$ there exists $\ell = \ell(K, \varepsilon) > 0$ such that, for all $r \in \mathbb{R}$ there exists $\tau \in [r, r+\ell]$ satisfying $d(f(x,t+\tau), f(x,t)) \leq \varepsilon$ for all $x \in K$ and for all $t \in \mathbb{R}$. This is the second condition to have $f \in APU(X \times \mathbb{R}, Y)$. After that, it suffices to prove that f is continuous on $X \times \mathbb{R}$.

Let C be a compact subset of $X \times \mathbb{R}$. We set pr_1 and pr_2 the two projections on $X \times \mathbb{R}$, and $C_1 := pr_1(C)$, $C_2 := pr_2(C)$. Note that C_1 and C_2 are compact and that we have $C \subset C_1 \times C_2$. We fix $\varepsilon > 0$. By using (11) with $K = C_1$, we know that the following assertion holds:

(13) There exists $\delta = \delta(C_1, \frac{\varepsilon}{3}) > 0$ such that, for all $x, z \in C_1$, if $d(x,z) \leq \delta$ then we have $d(f(x,t), f(z,t)) \leq \frac{\varepsilon}{3}$ for all $t \in \mathbb{R}$.

Since $C_1 \in \mathcal{P}_c(X)$, there exists a finite subset $\{z_1, \dots, z_m\}$ in C_1 such that $C_1 \subset \bigcup_{1 \leq i \leq m} \{z \in X : d(z, z_i) \leq \delta\}$. Since $f(z_i, .) \in AP^0(\mathbb{R}, Y)$, $f(z_i, .)$ is uniformly continuous on C_2, and consequently we have: for all $i = 1, \dots, m$ and for all $\sigma > 0$ there exists $\eta_i(\sigma) > 0$ such that, for all $s, t \in \mathbb{R}$, if $|s-t| \leq \eta_i(\sigma)$ then $d(f(z_i,s), f(z_i,t)) \leq \sigma$. We set $\eta(\sigma) := \min_{1 \leq i \leq m} \eta_i(\sigma) > 0$, and we obtain the following assertion:

(14) For all $\sigma > 0$ there exists $\eta(\sigma) > 0$ such that, for all $s, t \in \mathbb{R}$, if $|s-t| \leq \eta(\sigma)$ then we have $d(f(z_i,s), f(z_i,t)) \leq \sigma$ for all $i = 1, \dots, m$.

Now we fix $x \in C_1$ and $t \in C_2$. We choose z_i such that $d(x, z_i) \leq \delta$ where δ is the one of (13). We consider $z \in C_1$ such that $d(z, z_i) \leq \delta$ and $s \in C_2$ such that $|s-t| \leq \eta(\frac{\varepsilon}{3}$ where η is provided by (14). Then by using (13) and (14), we obtain the following inequalities:

$$d(f(x,t), f(z,s)) \leq$$

$$d(f(x,t), f(z_i,t)) + d(f(z_i,t), f(z_i,s)) + d(f(z_i,s), f(z,s)) \leq 3\frac{\varepsilon}{3} = \varepsilon.$$

That proves the continuity of the restriction of f to $C_1 \times C_2$. Consequently the restriction of f to C is also continuous, for all compact subset C of $X \times \mathbb{R}$. Since $X \times \mathbb{R}$ is a metric space, this proves the continuity of f on $X \times \mathbb{R}$. □

Corollary 3.13. *Let $\phi : X \to Y$ be a mapping. Then the two following assertions are equivalent.*

(i) *ϕ is continuous from X into Y.*

(ii) *The superposition operator $u \mapsto \phi \circ u$ is continuous from $AP^0(\mathbb{R}, X)$ into $AP^0(\mathbb{R}, Y)$.*

Following Remark 2.1, Corollary 3.13 is a straightforward consequence of Theorem 3.5 and Theorem 3.12.

Remark 3.14. All the results of this section remain valid if we replace $\mathbb{R}$ by $\mathbb{R}_+ = [0,\infty)$ or by a locally compact Abelian group.

Remark 3.15. When Ω is a nonempty open subset of X, we define

$$AP^0(\mathbb{R},\Omega) := \{u \in AP^0(\mathbb{R},X) : \overline{u(\mathbb{R})} \subset \Omega\}.$$

Since $\overline{u(\mathbb{R})}$ is compact when $u \in AP^0(\mathbb{R},X)$ and since $X \setminus \Omega$ is closed, when $u \in AP^0(\mathbb{R},\Omega)$, we can find $\rho > 0$ such that $v \in AP^0(\mathbb{R},\Omega)$ when $v \in AP^0(\mathbb{R},X)$ and $\|v-u\|_\infty < \rho$. And so $AP^0(\mathbb{R},\Omega)$ is an open subset of $AP^0(\mathbb{R},X)$. When $f \in C^0(X \times \mathbb{R},Y)$ satisfies the condition which defines $APU(X \times \mathbb{R},Y)$ where we replace $K \in \mathcal{P}_c(X)$ by $K \in \mathcal{P}_c(\Omega)$, we say that $f \in APU(\Omega \times \mathbb{R},Y)$. And then all the results of this section remain valid when we replace $AP^0(\mathbb{R},X)$ by $AP^0(\mathbb{R},\Omega)$ and $APU(X \times \mathbb{R},Y)$ by $APU(\Omega \times \mathbb{R},Y)$.

4 An Application to Evolution Equations

We consider a Banach space X, a (possibly) unbounded linear operator $A : \mathcal{D}(A) \subset X \to X$, a nonlinear mapping $f : X \times \mathbb{R}_+ \to X$, and the following semilinear evolution equation:

(15) $u'(t) = Au(t) + f(u(t),t)$.

We consider the following list of assumptions on the equation (15).

(16) A is the infinitesimal generator of a C^0-semigroup of linear operators $(S(t))_{t\geq 0}$ on X such that there exists $\omega > 0$ satisfying $\|S(t)x - S(t)y\| \leq e^{\omega t}\|x-y\|$ for all $t \geq 0$ and for all $x,y \in X$.

(17) $f \in APU(X \times \mathbb{R}_+,X)$.

(18) There exists $\gamma > 0$ such that, for all $\lambda > 0$, for all $x,y \in X$ and for all $t \geq 0$ we have $(1-\lambda\gamma)\|x-y\| \leq \|x-y+\lambda(f(x,t)-f(y,t))\|$.

All the notions used in these three assumptions are defined in [29]; (18) is called a condition of accretivity of $-f+\gamma I$, where I is the identity operator on X. Recall that a so-called mild solution of (15) is a continuous solution u of the following integral equation:

$$u(t) = S(t-s)x + \int_s^t S(t-r)f(u(r),r)dr, \quad t \geq s.$$

Theorem 4.1. *We assume (16)-(18) fulfilled and we assume that* $\omega - \gamma < 0$. *Then there exists a unique almost periodic mild solution of (15).*

This result is proven by Corollary 2.27 in [29], p. 146 by using an (apparently) additional assumption. Now we explain what is new here. In [29], the authors formulate a so-called condition (H5), Definition 2.18, p. 144, which says (by using our vocabulary and our notations) that the superposition $N_f^1 : AP^0(\mathbb{R}_+,X) \to AP^0(\mathbb{R}_+,X)$ is well-defined and continuous. By using our results of Section 3, this condition (H5) is translatable into a condition on f which is exactly (17). Moreover, inside their so-called condition (H4), Definition 2.17, p.144, they assume that f is continuous on $X \times \mathbb{R}_+$ which is redundant since this continuity is contained in (17) after our Theorem 3.12. Finally, in their statement of their Corollary 2.17, the authors of [29] assume (by translating in our notations) that $-N_f^1 + \gamma N_I^1$ is an accretive operator in the sense given in their Definition 1.5, p. 17. It is easy to verify that (18) implies their accretivity condition on the operators.

5 Differentiability and Almost Periodicity

In this section, X and Y are Banach spaces.

Theorem 5.1. *Let $f \in APU(X \times \mathbb{R}, Y)$ such that the partial Fréchet-differential with respect to the first vector variable $D_x f(x,t)$ exists for all $(x,t) \in X \times \mathbb{R}$. We also assume that $D_x f \in APU(X \times \mathbb{R}, \mathcal{L}(X,Y))$.*
Then the superposition operator N_f^1, defined in (4), is continuously differentiable from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$, and for all $u, v \in AP^0(\mathbb{R},X)$ we have
$DN_f^1(u).v = [t \mapsto D_x f(u(t),t).v(t)]$.

Proof. Since $D_x f \in APU(X \times \mathbb{R}, \mathcal{L}(X,Y))$, by using Theorem 3.5, we have:

(19) $N^1_{D_x f}$ is continuous from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R}, \mathcal{L}(X,Y))$.

We fix $u \in AP^0(\mathbb{R},X)$ and set

(20) $L(t) := D_x f(u(t),t)$ for all $t \in \mathbb{R}$.

$L \in AP^0(\mathbb{R}, \mathcal{L}(X,Y))$ in view of (19). We define now the mapping $F : X \times \mathbb{R} \to Y$ by

(21) $F(z,t) := L(t).z$ for all $(z,t) \in X \times \mathbb{R}$.

We consider the bilinear continuous operator $B : \mathcal{L}(X,Y) \times X \to Y$ defined by $B(T,x) := T.x$. Denoting by pr_1 and pr_2 the two projections on $X \times \mathbb{R}$, we see that $F = B \circ (L \circ pr_2, pr_1)$ and consequently F is continuous as a composition of continuous mappings. We fix $K \in \mathcal{P}_c(X)$ and $\varepsilon > 0$. Let $\rho > 0$ such that $\|x\| \le \rho$ for all $x \in K$. Since L is almost periodic, there exists $\ell = \ell(\frac{\varepsilon}{\rho}) > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\ell]$ satisfying $\|L(t+\tau) - L(t)\| \le \frac{\varepsilon}{\rho}$ for all $t \in \mathbb{R}$. Consequently, for all $(x,t) \in K \times \mathbb{R}$, we have $\|F(x,t+\tau) - F(x,t)\| \le \|L(t+\tau) - L(t)\|.\|x\| \le \frac{\varepsilon}{\rho}.\rho = \varepsilon$. And so we have proven that $F \in APU(X \times \mathbb{R}, Y)$. Then, by using Lemma 3.4, we obtain the following.

(22) For all $v \in AP^0(\mathbb{R},X)$, $\Lambda.v := [t \mapsto D_x f(u(t),t).v(t)] \in AP^0(\mathbb{R},Y)$.

By using the mean value theorem, Corollaire 1 in [2], Corollaire 1, p. 144, or result 8.6.2 in [22], p. 164, we have, for all $v \in AP^0(\mathbb{R},X)$ and for all $t \in \mathbb{R}$, the following inequality:

(23) $\|f(u(t)+v(t),t) - f(u(t),t) - D_x f(u(t),t).v(t)\|$
$\le \sup_{\zeta \in]u(t),u(t)+v(t)[} \|D_x f(\zeta,t) - D_x f(u(t),t)\|.\|v(t)\|$.

By using Lemma 3.7, and setting $K := \overline{u(\mathbb{R})} \in \mathcal{P}_c(X)$, we obtain:

(24) For all $\varepsilon > 0$ there exists $\delta = \delta(K,\varepsilon) > 0$ such that, for all $x \in K$ and for all $z \in X$, if $\|x - z\| \leq \delta$ then we have $\|D_x f(z,t) - D_x f(x,t)\| \leq \varepsilon$ for all $t \in \mathbb{R}$.

Fix now $\varepsilon > 0$ and consider $v \in AP^0(\mathbb{R},X)$ such that $\|v\|_\infty \leq \delta$ where δ is provided by (24). Then, for all $t \in \mathbb{R}$ and all $\zeta \in]u(t), u(t)+v(t)[$, we have $\|\zeta - u(t)\| \leq \|(u(t)+v(t)) - u(t)\| = \|v(t)\| \leq \|v\|_\infty \leq \delta$, which implies, (in view of(24)), the following assertion:

(25) For all $t \in \mathbb{R}$ and all $\zeta \in]u(t), u(t)+v(t)[$ and $\|D_x f(\zeta,t) - D_x f(u(t),t)\| \leq \varepsilon$.

From (23) and (25) we deduce that

$$\|f(u(t)+v(t),t) - f(u(t),t) - D_x f(u(t),t).v(t)\| \leq \varepsilon \|v(t)\|$$

for all $t \in \mathbb{R}$, and by taking the supremum on $t \in \mathbb{R}$, we obtain $\|N_f^1(u+v) - N_f^1(u) - \Lambda.v\|_\infty \leq \|v\|_\infty$ when $\|v\|_\infty \leq \delta$; this proves that N_f^1 is Fréchet-differentiable at u and that

(26) $DN_f^1(u).v = [t \mapsto D_x f(u(t),t).v(t)]$.

Now it suffices to proves the continuity of DN_f^1. By using (19) and Theorem 3.5, we know that the following assertion holds.

(27) For all $u \in AP^0(\mathbb{R},X)$ and all $\varepsilon > 0$ there exists $\eta = \eta(u,\varepsilon) > 0$ such that, for all $u_1 \in AP^0(\mathbb{R},X)$, if $\|u - u_1\|_\infty \leq \eta$ then $\|N^1_{D_x f}(u) - N^1_{D_x f}(u_1)\| \leq \varepsilon$.

We fix $u \in AP^0(\mathbb{R},X)$ and $\varepsilon > 0$. Let $u_1 \in AP^0(\mathbb{R},X)$ such that $\|u - u_1\|_\infty \leq \eta$, where η is provided by (27). Then, for all $v \in AP^0(\mathbb{R},X)$ such that $\|v\|_\infty \leq 1$, by using (27) we have

$$\|D_x f(u(t),t).v(t) - D_x f(u_1(t),t).v(t)\| \leq \|D_x f(u(t),t) - D_x f(u_1(t),t)\|.\|v(t)\|$$

$$\leq \|N_{D_x f}(u) - N_{D_x f}(u_1)\|.\|v\|_\infty \leq \varepsilon.1 = \varepsilon,$$

that implies, by taking the supremum on all $t \in \mathbb{R}$, that $\|DN_f^1(u).v - DN_f^1(u_1).v\|_\infty \leq \varepsilon$, and by taking the supremum on the $v \in AP^0(\mathbb{R},X)$ such that $\|v\|_\infty \leq 1$, we obtain $\|DN_f^1(u) - DN_f^1(u_1)\|_\infty \leq \varepsilon$ in norm of linear operators. And so the continuity of DN_f^1 follows. □

Theorem 5.1 was established in [20], Proposition 3 in Section 3 of Chapter 2, p. 17, by using a different proof. When X is a finite-dimensional space, a proof of Theorem 5.1 is given in [11], Lemma 7, p. 710. By induction from Theorem 5.1 we obtain the following result.

Theorem 5.2. *Let $f \in APU(X \times \mathbb{R}, Y)$ and $n \in \mathbb{N}_*$. We assume that the partial Fréchet-differential with respect the first vector variable of order n, $D_x^n f(x,t)$, exists for all $(x,t) \in X \times \mathbb{R}$, and that $D_x^k f \in APU(X \times \mathbb{R}, \mathcal{L}_k(X^k,Y))$ for all $k = 1,\dots,n$.*
Then the superposition operator N_f^1, defined in (4), is n-times continuously differentiable from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$, and moreover, for all $u, v_1, \dots, v_n \in AP^0(\mathbb{R},X)$ we have $D^n N_f^1(u).(v_1,\dots,v_n) = [t \mapsto D_x^n f(u(t),t).(v_1(t),\dots,v_n(t))]$.

By using Remark 2.1, a straightforward of this result is the following one.

Corollary 5.3. *Let $n \in \mathbb{N}_*$ and $\phi \in C^n(X,Y)$. Then the superposition operator $N^1_\phi : u \mapsto \phi \circ u$, from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$, is n-times continuously differentiable on $AP^0(\mathbb{R},X)$, and, for all $u, v_1, \ldots, v_n \in AP^0(\mathbb{R},X)$, we have $D^n N^1_\phi(u).(v_1,\ldots,v_n) = [t \mapsto D^n\phi(u(t)).(v_1(t),\ldots,v_n(t))]$.*

When $n = 1$, in the finite-dimensional setting, Corollary 5.3 is proven in [9], Proposition 1, p. 19.

6 An Application to Differential Equations

In this section, X is a Banach space. Let $f_0, f_1 : X \times \mathbb{R} \to X$ be two mappings and $\varepsilon \in \mathbb{R}$. We consider the following ordinary differential equation:

(28) $u'(t) = f_0(u(t),t) + \varepsilon. f_1(u(t),t)$.

When $\varepsilon = 0$ this equation becomes:

(29) $u'(t) = f_0(u(t),t)$.

We will assume that there exists a solution $u_* \in AP^1(\mathbb{R},X)$ of (29), that the partial Fréchet-differential of f_0 with respect to x exists and we consider the following condition on the linearization of (29) around u_*.

(30) For all $b \in AP^0(\mathbb{R},X)$ there exists a unique $v \in AP^1(\mathbb{R},X)$ such that $v'(t) = D_x f_0(u_*(t),t).v(t) + b(t)$ for all $t \in \mathbb{R}$.

We can find conditions on $D_x f_0(u_*(t),t)$ which ensure the validity of (30). For instance in [20], Théorème 3, Section 3, Chapter III, p. 43, the following assertion is proven: if X is a Hilbert space and if $[t \mapsto D_x f_0(u_*(t),t)] \in AP^0(\mathbb{R}, \mathcal{L}(X,X))$ and if there exists $\alpha > 0$ satisfying $\langle D_x f_0(u_*(t),t).x \mid x\rangle \geq \alpha\|x\|^2$ for all $(x,t) \in X \times \mathbb{R}$, then (30) is fulfilled. In the finite-dimensional setting, such a question is considered in [27], Lemma 13-1, p. 122.

Theorem 6.1. *We assume that $f_0, f_1 \in APU(X \times \mathbb{R}, X)$, that the partial Fréchet-differentials $D_x f_0(x,t)$ and $D_x f_1(x,t)$ exist for all $(x,t) \in X \times \mathbb{R}$, and that $D_x f_0, D_x f_1 \in APU(X \times \mathbb{R}, \mathcal{L}(X,X))$. We assume that there exists $u_* \in AP^1(\mathbb{R},X)$ which is a solution of (29) for which condition (30) is also fulfilled. Then there exist $\varepsilon_0 > 0$ and a continuously differentiable function $\varepsilon \mapsto u_\varepsilon$, from $(-\varepsilon_0, \varepsilon_0)$ into $AP^1(\mathbb{R},X)$, such that, for all $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, u_ε is an almost periodic solution of the differential equation (28) and such that $u_0 = u_*$.*

Proof. We introduce the operator $T : AP^1(\mathbb{R},X) \times \mathbb{R} \to AP^0(\mathbb{R},X)$ defined by $T(u,\varepsilon) := [t \mapsto u'(t) - f_0(u(t),t) - \varepsilon. f_1(u(t),t)]$. We consider the operator $\frac{d}{dt} : AP^1(\mathbb{R},X) \to AP^0(\mathbb{R},X)$ which is linear continuous. We also consider the canonical injection $in : AP^1(\mathbb{R},X) \to AP^0(\mathbb{R},X)$, $in(u) := u$, which is linear continuous. Then we see that $T(u,\varepsilon) = \frac{d}{dt}u - N^1_{f_0} \circ in(u) - \varepsilon.N^1_{f_1} \circ in(u)$ for all $(u,\varepsilon) \in AP^1(\mathbb{R},X) \times \mathbb{R}$. By using Theorem 5.2, we know that $N^1_{f_0}$ and $N^1_{f_1}$ are of class C^1 on $AP^0(\mathbb{R},X)$. Then we obtain that T is of class C^1 on $AP^1(\mathbb{R},X) \times \mathbb{R}$. We also verify that the following equation holds:

(31) $T(u_*,0) = 0$.

We calculate the partial Fréchet-differential of T with respect to its first (functional) variable $D_uT(u_*,0).v = \frac{d}{dt}v - DN^1_{f_0}(u_*).v - 0.DN^1_f(u_*).v$. And we see that $D_uT(u_*,0).v = b$ is equivalent to the differential equation $v'(t) = D_x f_0(u_*(t),t).v(t) + b(t)$ for all $t \in \mathbb{R}$. And consequently, (30) ensures that the following assertion holds:

(32) $D_uT(u_*,0) \in Isom(AP^1(\mathbb{R},X) \times \mathbb{R}, AP^0(\mathbb{R},X))$.

After (31) and (32) we can use the implicit function theorem, [1] Theorem 2.5.7, p. 107 and conclude that there exist $\varepsilon_0 > 0$ and a continuously differentiable function $\varepsilon \mapsto u_\varepsilon$, from $(-\varepsilon_0, \varepsilon_0)$ into $AP^1(\mathbb{R},X)$, such that $T(u_\varepsilon, \varepsilon) = 0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, that means that $u'_\varepsilon(t) = f_0(u_\varepsilon(t),t) + \varepsilon.f_1(u_\varepsilon(t),t)$ for all $t \in \mathbb{R}$, and moreover $u_0 = u_*$. □

Remark 6.2. Such a method based on the implicit function theorem is used for instance in [11] for the bounded solutions of second-order ordinary differential equations and in [8] for the periodic solutions of Hamiltonian systems.

7 Differentiable Almost Periodic Functions

In this section, X and Y are Banach spaces.

Lemma 7.1. *Let $f \in APU(X \times \mathbb{R}, Y) \cap C^1(X \times \mathbb{R}, Y)$ such that its Fréchet-differential $Df \in APU(X \times \mathbb{R}, \mathcal{L}(X \times \mathbb{R}, Y))$. We consider the new mapping $g : X \times X \times \mathbb{R} \to Y$ defined by $g(x_1,x_2,t) := Df(x_1,t).(x_2,1)$. Then the following assertions hold.*

(i) $g \in APU((X \times X) \times \mathbb{R}, Y)$.

(ii) *Let $n \in \mathbb{N}_*$, $n \geq 2$. If we assume moreover that $f \in C^n(X \times \mathbb{R}, Y)$ and that $D^i f \in APU(X \times \mathbb{R}, \mathcal{L}_i((X \times \mathbb{R})^i, Y))$ for all $i = 1,...,n$, then $g \in C^{n-1}(X \times X \times \mathbb{R}, Y)$ and $D^j g \in APU((X \times X) \times \mathbb{R}, \mathcal{L}_j((X \times X \times \mathbb{R})^j, Y))$ for all $j = 1,...,n-1$.*

Proof. **(i)** We consider the three projection mappings pr_1, pr_2 and pr_3 on the product space $X \times X \times \mathbb{R}$, and the bilinear continuous mapping $B : \mathcal{L}(X \times \mathbb{R}, Y) \times (X \times \mathbb{R}) \to Y$ defined by $B(\varphi,(x,t)) := \varphi(x,t)$. We see that the following equality holds.

(33) $g = B \circ (Df \circ (pr_1, pr_3), (pr_2, c))$

where $c : X \times X \times \mathbb{R} \to \mathbb{R}$ is the constant mapping $c(x_1,x_2,t) := 1$. Since all the mappings present in the right member of (32) are continuous, g is continuous on $X \times X \times \mathbb{R}$ as a composition of continuous mappings. Now we fix $C \in \mathcal{P}_c(X \times X)$ and $\varepsilon > 0$. We set $C_1 := pr_1(C)$ and $C_2 := pr_2(C)$ which are compact subsets of X. We chose $\rho > 0$ such that $\|x_2\| + 1 \leq \rho$ for all $x_2 \in C_2$. Since we have $Df \in APU(X \times \mathbb{R}, \mathcal{L}(X \times \mathbb{R}, Y))$, we know that there exists $\ell = \ell(C_2, \frac{\varepsilon}{\rho}) > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\ell]$ satisfying $\|Df(x_1,t+\tau) - Df(x_1,t)\| \leq \frac{\varepsilon}{\rho}$ for all $(x_1,t) \in C_2 \times \mathbb{R}$. Therefore we have

$$\|Df(x_1,t+\tau).(x_2,1) - Df(x_1,t).(x_2,1)\|$$

$$\leq \|Df(x_1,t+\tau) - Df(x_1,t)\|.(\|x_2\| + 1) \leq \frac{\varepsilon}{\rho}.\rho = \varepsilon$$

for all $(x_1,x_2,t) \in C_1 \times C_2 \times \mathbb{R}$. Since we have $C \subset C_1 \times C_2$, we have proven that $g \in APU((X \times X) \times \mathbb{R}, Y)$.

(ii) We define the mappings $\Phi : X \times X \times \mathbb{R} \to \mathcal{L}(X \times \mathbb{R}, Y)$ and $\Psi : X \times X \times \mathbb{R} \to X \times \mathbb{R}$ by setting $\Phi := Df \circ (pr_1, pr_3)$ and $\Psi := (pr_2, c)$, and so by using (32), we obtain the following equality.

(34) $g = B \circ (\Phi, \psi)$.

Then by using the chain rule for the high-order differentials, for instance this one given in Box 2.4 A in [1], p. 91, for all $p = 2,\dots,n$, for all $(x_1,x_2,t) \in X \times X \times \mathbb{R}$, and for all $z^i = (x_1^i, x_2^i, t^i) \in X \times X \times \mathbb{R}$ when $i = 1,\dots,n$, we have the following formula:

$$D^p g(x_1,x_2,t).(z^1,\dots z^p)$$

$$= \sum_{i=0}^{p} \sum_{\sigma \in \mathcal{S}_p^i} \frac{p!}{i!(p-i)!} B(D^i\Phi(x_1,x_2,t).(z^{\sigma(1)},\dots z^{\sigma(p)}), D^{p-i}\Psi(x_1,x_2,t).(z^{\sigma(i+1)},\dots z^{\sigma(p)}))$$

where $\mathcal{S}_p^i$ denotes the set of the bijections σ from $\{1,\dots,p\}$ into itself such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(p)$. We note that

$$D^i\Phi((x_1,x_2,t).(z^{\sigma(1)},\dots z^{\sigma(p)}) = D^{i+1} f(x_1,t).((x_1^{\sigma(1)}, t^{\sigma(1)}),\dots,(x_1^{\sigma(i)}, t^{\sigma(i)})),$$

and since Ψ is affine, we have $D^{p-i}\Psi(x_1,x_2,t) = 0$ when $i \le p-2$ and we have $D^1\Psi(x_1,x_2,t).(\delta x_1, \delta x_2, \delta t) = (\delta x_2, 0)$. And so the previous formula for $D^p g$ becomes the following one.

(35) $D^p g(x_1,x_2,t).(z^1,\dots z^p) = D^{p+1} f(x_1,t).((x_1^1,t^1),\dots,(x_1^p,t^p),(x_2,1))$
$+ \sum_{i=1}^{p} D^p f(x_1,t).((x_1^1,t^1),\dots,(x_1^{i-1},t^{i-1}),(x_2,0),(x_1^{i+1},t^{i+1}),\dots,(x_1^p,t^p))$.

Now we fix $C \in \mathcal{P}_c(X \times X)$ and $\varepsilon > 0$, and we consider C_1 and C_2 defined like in the proof of (i). We chose $\rho > 0$ such that $\|x_2\| \le \rho$ for all $x_2 \in C_2$.
Since $D^{p+1} f \in APU(X \times \mathbb{R}, \mathcal{L}_{p+1}((X \times \mathbb{R})^{p+1}, Y))$ and $D^p f \in APU(X \times \mathbb{R}, \mathcal{L}_p((X \times \mathbb{R})^p, Y))$, we have $(D^{p+1} f, D^p f) \in APU(X \times \mathbb{R}, \mathcal{L}_{p+1}((X \times \mathbb{R})^{p+1}, Y) \times \mathcal{L}_p((X \times \mathbb{R})^p, Y))$, and consequently there exists $\ell = \ell(C_1, \frac{\varepsilon}{(p+1)(\rho+1)}) > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\ell]$ satisfying $\|D^{p+1} f(x_1,t+\tau) - D^{p+1} f(x_1,t)\|_{\mathcal{L}_{p+1}} \le \frac{\varepsilon}{(p+1)(\rho+1)}$ and $\|D^p f(x_1,t+\tau) - D^p f(x_1,t)\|_{\mathcal{L}_p} \le \frac{\varepsilon}{(p+1)(\rho+1)}$ for all $x_1 \in C_1$ and for all $t \in \mathbb{R}$. Consequently, by using (35), for all $z^1,\dots,z^p \in X \times X \times \mathbb{R}$ such that $\|z^i\| \le 1$ when $i = 1,\dots,p$, for all $(x_1,x_2,t) \in C_1 \times C_2 \times \mathbb{R}$, the following inequalities hold:

$$\|D^p g(x_1,x_2,t+\tau).(z^1,\dots z^p) - D^p g(x_1,x_2,t).(z^1,\dots z^p)\|$$

$$\le \|(D^{p+1} f(x_1,t+\tau) - D^{p+1} f(x_1,t)).((x_1^1,t^1),\dots,(x_1^p,t^p),(x_2,1)\|$$

$$+ \sum_{i=1}^{p} \|(D^p f(x_1,t+\tau) - D^p f(x_1,t)).$$

$$((x_1^1,t^1),\dots,(x_1^{i-1},t^{i-1}),(x_2,0),(x_1^{i+1},t^{i+1}),\dots,(x_1^p,t^p))\|$$

$$\leq \|(D^{p+1}f(x_1,t+\tau)-D^{p+1}f(x_1,t))\|.\prod_{i=1}^{p}(\|x_1^i\|+|t^i|).(\|x_2\|+1)$$

$$+\sum_{i=1}^{p}\|D^p f(x_1,t+\tau)-D^p f(x_1,t)\|.\prod_{1\leq j\leq p, j\neq i}(\|x_1^j\|+|t^j|).\|x_2\|$$

$$\leq \frac{\varepsilon}{(p+1)(\rho+1)}.(\rho+1)+\sum_{i=1}^{p}\frac{\varepsilon}{(p+1)(\rho+1)}.(\rho+1)=\varepsilon.$$

And so we have proven that $D^p g \in APU((X\times X)\times\mathbb{R}, \mathcal{L}_p((X\times X\times\mathbb{R})^p, Y)))$. □

Theorem 7.2. *Let $n\in\mathbb{N}_*$ and $f\in APU(X\times\mathbb{R},Y)\cap C^n(X\times\mathbb{R},Y)$ such that $D^k f\in APU(X\times\mathbb{R},\mathcal{L}_k((X\times\mathbb{R})^k,Y))$ for all $k=1,\dots,n$. Then the superposition operator $N_f^2 : AP^n(\mathbb{R},X)\to AP^n(\mathbb{R},Y)$, defined by $N_f^2(u) := [t\mapsto f(u(t),t)]$, is well-defined and continuous on $AP^n(\mathbb{R},X)$.*

Proof. We proceed by induction on n.
First Step: $n=1$. We denote by $in_1 : AP^1(\mathbb{R},X)\to AP^0(\mathbb{R},X)$ the canonical injection. By using Theorem 3.5 we know that $N_f^1(in_1(u))\in AP^0(\mathbb{R},Y)$ when $u\in AP^0(\mathbb{R},X)$. The function $t\mapsto f(u(t),t)$ is differentiable as a composition of differentiable functions, and we have

$$\frac{d}{dt}f(u(t),t)=Df(u(t),t).(u'(t),1)=g(u(t),u'(t),t)$$

where g is defined in Lemma 7.1. Since $(in_1(u),u')\in AP^0(\mathbb{R},X\times X)$, by using Lemma 3.4, we know that $[t\mapsto f(u(t),t)]=N_g^1(in_1(u),u')\in AP^0(\mathbb{R},Y)$, and so N_f^2 is well-defined on $AP^1(\mathbb{R},X)$.

We note that, for all $u,u_1\in AP^1(\mathbb{R},X)$, the following equality holds:

(36) $\|N_f^2(u)-N_f^2(u_1)\|_{C^1}=\|N_f^1\circ in_1(u)-N_f^1\circ in_1(u_1)\|_\infty$
$+\|N_g^1\circ(in_1,\frac{d}{dt})(u)-N_g^1\circ(in_1,\frac{d}{dt})(u_1)\|_\infty.$

It is easy to see that $in_1 : AP^1(\mathbb{R},X)\to AP^0(\mathbb{R},X)$ and $\frac{d}{dt} : AP^1(\mathbb{R},X)\to AP^0(\mathbb{R},X)$ are linear continuous. By using Theorem 3.5, $N_f^1 : AP^0(\mathbb{R},X)\to AP^0(\mathbb{R},Y)$ is continuous. By using Lemma 7.1, we know that $N_g^1 : AP^0(\mathbb{R},X\times X)\to AP^0(\mathbb{R},Y)$ is continuous. Consequently $N_f^1\circ in_1 : AP^1(\mathbb{R},X)\to AP^0(\mathbb{R},Y)$ and $N_g^1\circ(in_1,\frac{d}{dt}) : AP^1(\mathbb{R},X)\to AP^0(\mathbb{R},Y)$ are continuous as compositions of continuous operators. And so, by using (36), we deduce that N_f^2 is continuous from $AP^1(\mathbb{R},X)$ into $AP^1(\mathbb{R},Y)$.

Second Step: the induction assumption. We assume that the result is valid for an integer $n\in\mathbb{N}_*$.

Third Step: we prove the result for $n+1$. We consider the canonical injection $in_{n+1} : AP^{n+1}(\mathbb{R},X)\to AP^n(\mathbb{R},X)$. After Second Step we know that the operator $N_f^{2,n} : AP^n(\mathbb{R},X)\to AP^n(\mathbb{R},Y)$, defined by $N_f^{2,n}(w) := [t\mapsto f(w(t),t)]$ is well-defined and continuous. For all $u\in AP^{n+1}(\mathbb{R},X)$, we have $in_{n+1}(u)\in AP^n(\mathbb{R},X)$ and consequently we obtain
$N_f^{2,n}(in_{n+1}(u))\in AP^n(\mathbb{R},Y)$. By using Lemma 7.1, ii, we know that $g\in APU((X\times X)\times\mathbb{R},Y)\cap C^n(X\times X\times\mathbb{R},Y)$, and $D^j g\in APU(((X\times X)\times\mathbb{R},\mathcal{L}_j((X\times X\times\mathbb{R})^j,Y))$ for all $j=1,\dots,n$. Consequently by using Second Step, we know that $N_g^2 : AP^n(\mathbb{R},X\times X)\to AP^n(\mathbb{R},Y)$, defined by $N_g^2(u_1,u_2) := [t\mapsto g(u_1(t),u_2(t),t)]$, is well-defined and continuous.

And so, for all $u \in AP^{n+1}(\mathbb{R},X)$ we deduce that $[t \mapsto \frac{d^n}{dt^n} g(u(t),u'(t),t)] \in AP^0(\mathbb{R},Y)$, and we note that

$$\frac{d^n}{dt^n} g(u(t),u'(t),t) = \frac{d^n}{dt^n}(\frac{d}{dt} f(u(t),t)) = \frac{d}{dt}(\frac{d^n}{dt^n} f(u(t),t)),$$

that implies the following equality:

(37) $\frac{d^n}{dt^n} g(u(t),u'(t),t) = \frac{d^{n+1}}{dt^{n+1}} f(u(t),t).$

From this equality we obtain that $[t \mapsto f(u(t),t)] \in AP^{n+1}(\mathbb{R},Y)$ when $u \in AP^{n+1}(\mathbb{R},X)$, that permits to say that the operator $N_f^{2,n+1} : AP^{n+1}(\mathbb{R},X) \to AP^{n+1}(\mathbb{R},Y)$, defined by $N_f^{2,n+1}(u) := [t \mapsto f(u(t),t)]$, is well-defined.

Now we treat the continuity of $N_f^{2,n+1}$. For all $u,u_1 \in AP^{n+1}(\mathbb{R},X)$ we note that the following majorizations hold:

$$\|N_f^{2,n+1}(u) - N_f^{2,n+1}(u_1)\|_{C^{n+1}}$$

$$= \|N_f^{2,n}(in_{n+1}(u)) - N_f^{2,n}(in_{n+1}(u_1))\|_{C^n} + \sup_{t\in\mathbb{R}} \|\frac{d^{n+1}}{dt^{n+1}} f(u(t),t) - \frac{d^{n+1}}{dt^{n+1}} f(u_1(t),t)\|$$

$$= \|N_f^{2,n} \circ in_{n+1}(u) - N_f^{2,n} \circ in_{n+1}(u_1)\|_{C^n} + \sup_{t\in\mathbb{R}} \|\frac{d^n}{dt^n} g(u(t),u'(t),t) - \frac{d^n}{dt^n} g(u_1(t),u_1'(t),t)\|$$

$$\leq \|N_f^{2,n} \circ in_{n+1}(u) - N_f^{2,n} \circ in_{n+1}(u_1)\|_{C^n} + \|N_g^{2,n}(in_{n+1}(u), \frac{d}{dt}u)) - N_g^{2,n}(in_{n+1}(u_1), \frac{d}{dt}u_1))\|_{C^n}.$$

And since $N_f^{2,n}$, $N_g^{2,n}$, in_{n+1} and $\frac{d}{dt}$ are continuous operators, the compositions $N_f^{2,n} \circ in_{n+1}$ and $N_g^{2,n} \circ (in_{n+1}, \frac{d}{dt})$ are also continuous. Then by using the last previous inequalities, we deduce that $N_f^{2,n+1} : AP^{n+1}(\mathbb{R},X) \to AP^{n+1}(\mathbb{R},Y)$ is continuous. □

Before to treat the differentiability of the nonlinear operator N_f^2 we need an additional lemma of differential calculus.

Lemma 7.3. *Let $\Psi \in C^{n+1}(X \times \mathbb{R},Y)$, with $n \in \mathbb{N}_*$, and let $p,q \in C^n(\mathbb{R},X)$. Then, for all $(\varepsilon,t) \in \mathbb{R}\times\mathbb{R}$, the following equality holds:*

$$\frac{d}{d\varepsilon}\frac{d^n}{dt^n}\Psi(p(t)+\varepsilon.q(t),t) = \frac{d^n}{dt^n}\frac{d}{d\varepsilon}\Psi(p(t)+\varepsilon.q(t),t).$$

Proof. We proceed by induction on $n \in \mathbb{N}_*$.
First Step: $n = 1$. By doing a straightforward calculation, we verify that $\frac{d}{d\varepsilon}\frac{d}{dt}\Psi(p(t)+\varepsilon.q(t),t)$ and $\frac{d}{dt}\frac{d}{d\varepsilon}\Psi(p(t)+\varepsilon.q(t),t)$ are both equal to
$D^2\Psi(p(t)+\varepsilon.q(t),t).((q(t),0),(p'(t)+\varepsilon.q'(t),1)) + D\Psi(p(t)+\varepsilon.q(t),t).((q'(t),0))$.

Second Step: induction assumption on $n-1$.

Third Step: the case n. We use the induction assumption on the mapping $\Psi_1 \in C^n((X\times X)\times\mathbb{R},Y)$ defined by $\Psi_1((p_1,p_2),t) := D\Psi(p_1,t).(p_2,1)$. And so we have the equality:

$$\frac{d}{d\varepsilon}\frac{d^{n-1}}{dt^{n-1}}\Psi_1((p(t),p'(t))+\varepsilon.(q(t),q'(t)),t) = \frac{d^{n-1}}{dt^{n-1}}\frac{d}{d\varepsilon}\Psi_1((p(t),p'(t))+\varepsilon.(q(t),q'(t)),t).$$

Note that

$$\Psi_1((p(t),p'(t))+\varepsilon.(q(t),q'(t)),t) = D\Psi(p(t)+\varepsilon.q(t),t).((p'(t)+\varepsilon.q'(t),1)$$

$$= \frac{d}{dt}\Psi(p(t)+\varepsilon.q(t),t),$$

and so we obtain the following equality:

$$\frac{d}{d\varepsilon}\frac{d^{n-1}}{dt^{n-1}}\frac{d}{dt}\Psi(p(t)+\varepsilon.q(t),t) = \frac{d^{n-1}}{dt^{n-1}}\frac{d}{d\varepsilon}\frac{d}{dt}\Psi(p(t)+\varepsilon.q(t),t).$$

Then by using the symmetry of the second differential of $(t,\varepsilon)\mapsto\Psi(p(t)+\varepsilon.q(t),t)$, we deduce from the previous equality the following ones:

$$\frac{d}{d\varepsilon}\frac{d^n}{dt^n}\Psi(p(t)+\varepsilon.q(t),t) = \frac{d^{n-1}}{dt^{n-1}}\frac{d}{dt}\frac{d}{d\varepsilon}\Psi(p(t)+\varepsilon.q(t),t) = \frac{d^n}{dt^n}\frac{d}{d\varepsilon}\Psi(p(t)+\varepsilon.q(t),t).$$

□

Remark 7.4. This last lemma is in the spirit of a classical result on second-order partial derivatives like it is exposed in [25], Theorem 3.3, p. 92. Note that we cannot simplify the proof of Lemma 7.3 by saying that the function $(\varepsilon,t)\mapsto\psi(p(t)+\varepsilon.q(t),t)$ is $(n+1)$-times differentiable and by using the symmetry of the total differential of order $n+1$.

Theorem 7.5. *Let $f\in APU(X\times\mathbb{R},Y)\cap C^{n+1}(X\times\mathbb{R},Y)$ such that $D^k f\in APU(X\times\mathbb{R},\mathcal{L}_k((X\times\mathbb{R})^k,Y))$ for all $k=1,\dots,n+1$. Then the superposition operator $N_f^2 : AP^n(\mathbb{R},X)\to AP^n(\mathbb{R},Y)$, defined by $N_f^2(u) := [t\mapsto f(u(t),t)]$, is continuously differentiable on $AP^n(\mathbb{R},X)$, and for all $u,v\in AP^n(\mathbb{R},X)$ we have: $DN_f^2(u).v = [t\mapsto D_x f(u(t),t).v(t)]$.*

Proof. By using the high-order chain rule, [1] p. 92, we know that, when $u\in AP^n(\mathbb{R},X)$ and $1\le k\le n$, by setting $U(t) := (u(t),t)$, $\frac{d^k}{dt^k}f(u(t),t) = D^k(f\circ U)(t).(1,\dots,1)$ is equal to a linear combination of terms if the following form:

$$D^i f(U(t)).(D^{j_1}U(t).(1,\dots,1),\dots,D^{j_i}U(t).(1,\dots,1))$$

$$= D^i f(U(t)).(\frac{d^{j_1}}{dt^{j_1}}U(t),\dots,\frac{d^{j_i}}{dt^{j_i}}U(t)) = D^i f(u(t),t).((\frac{d^{j_1}}{dt^{j_1}}u(t),\theta_{j_1}),\dots,(\frac{d^{j_i}}{dt^{j_i}}u(t),\theta_{j_i}))$$

where θ_{j_m} is equal to zero or to 1.

The operator $u\mapsto\frac{d^{j_m}}{dt^{j_m}}u$ is continuously differentiable from $AP^n(\mathbb{R},X)$ into $AP^0(\mathbb{R},X)$ since it is linear continuous. The operator $u\mapsto\theta_{j_m}$ is continuously differentiable from $AP^n(\mathbb{R},X)$ into $AP^0(\mathbb{R},X)$ since it is constant. When $j\le k\le n$, we have $D^j f$ continuously differentiable and by using Theorem 5.1 we know that $N^1_{D^j f}$ is continuously differentiable from $AP^0(\mathbb{R},X)$ into $AP^0(\mathbb{R},X)$. The canonical injection in_n from $AP^n(\mathbb{R},X)$ into $AP^0(\mathbb{R},X)$ is continuously differentiable since it is linear continuous. And then the operator $u\mapsto[t\mapsto D^j f(u(t),t)]$ which is equal to $N^1_{D^j f}\circ in_n$ is continuously differentiable as a composition of continuously differentiable operators. The operator $M_i : \mathcal{L}_i((X\times\mathbb{R})^i,Y)\times$

$(X \times \mathbb{R})^i \to Y$, defined by $M_i(T,(z^1,...,z^i)) := T.(z^1,..;,z^i)$, is continuously differentiable since it is n-linear continuous. Consequently the operator

$$u \mapsto [t \mapsto D^i f(u(t),t).((\frac{d^{j_1}}{dt^{j_1}}u(t),\theta_{j_1}),...,\frac{d^{j_i}}{dt^{j_i}}u(t),\theta_{j_i})]$$

is continuously differentiable from $AP^n(\mathbb{R},X)$ into $AP^0(\mathbb{R},Y)$ as a composition of continuously differentiable operators. And since a combination of continuously differentiable operators is continuously differentiable, we can assert that the following assertion holds.

(38) The operator $S_k : AP^n(\mathbb{R},X) \to AP^0(\mathbb{R},Y)$, defined by $S_k(u) := [t \mapsto \frac{d^k}{dt^k} f(u(t),t)]$, is continuously differentiable.

Now we fix $u \in AP^n(\mathbb{R},X)$. From (38) we know that the following assertion holds:

(39) For all $k = 1,...,n$ and for all $\varepsilon > 0$ there exists $\eta^k_\varepsilon > 0$ such that , for all $v \in AP^n(\mathbb{R},X)$, if $\|v\|_{C^n} \le \eta^k_\varepsilon$ then we have $\sup_{t\in\mathbb{R}} \|\frac{d^k}{dt^k} f(u(t)+v(t),t) - \frac{d^k}{dt^k} f(u(t),t) - (DS_k(u).v)(t)\| \le \varepsilon.\|v\|_{C^n}$.

By using Theorem 5.1 and the Fréchet-differentiability of $N^1_f \circ in_n$ note that the assertion (39) remains valid for $k = 0$. Since S_k is Fréchet-differentiable at u, we know that $DS_k(u).v = \frac{d}{d\varepsilon}_{|\varepsilon=0} S_k(u+\varepsilon.v)$ which means that

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}} \|\frac{1}{\varepsilon}(\frac{d^k}{dt^k} f(u(t)+\varepsilon.v(t),t) - \frac{d^k}{dt^k} f(u(t),t)) - (DS_k(u).v)(t)\| = 0$$

and so we have, for all $t \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0}(\frac{1}{\varepsilon}(\frac{d^k}{dt^k} f(u(t)+\varepsilon.v(t),t) - \frac{d^k}{dt^k} f(u(t),t)) - DS_k(u).v)(t)) = 0$$

that gives us the following equality

(40) $(DS_k(u).v)(t) = \frac{d}{d\varepsilon}_{|\varepsilon=0} \frac{d^k}{dt^k} f(u(t)+\varepsilon.v(t),t)$ for all $t \in \mathbb{R}$.

And then, by using Lemma 7.3, and by noting that we have $\frac{d}{d\varepsilon}_{|\varepsilon=0} f(u(t)+\varepsilon.v(t),t) = D_x f(u(t),t).v(t)$, we obtain

(41) $(DS_k(u).v)(t) = \frac{d^k}{dt^k}(D_x f(u(t),t).v(t))$ for all $t \in \mathbb{R}$.

And so by using (39), (41) and Theorem 5.1, by setting $\eta_\varepsilon := \min_{0\le k\le n} \eta^k_{\varepsilon/n+1} > 0$, we obtain that for all $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for all $v \in AP^n(\mathbb{R},X)$, if $\|v\|_{C^n} \le \eta_\varepsilon$ then we have

$$\|N^2_f(u+v) - N^2_f(u) - \Lambda(u).v\|_{C^n} =$$

$$\sum_{k=0}^{n}\sup_{t\in\mathbb{R}} \|\frac{d^k}{dt^k} f(u(t)+v(t),t) - \frac{d^k}{dt^k} f(u(t),t)) - \frac{d^k}{dt^k}(D_x f(u(t),t).v(t))\| \le \varepsilon$$

that proves the Fréchet-differentiability of N_f^2 at u and that we have $DN_f^2(u).v = [t \mapsto D_x f(u(t),t).v(t)]$. To establish the continuity of N_f^2 we begin to note that, when $u, u_1, v \in AP^n(\mathbb{R}, X)$ with $\|v\|_{C^n} \leq 1$, the following inequalities hold:

$$\|DN_f^2(u).v - DN_f^2(u_1).v\|_{C^n} = \sum_{k=0}^{n} \sup_{t\in\mathbb{R}} \|\frac{d^k}{dt^k}(D_x j(u(t),t).v(t)) - \frac{d^k}{dt^k}(D_x j(u_1(t),t).v(t))\|$$

$$= \sum_{k=0}^{n} \sup_{t\in\mathbb{R}} \|(DS_k(u).v)(t) - (DS_k(u_1).v)(t)\| = \sum_{k=0}^{n} \|DS_k(u).v - DS_k(u_1).v\|_\infty$$

$$\leq \sum_{k=0}^{n} \|DS_k(u) - DS_k(u_1)\|_{\mathcal{L}(AP^n(\mathbb{R},X),AP^0(\mathbb{R},X))}.$$

And by taking the supremum on the $v \in AP^n(\mathbb{R}, X)$ such that $\|v\|_{C^n} \leq 1$, we obtain the following inequality:

$$\|DN_f^2(u).v - DN_f^2(u_1).v\|_{\mathcal{L}(AP^n(\mathbb{R},X),AP^n(\mathbb{R},X))} \leq \sum_{k=0}^{n} \|DS_k(u) - DS_k(u_1)\|_{\mathcal{L}(AP^n(\mathbb{R},X),AP^0(\mathbb{R},X))}.$$

And since the DS_k are continuous we deduce from the last inequality that DN_f^2 is continuous. □

8 Asymptotically Almost Periodic Functions

In this section, X and Y are Banach spaces.

Lemma 8.1. *Let $f \in AAPU(X \times \mathbb{R}_+, Y)$ and $K \in \mathcal{P}_c(X)$. Then the restriction of f to $K \times \mathbb{R}_+$ is uniformly continuous.*

Proof. We fix $\varepsilon > 0$ and we consider $T = T(K, \frac{\varepsilon}{3}) \geq 0$ and $\ell = \ell(K, \frac{\varepsilon}{3}) > 0$ provided by the definition of $f \in AAPU(X \times \mathbb{R}_+, Y)$; see Section 2. Since $[0, T + \ell + 2]$ is compact, the product $K \times [0, T + \ell + 2]$ is also compact, and by using the classical Heine theorem, we know that the restriction of f at this product is uniformly continuous, that permits to obtain the following assertion:

(42) There exists $\eta \in (0,1)$ such that, for all $x_1, x_2 \in K$ and for all $t_1, t_2 \in [0, T + \ell + 2]$, if $\|x_1 - x_2\| \leq \eta$ and if $|t_1 - t_2| \leq \eta$, then we have $\|f(x_1,t_1) - f(x_2,t_2)\| \leq \frac{\varepsilon}{3}$.

Now we fix $x_1, x_2 \in K$ such that $\|x_1 - x_2\| \leq \eta$ and $t_1, t_2 \in \mathbb{R}_+$ such that $|t_1 - t_2| \leq \eta$.

First case: $t_1, t_2 \in [0, T]$. Then by using (42) we have $\|f(x_1,t_1) - f(x_2,t_2)\| \leq \frac{\varepsilon}{3} \leq \varepsilon$.

Second case: $t_1 \in [0, T]$ and $t_2 > T$. Note that we have $t_2 - t_1 = |t_1 - t_2| \leq \eta < 1$, that implies $t_2 \leq t_1 + 1 \leq T + 1 \leq T + \ell + 2$, and then by using (42) we obtain $\|f(x_1,t_1) - f(x_2,t_2)\| \leq \frac{\varepsilon}{3} \leq \varepsilon$.

Third case: $t_1 > T$ and $t_2 > T$. We set $r := -t_1 + T + 1$, and then there exists $\tau \in [r, r + \ell]$ such that $\|f(z, t + \tau) - f(z,t)\| \leq \frac{\varepsilon}{3}$ for all $z \in K$ and for all $t \geq T$.
We note that $-t_1 + T + 1 \leq \tau \leq -t_1 + T + 1 + \ell$ that implies $T + 1 \leq t_1 + \tau \leq T + 1 + \ell \leq T + \ell + 2$, and we also note that $t_2 + \tau \leq t_1 + \eta + \tau \leq t_1 + 1 + \tau \leq T + \ell + 2$. Then by using (42) we obtain the following inequality:

(43) $\| f(x_1,t_1+\tau) - f(x_2,t_2+\tau)\| \leq \frac{\varepsilon}{3}$.

By taking into account the role of T, since $t_1 \geq T$ and $t_2 \geq T$ we obtain the following inequalities:

(44) $\| f(x_1,t_1+\tau) - f(x_1,t_1)\| \leq \frac{\varepsilon}{3}$ and $\| f(x_2,t_2+\tau) - f(x_2,t_2)\| \leq \frac{\varepsilon}{3}$.

By using (43) and (44) we have:

$$\| f(x_1,t_1) - f(x_2,t_2)\| \leq \| f(x_1,t_1) - f(x_1,t_1+\tau)\| + \| f(x_1,t_1+\tau) - f(x_2,t_2+\tau)\|$$

$$+\| f(x_2,t_2+\tau) - f(x_2,t_2)\| \leq 3\frac{\varepsilon}{3} = \varepsilon.$$

□

The previous proof is very similar to the proof of proposition 10, p.42, in [42].

Lemma 8.2. *Let $f \in AAPU(X \times \mathbb{R}_+, Y)$. Then for all $K \in \mathcal{P}_c(X)$ and all $\varepsilon > 0$ there exists $\delta = \delta(K,\varepsilon) > 0$ such that, for all $x \in K$ and all $z \in X$, if $\|x - z\| \leq \delta$ then we have $\| f(x,t) - f(z,t)\| \leq \varepsilon$, for all $t \in \mathbb{R}_+$.*

The proof of Lemma 8.2 is similar to this one of Lemma 3.7, by replacing the use of Lemma 3.1 by the use of Lemma 8.1.

Lemma 8.3. *Let $f \in AAPU(X \times \mathbb{R}_+, Y)$ and $u \in AAP(\mathbb{R}_+, X)$. Then we have $[t \mapsto f(u(t),t)] \in AAP(\mathbb{R}_+, Y)$.*

This result is due to Zaidman [28]; it permits us to define the following superposition operator

(45) $N_f^3 : AAP(\mathbb{R}_+, X) \to AAP(\mathbb{R}_+, Y)$, $N_f^3(u) := [t \mapsto f(u(t),t)]$.

Theorem 8.4. *Let $f : X \times \mathbb{R}_+ \to Y$ be a mapping. Then the two following assertions are equivalent.*

(i) $f \in AAPU(X \times \mathbb{R}_+, Y)$.

(ii) *The superposition operator N_f^3, defined in (45), is continuous from $AAP(\mathbb{R}_+, X)$ into $AAP(\mathbb{R}_+, Y)$.*

Proof. (**i** $\Longrightarrow$ **ii**). We fix $u \in AAP(\mathbb{R}_+, X)$ and $\varepsilon > 0$. We set $K := \overline{u(\mathbb{R}_+)}$. We know that $K \in \mathcal{P}_c(X)$, [42], Lemma p. 37. Then we consider $\delta = \delta(K,\varepsilon) > 0$ provided by Lemma 8.2. When $v \in AAP(\mathbb{R}_+, X)$ is such that $\|v - u\|_\infty \leq \delta$, then by using Lemma 8.2, we obtain $\| f(v(t),t) - f(u(t),t)\| \leq \varepsilon$ for all $t \geq 0$, that means: $\|N_f^3(v) - N_f^3(u)\|_\infty \leq \varepsilon$.

(**ii** $\Longrightarrow$ **i**). Since $N_f^3(AAP(\mathbb{R}_+, X) \subset AAP(\mathbb{R}_+, Y)$, when $x \in X$ we define $u_x(t) := x$ for all $t \geq 0$, and since we have $u_x \in AAP(\mathbb{R}_+, X)$, we obtain that $t \mapsto f(x,t) = f(u_x(t),t) = N_f^3(u_x)(t)$ lies to $AAP(\mathbb{R}_+, Y)$. And consequently we obtain the following assertion.

(46) For all $x \in X$, $f(x,.) \in AAP(\mathbb{R}_+, Y)$.

We consider the operator $U : x \mapsto u_x$ from X into $AAP(\mathbb{R}_+, X)$. Since U is a linear isometry, U is continuous and consequently the composition $N_f^3 \circ U$ is continuous from X into $AAP(\mathbb{R}_+, Y)$. Note that $(N_f^3 \circ U(x))(t) = f(u_x(t), t) = f(x,t)$ and so we have:

(47) The mapping $x \mapsto f(x,.)$ is continuous from X into $AAP(\mathbb{R}_+, Y)$.

Consequently, when we fix $K \in \mathcal{P}_c(X)$, $x \mapsto f(x,.)$ is uniformly continuous on K that gives us the following assertion.

(48) For all $K \in \mathcal{P}_c(X)$ and for all $\varepsilon > 0$, there exists $\eta = \eta(K, \varepsilon) > 0$ such that, for all $x, z \in K$, if $\|x - z\| \leq \eta$ implies $\| f(x,t) - f(z,t)\| \leq \frac{\varepsilon}{3}$ for all $t \in \mathbb{R}_+$.

We fix $K \in \mathcal{P}_c(X)$ and we consider a finite list of elements $x_1, x_2, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=12}^n \overline{B}(x_i, \eta)$ where η is provided by (48). After (46) we know that $f(x_i,.) \in AAP(\mathbb{R}_+, Y)$ for all $i = 1, \ldots, n$. It is easy to see that $(0, \ldots, 0, f(x_i,.), 0, \ldots, 0) \in AAP(\mathbb{R}_+, Y^n)$, and since a finite sum of asymptotically almost periodic functions is asymptotically almost periodic, we obtain that $(f(x_1,.), \ldots, f(x_i,.), \ldots f(x_n,.)) = \sum_{i=1}^n (0, \ldots, 0, f(x_i,.), 0, \ldots, 0) \in AAP(\mathbb{R}_+, Y^n)$, and so we have the following assertion.

(49) For all $\varepsilon > 0$ there exist $\ell > 0$ and $T \geq 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\tau]$ satisfying $\| f(x_i, t+\tau) - f(x_i, t)\| \leq \frac{\varepsilon}{3}$ for all $t \geq T$ and for all $i = 1, .., n$.

Let $x \in K$. We choose j between 1 and n such that $\|x - x_j\| \leq \eta$. Then when $t \geq T$, by using (48) and (49) we obtain the following inequalities:

$$\| f(x, t+\tau) - f(x,t)\| \leq \| f(x, t+\tau) - f(x_j, t+\tau)\| + \| f(x_j, t+\tau) - f(x_j, t)\|$$

$$+ \| f(x_j, t) - f(x,t)\| \leq 3\frac{\varepsilon}{3} = \varepsilon,$$

that provides the following assertion

(50) For all $K \in \mathcal{P}_c(X)$ and for all $\varepsilon > 0$ there exist $\ell > 0$ and $T \geq 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r+\tau]$ satisfying $\| f(x, t+\tau) - f(x,t)\| \leq \frac{\varepsilon}{3}$ for all $t \geq T$ and for all $x \in K$.

Note that (50) is the second condition on f to belong to $AAPU(X \times \mathbb{R}_+, Y)$. Now it suffices to prove that f is continuous on $X \times \mathbb{R}$. Let $((x_n, t_n))_n$ be a sequence of elements of $X \times \mathbb{R}_+$ which converges toward (x_*, t_*). For all $n \in \mathbb{N}$, we have

$$\| f(x_n, t_n) - f(x_*, t_*)\| \leq \| f(x_n, t_n) - f(x_*, t_n)\| + \| f(x_*, t_n) - f(x_*, t_*)\|$$

$$\leq \| f(x_n, .) - f(x_*, .)\|_\infty + \| f(x_*, t_n) - f(x_*, t_*)\|.$$

By using (47) we have $\lim_{n\to\infty} \| f(x_n,.) - f(x_*,.)\|_\infty = 0$, and by using (46) we know that $f(x_*,.)$ is continuous and so we have $\lim_{n\to\infty} \| f(x_*, t_n) - f(x_*, t_*)\| = 0$. Then by using the last previous inequalities we obtain $\lim_{n\to\infty} f(x_n, t_n) = f(x_*, t_*)$ that proves the continuity of f. □

We note that the previous proof of $(i \Longrightarrow ii)$ is similar to the second proof of Theorem 3.5 given in Section 3.

Theorem 8.5. *Let $f \in AAPU(X \times \mathbb{R}_+, Y)$ such that the partial Fréchet-differential $D_x f(x,t)$ exists for all $(x,t) \in X \times \mathbb{R}_+$ and such that $D_x f \in AAPU(X \times \mathbb{R}_+, \mathcal{L}(X,Y))$.*
Then the superposition operator N_f^3, defined in (45), is continuously Fréchet-differentiable from $AAP(\mathbb{R}_+, X)$ into $AAP(\mathbb{R}_+, Y)$, and we have $DN_f^3(u).v = [t \mapsto D_x f(u(t),t).v(t)]$ for all $u, v \in AAP(\mathbb{R}_+, X)$.

The proof of Theorem 8.5 is similar to this one of Theorem 5.1 by replacing the use of Theorem 3.5 by the use of Theorem 8.4, and by replacing the use of Lemma 3.4 by the use of Lemma 8.3.

9 Almost Automorphic Functions

In this section, X is a complete metric space and Y is a Banach space.

Lemma 9.1. *Let Ω be a subset of X. If $\phi \in C^0(\Omega, Y)$ and if $v \in AA(\mathbb{R}, X)$ is such that $\overline{v(\mathbb{R})}$ is contained in Ω, then we have $\phi \circ v \in AA(\mathbb{R}, Y)$.*

Lemma 9.1 is Theorem 2.1.5, p. 14 in [33].

Lemma 9.2. *Let $f \in AAU(X \times \mathbb{R}, Y)$, $K \in \mathcal{P}_c(X)$, and $n \in \mathbb{N}_*$. Then there exist $N_n \in \mathbb{N}_*$, $c_j^n \in C^0(K, \mathbb{R})$ and $a_j^n \in AA(\mathbb{R}, Y)$ for all $j = 1, \dots, N_n$ such that the following approximation holds.*

(51) $\| \sum_{j=1}^{N_n} c_j^n(x) a_j^n(t) - f(x,t) \| \leq \frac{1}{n}$ *for all $x \in K$ and for all $t \in \mathbb{R}$.*

Proof. By using Remark 2.2, we have $\Phi \in C^0(X, AA(\mathbb{R}, Y))$ where $\Phi(x) := [t \mapsto f(x,t)]$. Consequently, $\Phi(K)$ is a compact subset of $AA(\mathbb{R}, Y)$. Then by using the Schauder's approximation theorem, cf. Remarque 1, p. 90, in [18] or p. 116-117 in [26], we can assert that there exists $\Phi_n : K \to Y$, $\Phi_n(x) := \sum_{j=1}^{N_n} c_j^n(x) a_j^n$, where $c_j^n \in C^0(K, \mathbb{R})$ and $a_j^n \in AA(\mathbb{R}, Y)$ for all $j = 1, \dots, N_n$, such that $\|\Phi(x) - \Phi_n(x)\|_\infty \leq \frac{1}{n}$ for all $x \in K$, that proves the lemma. □

Remark 9.3. The converse of Lemma 9.2 is obviously true, but we will not use it.

Lemma 9.4. *Let $f \in AAU(X \times \mathbb{R}, Y)$ and $u \in AA(\mathbb{R}, X)$. Then we have $[t \mapsto f(u(t),t)] \in AA(\mathbb{R}, Y)$.*

Proof. We set $K := \overline{u(\mathbb{R})}$ which is compact [33], Theorem 2.1.3, (v), p. 12. Let $c \in C^0(K, \mathbb{R})$ and $a \in AA(\mathbb{R}, Y)$. We denote by $p : \mathbb{R} \times Y \to Y$ the bilinear continuous mapping $p(t,y) := ty$, and we define the function $v : \mathbb{R} \to \mathbb{R} \times Y$ by setting $v(t) := (c(u(t)), a(t))$. By using Lemma 9.1 we know that $c \circ u$ is almost automorphic, and by using the definition of the almost automorphy, we see that $v \in AA(\mathbb{R}, \mathbb{R} \times Y)$. Since $c(u(t))a(t) = p \circ v(t)$ for all $t \in \mathbb{R}$, we have proven the following assertion:

(52) $[t \mapsto c(u(t))a(t)] \in AA(\mathbb{R}, Y)$ when $c \in C^0(K, \mathbb{R})$ and $a \in AA(\mathbb{R}, Y)$.

We consider the mapping $f_n(x,t) := \sum_{j=1}^{N_n} c_j^n(x) a_j^n(t)$ provided by Lemma 9.2. Since a finite sum of almost automorphic functions is almost automorphic too ([33], Theorem 2.1.3, p. 12), by using (52) we obtain that $[t \mapsto f_n(u(t),t)] \in AA(\mathbb{R}, Y)$. By using (51) we obtain that

$\|f_n(u(t),t) - f(u(t),t)\| \leq \frac{1}{n}$ for all $t \in \mathbb{R}$, and consequently $[t \mapsto f(u(t),t)]$ is almost automorphic as a uniform limit of a sequence of almost automorphic functions, ([33], Theorem 2.1.10). □

When $f \in AAU(X \times \mathbb{R}, Y)$ Lemma 9.4 permits us to define the following superposition operator:

(53) $N_f^4 : AA(\mathbb{R},X) \to AA(\mathbb{R},Y)$, $N_f^4(u) := [t \mapsto f(u(t),t)]$.

Lemma 9.5. *Let $f \in AAU(X \times \mathbb{R}, Y)$. Then for all $K \in \mathcal{P}_c(X)$ and for all $\varepsilon > 0$, there exists $\delta = \delta(K,\varepsilon) > 0$ such that, for all $x \in K$ and for all $z \in X$, if $d(x,z) \leq \delta$ then we have $d(f(x,t), f(z,t)) \leq \varepsilon$ for all $t \in \mathbb{R}$.*

The proof of Lemma 9.5 is similar to this one of Lemma 3.7 by obtaining (8) from (2).

Theorem 9.6. *Let $f : X \times \mathbb{R} \to Y$ be a mapping. Then the two following assertions are equivalent.*

(i) $f \in AAU(X \times \mathbb{R}, Y)$.

(ii) *The superposition operator N_f^4, defined in (53), is continuous from $AA(\mathbb{R},X)$ into $AA(\mathbb{R},Y)$.*

Proof. $(i \Longrightarrow ii)$. The proof of this implication is similar to the second proof of Theorem 3.5 by replacing the use of Lemma 3.7 by this one of Lemma 9.5.
$(ii \Longrightarrow i)$. For all $x \in X$, we consider the constant function $u_x : \mathbb{R} \to X$ defined by $u_x(t) := x$. Then we have $u_x \in AA(\mathbb{R},X)$, and since $N_f^4(AA(\mathbb{R},X)) \subset AA(\mathbb{R},Y)$, we obtain that $f(x,.) = N_f^4(u_x) \in AA(\mathbb{R},Y)$, and so (1) is satisfied. Since the mapping $U : X \to AA(\mathbb{R},X)$, defined by $U(x) := u_x$, is continuous, and since N_f^4 is continuous, the composition $N_f^4 \circ U$ is also continuous on X. Then by using the Heine theorem, for all $K \in \mathcal{P}_c(X)$, the mapping $[x \mapsto f(x,.) = N_f^4 \circ U(x)]$ is uniformly continuous on K, that is exactly (2). And so we have $f \in AAU(X \times \mathbb{R}, Y)$. □

Theorem 9.7. *We assume that X and Y are Banach spaces. Let $f \in AAU(X \times \mathbb{R}, Y)$ such that the partial Fréchet-differential $D_x f(x,t)$ exists for all $(x,t) \in X \times \mathbb{R}$, and such that $D_x f \in AAU(X \times \mathbb{R}, \mathcal{L}(X,Y))$. Then the superposition operator N_f^4, defined in (53), is continuously Fréchet-differentiable from $AA(\mathbb{R},X)$ into $AA(\mathbb{R},Y)$, and we have $DN_f^4(u).v = [t \mapsto D_x f(u(t),t).v(t)]$ for all u, $v \in AA(\mathbb{R},X)$.*

The proof of Theorem 9.7 is similar to this one of Theorem 5.1 by replacing the use of Theorem 3.5 by this one of Theorem 9.6, and the use of Lemma 3.4 by this one of condition (2).

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