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ON THE FINE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR Δ_v OVER THE SEQUENCE SPACE c_0

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Abstract

The purpose of the paper is to determine fine spectrum of newly introduced operator Δ_v on the sequence space c_0. The operator Δ_v on c_0 is defined by $\Delta_v x = (v_n x_n - v_{n-1}x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = 0$, where $v = (v_k)$ is either constant or strictly decreasing sequence of positive real numbers such that $\lim_{k\to\infty} v_k = L > 0$ and $\sup_k v_k \leq 2L$. In this paper, it is shown that spectrum $\sigma(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}$, the point spectrum $\sigma_p(\Delta_v, c_0) = \phi$ if v is a constant and $\sigma_p(\Delta_v, c_0) = \{v_n\}$ if v is a strictly decreasing sequence. We have also obtained the results on continuous spectrum $\sigma_c(\Delta_v, c_0)$, residual spectrum $\sigma_r(\Delta_v, c_0)$ and fine spectrum of the operator Δ_v on c_0.

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1 Introduction

Let $v = (v_k)$ be either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L > 0 \text{ and} \tag{1.1}$$

$$\sup_k v_k \leq 2L. \tag{1.2}$$

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We introduce the operator Δ_v on the sequence space c_0 as follows,

$\Delta_v : c_0 \to c_0$ is defined by,

$$\Delta_v x = \Delta_v(x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0.$$

It is easy to verify that the operator Δ_v can be represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \dots \\ -v_0 & v_1 & 0 & \dots \\ 0 & -v_1 & v_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.3}$$

The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c is determined by B. Altay and F. Basar [3]. Complete study of the spectrum such as the point spectrum, the continuous spectrum, the residual spectrum of the operator Δ on the sequence spaces c_0 and c is made by these authors. The fine spectrum of the Cesaro operator on the sequence space l_p is studied by Gonzalez [9], where $1 < p < \infty$ while weighted mean matrices of operators on l_p is investigated by Cartlidge [7]. The spectrum of the Cesaro operator on the sequence spaces c_0 and bv is also investigated by Reade [6] and Okutoyi [8] respectively. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c is studied by Yildirim [10]. Recently, the fine spectra of the cesaro operator over the sequence space c_0 is determined by Akhmedov and Basar [1].

In this paper, we determine the spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the operator Δ_v on the sequence space c_0. The results of this paper not only generalize the corresponding results of [3] but also give results for some more operators.

2 Preliminaries, Background and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ be a bounded linear operator. We denote the range of T as $R(T)$, where

$$R(T) = \{y \in Y : \ y = Tx, \ x \in X\},$$

and the set of all bounded linear operators on X into itself is denoted by $B(X)$. Further, the adjoint $T^{\star}$ of T is a bounded linear operator on the dual $X^{\star}$ of X defined by $(T^{\star}\phi)(x) = \phi(Tx)$, for all $\phi \in X^{\star}$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\alpha = (T - \alpha I)$, where α is a complex number and I is the identity operator on $D(T)$. The inverse of T_α (if exists) is denoted by T_α^{-1}, i.e.

$$T_\alpha^{-1} = (T - \alpha I)^{-1}$$

and call it the resolvent operator of T. It is easy to verify that T_α^{-1} is linear if T_α is linear.

Many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. We are interested in the set of all α in the complex plane such that T_α^{-1}

exists/ T_α^{-1} is bounded/ domain of T_α^{-1} is dense in X. In this paper, we have made an effort to find the characterizations of spectrum and fine spectrum of the operator Δ_v on sequence space c_0.

Firstly, we give below definitions and known results which will be used in the sequel.

Definition 2.1. ([4], pp. 371) Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$. A regular value of T is a complex number α such that
(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values α of T. Its complement $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point spectrum* $\sigma_p(T,X)$ is the set of all $\alpha \in \mathbf{C}$ such that T_α^{-1} does not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set of all $\alpha \in \mathbf{C}$ such that T_α^{-1} exists and satisfies (R3) but not (R2), that is, T_α^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T,X)$ is the set of all $\alpha \in \mathbf{C}$ such that T_α^{-1} exists but do not satisfy (R3), that is, the domain of T_α^{-1} is not dense in X. The condition (R2) may or may not holds good.

Goldberg's classification of operator $T_\alpha = (T - \alpha I)$ (see [11], pp. 58) : Let X be a Banach space and $T_\alpha = (T - \alpha I) \in B(X)$, where α is a complex number. Again let $R(T_\alpha)$ and T_α^{-1} be denote the range and inverse of the operator T_α respectively, then following possibilities may occur;
(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $\overline{R(T_\alpha)} \neq X$,
and
(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

Remark 2.2. Combining (A), (B), (C) and (1), (2), (3); we get nine different states. These are labelled by A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3. We use $\alpha \in B_2\sigma(T,X)$ means the operator $T_\alpha \in B_2$, i.e. $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$ and T_α is injective but T_α^{-1} is discontinuous. Similarly others. If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then α belongs to the resolvent set $\rho(T,X)$ of T on X. The other classification gives rise to the fine spectrum of T.

By w, we denote the space of all real or complex valued sequences. We write l_1 for the space of all absolutely summable sequences, i.e. $l_1 = \{x = (x_k) : \sum |x_k| < \infty\}$ and c_0 denote the space of all null sequences.

Definition 2.3. ([5], pp. 220-221) Let λ, μ be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk}, where

$n,k \in \mathbf{N} = \{0,1,2,\cdots\}$. For every $x = (x_k) \in \lambda$ and every integer n we write

$$A_n(x) = \sum_k a_{nk} x_k, \tag{2.1}$$

where the sum without limits is always taken from $k=0$ to $k=\infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say that $A \in (\lambda,\ \mu)$ if and only if $Ax \in \mu$ whenever $x \in \lambda$.

Lemma 2.4. *([2], pp. 129) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*
(1) the rows of A in l_1 and their l_1 norms are bounded,
(2) the columns of A are in c_0.

Note: The operator norm of T is the supremum of the l_1 norms of the rows.

Lemma 2.5. *([11], pp. 59) T has a dense range if and only if T^* is one to one, where T^* denotes the adjoint operator of the operator T.*

Lemma 2.6. *([11], pp. 60) The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

3 The Fine Spectrum of the Operator Δ_v on the Sequence Space c_0

In this section, we compute the spectrum and the fine spectrum of the operator Δ_v on the sequence space c_0. Throughout this paper, the sequence $v = (v_K)$ satisfies equations (1.1) and (1.2).

Theorem 3.1. *$\Delta_v : c_0 \to c_0$ is a bounded linear operator and $\|\Delta_v\|_{(c_0,c_0)} = 2\ \sup_k(v_k)$.*

Proof. Proof is simple. So we omit. □

Theorem 3.2. *The Spectrum of Δ_v on c_0 is given by*

$$\sigma(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.$$

Proof. The proof of this theorem is divided into two parts. In the first part, we show that $\sigma(\Delta_v, c_0) \subseteq \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}$, which is equivalent to show that

$$\alpha \in \mathbf{C} \text{ with } \left|1 - \frac{\alpha}{L}\right| > 1 \text{ implies } \alpha \notin \sigma(\Delta_v, c_0).$$

In the second part, we establish the reverse inequality, i.e.

$$\left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \subseteq \sigma(\Delta_v, c_0).$$

Let $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$. Clearly, $\alpha = L$ as well as $\alpha = v_k$, for all k do not satisfied. So, $\alpha \neq L$ and $\alpha \neq v_k$, for all k. We get $(\Delta_v - \alpha I) = (a_{nk})$ is a triangle and hence has an

inverse. Thus, $(\Delta_v - \alpha I)^{-1} = (b_{nk})$,
where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(v_0-\alpha)} & 0 & 0 & \cdots \\ \frac{v_0}{(v_0-\alpha)(v_1-\alpha)} & \frac{1}{(v_1-\alpha)} & 0 & \cdots \\ \frac{v_0 v_1}{(v_0-\alpha)(v_1-\alpha)(v_2-\alpha)} & \frac{v_1}{(v_1-\alpha)(v_2-\alpha)} & \frac{1}{(v_2-\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have $(\Delta_v - \alpha I)^{-1} \in (c_0, c_0)$ if and only if
(1) series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent, for each $n \in \mathbf{N}$ and $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$;
(2) $\lim_{n\to\infty} |b_{nk}| = 0$, for each $k \in \mathbf{N}$.

Now, we will show that the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent, for each $n \in \mathbf{N}$.

Let $S_n = \sum_{k=0}^{\infty} |b_{nk}|$. Then,

$$S_n = \left|\frac{v_0 v_1 \cdots v_{n-1}}{(v_0-\alpha)(v_1-\alpha)\cdots(v_n-\alpha)}\right| + \cdots + \left|\frac{v_{n-1}}{(v_{n-1}-\alpha)(v_n-\alpha)}\right| + \left|\frac{1}{v_n-\alpha}\right|.$$

Clearly, for each $n \in \mathbf{N}$, the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent.

Next, we show that $\sup_n S_n$ is finite. Let $\beta = \lim_{n\to\infty} \left|\frac{v_{n-1}}{v_n-\alpha}\right|$. As modulus function is continuous, so

$$\beta = \frac{1}{\left|1-\frac{\alpha}{L}\right|}. \tag{3.1}$$

Which shows that $0 < \beta < 1$ and gives

$$\lim_{n\to\infty} \frac{1}{|v_n-\alpha|} = \lim_{n\to\infty} \left(\left|\frac{v_{n-1}}{v_n-\alpha}\right| \left|\frac{1}{v_{n-1}}\right|\right) = \frac{\beta}{L}. \tag{3.2}$$

We have,

$$S_n = \left|\frac{v_{n-1}}{v_n-\alpha}\right| S_{n-1} + \left|\frac{1}{v_n-\alpha}\right|.$$

Taking limit both sides of above equation and using equations (3.1) and (3.2), we get

$$\lim_{n\to\infty} S_n = \frac{\beta}{L}\left(\frac{1}{1-\beta}\right) < \infty.$$

Since (S_n) is a sequence of positive real numbers and $\lim_{n\to\infty} S_n < \infty$, so $\sup_n S_n < \infty$.

Again since $\beta = \lim_{n\to\infty}\left|\frac{v_{n-1}}{v_n-\alpha}\right| < 1$, therefore $\left|\frac{v_{n-1}}{v_n-\alpha}\right| < 1$, for large n and consequently

$$\lim_{n\to\infty}|b_{n0}| = \lim_{n\to\infty}\left|\frac{v_0v_1\cdots v_{n-1}}{(v_0-\alpha)(v_1-\alpha)\cdots(v_n-\alpha)}\right| = 0.$$

Similarly, we can show that $\lim_{n\to\infty}|b_{nk}| = 0$, for all $k = 1,2,3,\cdots$.
Thus,

$$(\Delta_v - \alpha I)^{-1} \in B(c_0) \text{ for } \alpha \in \mathbf{C} \text{ with } \left|1-\frac{\alpha}{L}\right| > 1. \tag{3.3}$$

Now, we show that the domain of the operator $(\Delta_v - \alpha I)^{-1}$ is dense in c_0, this statement holds if and only if the range of the operator $(\Delta_v - \alpha I)$ is dense in c_0. Since we have $(\Delta_v - \alpha I)^{-1} \in (c_0, c_0)$, this is equivalent to the range of the operator $(\Delta_v - \alpha I)$ is dense in c_0.
This shows that,

$$\sigma(\Delta_v, c_0) \subseteq \left\{\alpha \in \mathbf{C} : \left|1-\frac{\alpha}{L}\right| \le 1\right\}. \tag{3.4}$$

Conversely, it is required to show

$$\left\{\alpha \in \mathbf{C} : \left|1-\frac{\alpha}{L}\right| \le 1\right\} \subseteq \sigma(\Delta_v, c_o). \tag{3.5}$$

We prove inclusion (3.5) under the assumption that $\alpha \neq L$ as well as $\alpha \neq v_k$, for all k, i.e. one of the conditions of Definition 2.1 fails. Let $\alpha \in \mathbf{C}$ with $\left|1-\frac{\alpha}{L}\right| \le 1$. Clearly $(\Delta_v - \alpha I)$ is a triangle and hence $(\Delta_v - \alpha I)^{-1}$ exists. So condition (R1) is satisfied but (R2) fails as can be seen below:

Now suppose $\alpha \in \mathbf{C}$ with $\left|1-\frac{\alpha}{L}\right| < 1$. Then $\beta = \lim_{n\to\infty}\left|\frac{v_{n-1}}{v_n-\alpha}\right| > 1$. This means that $\left|\frac{v_{n-1}}{v_n-\alpha}\right| > 1$, for large n and consequently

$$\lim_{n\to\infty}|b_{n0}| = \lim_{n\to\infty}\left|\frac{v_0v_1\cdots v_{n-1}}{(v_0-\alpha)(v_1-\alpha)\cdots(v_n-\alpha)}\right| \neq 0.$$

Hence

$$(\Delta_v - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbf{C} \text{ with } \left|1-\frac{\alpha}{L}\right| < 1. \tag{3.6}$$

Now, we consider $\alpha \in \mathbf{C}$ with $\left|1-\frac{\alpha}{L}\right| = 1$, i.e. $|L-\alpha| = L$, which implies $|v_k - \alpha| \le |v_k|$, for all k, therefore $\frac{1}{|v_k|} \le \frac{1}{|v_k-\alpha|}$, for all k. Using this inequality, we have

$$S_n = \left|\frac{v_0v_1\cdots v_{n-1}}{(v_0-\alpha)(v_1-\alpha)\cdots(v_n-\alpha)}\right| + \cdots + \left|\frac{v_{n-1}}{(v_{n-1}-\alpha)(v_n-\alpha)}\right| + \left|\frac{1}{v_n-\alpha}\right| \ge \frac{(n+1)}{v_n}.$$

Therefore, $\sup_n S_n \ge \sup_n \left[\frac{(n+1)}{v_n}\right] = \infty$. So condition (R2) fails. Hence

$$(\Delta_v - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbf{C} \text{ with } \left|1-\frac{\alpha}{L}\right| = 1. \tag{3.7}$$

Now, we prove the inclusion (3.5) under the assumption that $\alpha = L$ as well as $\alpha = v_k,$ for all k. We have

$$(\Delta_v - v_k I)x = \begin{pmatrix} (v_0 - v_k)x_0 \\ -v_0 x_0 + (v_1 - v_k)x_1 \\ -v_1 x_1 + (v_2 - v_k)x_2 \\ \vdots \\ -v_{k-2}x_{k-2} + (v_{k-1} - v_k)x_{k-1} \\ -v_{k-1}x_{k-1} \\ -v_k x_k + (v_{k+1} - v_k)x_{k+1} \\ \vdots \end{pmatrix}.$$

If $(v_k = L)$ is a constant sequence, then

$$(\Delta_v - v_k I)x = \theta \quad \Rightarrow x_0 = 0,\ x_1 = 0,\ x_2 = 0, \cdots.$$

This shows that, the operator $(\Delta_v - \alpha I)$ is one to one, but $R(\Delta_v - \alpha I)$ is not dense in c_0. So condition (R3) fails. Hence $L \in \sigma(\Delta_v, c_0)$.

Furthermore, If (v_k) is a strictly decreasing sequence, then for fixed k,

$$(\Delta_v - v_k I)\, x = \theta$$

$\Rightarrow x_0 = 0,\ x_1 = 0, \cdots, x_{k-1} = 0,\ x_{n+1} = \left(\dfrac{v_n}{v_{n+1} - v_k}\right) x_n,$ for all $n \geq k$. This shows that $(\Delta_v - v_k I)$ is not injective. So condition (R1) fails. Hence $v_k \in \sigma\,(\Delta_v, c_0)$, for all $k \in \mathbf{N}$.

When $\alpha = L$, then $|v_k - \alpha| < |v_k|$, for all k and hence $\dfrac{1}{|v_k|} < \dfrac{1}{|v_k - \alpha|}$, for all k.

Thus,

$$S_n = \left| \frac{v_0 v_1 \cdots v_{n-1}}{(v_0 - \alpha)(v_1 - \alpha)\cdots(v_n - \alpha)} \right| + \cdots + \left| \frac{v_{n-1}}{(v_{n-1} - \alpha)(v_n - \alpha)} \right| + \left| \frac{1}{v_n - \alpha} \right| > \frac{(n+1)}{v_n}.$$

Therefore, $\quad \sup\limits_n S_n > \sup\limits_n \left[\dfrac{(n+1)}{v_n} \right] = \infty$. So condition (R2) fails. Hence

$$(\Delta_v - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha = L. \tag{3.8}$$

So, $L \in \sigma(\Delta_v, c_0)$. Thus, in this case also $v_k \in \sigma(\Delta_v, c_0)$, for all $k \in \mathbf{N}$ and $L \in \sigma(\Delta_v, c_0)$.

This shows that,

$$\left\{ \alpha \in \mathbf{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\} \subseteq \sigma(\Delta_v, c_0) \tag{3.9}$$

From inclusions (3.4) and (3.9), we get

$$\sigma(\Delta_v, c_0) = \left\{ \alpha \in \mathbf{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\}.$$

This completes the proof. □

Theorem 3.3. *The point spectrum of the operator Δ_v over c_0 is given by*

$$\sigma_p(\Delta_v, c_0) = \begin{cases} \phi, \ if\ (v_k) \ \textit{is a constant sequence.} \\ \\ \{v_0, v_1, v_2, \cdots\}, \ if\ (v_k) \ \textit{is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Suppose (v_k) is a constant sequence. Consider $\Delta_v x = \alpha x$, for $x \neq \theta = (0,0,\cdots)$ in c_0, which gives

$$\left.\begin{aligned} v_0 x_0 &= \alpha x_0 \\ -v_0 x_0 + v_1 x_1 &= \alpha x_1 \\ -v_1 x_1 + v_2 x_2 &= \alpha x_2 \\ &\vdots \\ -v_{k-1} x_{k-1} + v_k x_k &= \alpha x_k \\ &\vdots \end{aligned}\right\} \tag{3.10}$$

Let x_t is the first non-zero entry of the sequence $x = (x_n)$, so we get $-v_{t-1}x_{t-1} + v_t x_t = \alpha x_t$ which implies $\alpha = v_t$ and from the equation

$$-v_t x_t + v_{t+1} x_{t+1} = \alpha x_{t+1}.$$

we get $x_t = 0$, which is a contradiction to our assumption.
Therefore,

$$\sigma_p(\Delta_v, c_0) = \phi.$$

Case(ii): Suppose (v_k) is a strictly decreasing sequence. Consider $\Delta_v x = \alpha x$, for $x \neq \theta = (0,0,\cdots)$ in c_0, which gives system of equations (3.10).

Suppose $\alpha = v_0$, then

$$\begin{aligned} x_k &= \left(\frac{v_{k-1}}{v_k - v_0}\right) x_{k-1} \\ &= \left[\frac{v_{k-1} v_{k-2} \cdots v_0}{(v_k - v_0)(v_{k-1} - v_0)\cdots(v_1 - v_0)}\right] x_0, \text{ for all } k \geq 1. \end{aligned}$$

We take $x_0 \neq 0$, then get non-zero solution of the equation $(\Delta_v - v_0 I)x = \theta$.

Similarly if $\alpha = v_k$, for all $k \geq 1$, then $x_{k-1} = 0$, $x_{k-2} = 0$, $\cdots$, $x_0 = 0$ and

$$\begin{aligned} x_{n+1} &= \left(\frac{v_n}{v_{n+1} - v_k}\right) x_n, \text{ for all } n \geq k \\ &= \left[\frac{v_n v_{n-1} \cdots v_k}{(v_{n+1} - v_k)(v_n - v_k)\cdots(v_{k+1} - v_k)}\right] x_k, \text{ for all } n \geq k. \end{aligned}$$

We take $x_k \neq 0$, then get non-zero solution of the equation $(\Delta_v - v_k I)x = \theta$.
Thus,

$$\sigma_p(\Delta_v, c_0) = \{v_0, v_1, v_2, \cdots\}.$$

This completes the proof. □

Let $T : c_0 \to c_0$ be a bounded linear operator having matrix representation A and the dual space of c_0 denoted by c_0^*, then we known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose of the matrix A. Further, it is also known that c_0^* is isomorphic to l_1, the space of all absolutely summable sequences, with the norm $\|x\| = \sum_{k=0}^{\infty} |x_k|$.

We now obtain point spectrum of the dual operator Δ_v^* of Δ_v over the space c_0^*.

Theorem 3.4. *The point spectrum of the operator Δ_v^* over c_0^* is*

$$\sigma_p(\Delta_v^*, c_0^*) = \left\{ \alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1 \right\}.$$

Proof. Suppose $\Delta_v^* f = \alpha f$ for $\theta \neq f \in c_0^* \cong l_1$, where

$$\Delta_v^* = \begin{pmatrix} v_0 & -v_0 & 0 & \dots \\ 0 & v_1 & -v_1 & \dots \\ 0 & 0 & v_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}.$$

This gives,

$$f_k = \left(\frac{v_{k-1} - \alpha}{v_{k-1}} \right) f_{k-1}, \text{ for all } k \geq 1.$$

Hence

$$|f_k| = \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| |f_{k-1}|, \text{ for all } k \geq 1. \tag{3.11}$$

But,

$$\begin{aligned} |v_{k-1} - \alpha| &\leq v_{k-1} - L + |L - \alpha| \\ \Rightarrow \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| &< 1, \text{ for all } k = 1, 2, 3, \cdots, \text{ provided } \left|1 - \frac{\alpha}{L}\right| < 1. \end{aligned}$$

Using equation (3.11), we get

$$|f_k| < |f_{k-1}|, \text{ for all } k = 1, 2, 3, \cdots \text{ and consequently } \lim_{k \to \infty} \frac{|f_k|}{|f_{k-1}|} < 1.$$

$$\text{So} \quad \sum_{k=0}^{\infty} |f_k| < \infty.$$

Hence

$$\left|1 - \frac{\alpha}{L}\right| < 1 \quad \Rightarrow \quad \sum_{k=0}^{\infty} |f_k| < \infty.$$

Converse follows from the fact that,

$$\begin{aligned} \sum_{k=0}^{\infty} |f_k| < \infty \quad &\Rightarrow \quad \lim_{k \to \infty} \frac{|f_k|}{|f_{k-1}|} < 1 \\ &\Rightarrow \quad \lim_{k \to \infty} \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| < 1 \\ &\Rightarrow \quad \left|1 - \frac{\alpha}{L}\right| < 1. \end{aligned}$$

Hence

$$\sum_{k=0}^{\infty} |f_k| < \infty \quad \Rightarrow \quad \left|1 - \frac{\alpha}{L}\right| < 1.$$

Therefore,

$$\left|1 - \frac{\alpha}{L}\right| < 1 \quad \Leftrightarrow \quad \sum_{k=0}^{\infty} |f_k| < \infty.$$

This means that, $f \in c_0^*$ if and only if $f_0 \neq 0$ and $\left|1 - \frac{\alpha}{L}\right| < 1$.
Thus,

$$\sigma_p(\Delta_v^*, c_0^*) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\}.$$

This completes the proof. □

Next theorem gives the characterization of residual spectrum $\sigma_r(\Delta_v, c_0)$ of operator Δ_v on sequence space c_0.

Theorem 3.5. *Residual spectrum $\sigma_r(\Delta_v, c_0)$ of operator Δ_v over c_0 is*

$$\sigma_r(\Delta_v, c_0) = \begin{cases} \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\}, \ if\ (v_k) \textit{ is a constant sequence.} \\ \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\} \setminus \{v_0, v_1, v_2, \cdots\}, \ if \\ (v_k) \textit{ is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Let (v_k) be a constant sequence. For $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$, the operator $(\Delta_v - \alpha I)$ is a triangle except for $\alpha = L$ and consequently the operator $(\Delta_v - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_v - \alpha I)$ is one to one for $\alpha = L$ and hence has an inverse.

But by Theorem 3.4, the operator $(\Delta_v - \alpha I)^* = \Delta_v^* - \alpha I$ is not one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$. Hence by Lemma 2.5, the range of the operator $(\Delta_v - \alpha I)$ is not dense in c_0. Thus,

$$\sigma_r(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\}.$$

Case(ii): Let (v_k) be a strictly decreasing sequence with $\lim_{k\to\infty} v_k = L$. For $\alpha \in \mathbf{C}$ such that $\left|1 - \frac{\alpha}{L}\right| < 1$, the operator $(\Delta_v - \alpha I)$ is a triangle except for $\alpha = v_k$, for some $k \in \mathbf{N}$ and consequently the operator $(\Delta_v - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_v - \alpha I)$ is not one to one for $\alpha = v_k$, for some $k \in \mathbf{N}$. So $(\Delta_v - \alpha I)^{-1}$ does not exist.

But by Theorem 3.4, $(\Delta_v - \alpha I)^* = \Delta_v^* - \alpha I$ is not one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$. Hence by Lemma 2.5, the range of the operator $(\Delta_v - \alpha I)$ is not dense in c_0. Thus,

$$\sigma_r(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\} \setminus \{v_0, v_1, v_2, \cdots\}.$$

This completes the proof. □

Now we established the result for continuous spectrum $\sigma_c(\Delta_v, c_0)$ of operator Δ_v on sequence space c_0.

Theorem 3.6. *Continuous spectrum $\sigma_c(\Delta_v, c_0)$ of operator Δ_v over c_0 is*

$$\sigma_c(\Delta_v, c_0) = \begin{cases} \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| = 1\right\}, \ if\ (v_k) \text{ is a constant sequence.} \\ \\ \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| = 1\right\} \setminus \{v_0\}, \ if\ (v_k) \text{ is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Let (v_k) be a constant sequence. For $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| = 1$, the operator $(\Delta_v - \alpha I)$ is a triangle and has an inverse. The inverse of the operator $(\Delta_v - \alpha I)$ is discontinuous by statement (3.7). Therefore, $(\Delta_v - \alpha I)$ has an unbounded inverse.

But by Theorem 3.4, the operator $(\Delta_v - \alpha I)^* = \Delta_v^* - \alpha I$ is one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| = 1$. Hence by Lemma 2.5, the range of the operator $(\Delta_v - \alpha I)$ is dense in c_0. Thus,

$$\sigma_c(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| = 1\right\}.$$

Case(ii): Let (v_k) be a strictly decreasing sequence with $\lim_{k\to\infty} v_k = L$. For $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| = 1$, the operator $(\Delta_v - \alpha I)$ is a triangle except may be $\alpha = v_0$ and consequently the operator $(\Delta_v - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_v - \alpha I)$ is not one to one for $\alpha = v_0$. So $(\Delta_v - \alpha I)^{-1}$ does not exist. The inverse of the operator $(\Delta_v - \alpha I)$ is discontinuous by statement (3.7). Therefore, $(\Delta_v - \alpha I)$ has an unbounded inverse.

But by Theorem 3.4, the operator $(\Delta_v - \alpha I)^* = \Delta_v^* - \alpha I$ is one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| = 1$. Hence by Lemma 2.5, the range of the operator $(\Delta_v - \alpha I)$ is dense in c_0. Thus,

$$\sigma_c(\Delta_v, c_0) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| = 1\right\} \setminus \{v_0\}.$$

This completes the proof. □

Theorem 3.7. *If α satisfies $\left|1 - \frac{\alpha}{L}\right| > 1$, then $(\Delta_v - \alpha I) \in A_1$.*

Proof. It is required to show that the operator $(\Delta_v - \alpha I)$ is bijective and has a continuous inverse for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$. Since $\alpha \neq v_k$, therefore $(\Delta_v - \alpha I)$ is a triangle. Hence it has an inverse. The inverse of the operator $(\Delta_v - \alpha I)$ is continuous for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$ by statement (3.3). Also the equation $(\Delta_v - \alpha I)x = y$ gives $x = (\Delta_v - \alpha I)^{-1} y$, i.e.

$$x_n = \left((\Delta_v - \alpha I)^{-1} y\right)_n, \ n \in \mathbb{N}.$$

Thus, for every $y \in c_0$, we can find $x \in c_0$ such that

$$(\Delta_v - \alpha I)x = y, \ \text{ since } \ (\Delta_v - \alpha I)^{-1} \in (c_0, c_0).$$

This shows that, the operator $(\Delta_v - \alpha I)$ is onto, i.e. $R(\Delta_v - \alpha I) = c_0$ and hence $(\Delta_v - \alpha I) \in A_1$. □

Theorem 3.8. *Let (v_k) be a constant sequence, say $v_k = L$ and $\alpha = L$. Then $\alpha \in C_1\sigma(\Delta_v, c_0)$.*

Proof. We have,

$$\sigma_p\left(\Delta_v^*, c_0^*\right) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\}.$$

For $\alpha = L$, the operator $(\Delta_v - \alpha I)^*$ is not one to one. By Lemma 2.5, $R(\Delta_v - \alpha I)$ is not dense in c_0, i.e. $\overline{R(\Delta_v - \alpha I)} \neq c_0$. Again by Theorem 3.3, $\alpha = L$ does not belong to the set $\sigma_p(\Delta_v, c_0)$, therefore the operator $(\Delta_v - \alpha I)$ has an inverse.

To show the operator $(\Delta_v - \alpha I)^{-1}$ is continuous. By Lemma 2.6, it is enough to show that $(\Delta_v - \alpha I)^*$ is onto, i.e. for given $y = (y_n) \in l_1$, we have to find $x = (x_n) \in l_1$ such that $(\Delta_v - \alpha I)^* x = y$. Now, $(\Delta_v - \alpha I)^* x = y$, i.e.

$$\begin{aligned} -vx_1 &= y_0 \\ -vx_2 &= y_1 \\ &\vdots \\ -vx_n &= y_{n-1} \\ &\vdots \end{aligned}$$

Thus, $-vx_n = y_{n-1}$, for all $n \geq 1$ which implies $\sum_{n=0}^{\infty} |x_n| < \infty$, since $y \in l_1$.

This shows that, $(\Delta_v - \alpha I)^*$ is onto and hence $\alpha \in C_1\sigma(\Delta_v, c_0)$. □

Theorem 3.9. *Let (v_k) be a constant sequence, say $v_k = L$ and $\alpha \neq L$, $\alpha \in \sigma_r(\Delta_v, c_0)$. Then $\alpha \in C_2\sigma(\Delta_v, c_0)$.*

Proof. Since $\alpha \neq L$, therefore the operator $(\Delta_v - \alpha I)$ is a triangle. Hence it has an inverse. For $L \neq \alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$, the inverse of the operator $(\Delta_v - \alpha I)$ is discontinuous by statement (3.6). Thus, $(\Delta_v - \alpha I)$ is injective and $(\Delta_v - \alpha I)^{-1}$ is discontinuous.

Again by Theorem 3.4, $(\Delta_v - \alpha I)^*$ is not one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$. But Lemma 2.5 yields the fact that the range of the operator $(\Delta_v - \alpha I)$ is not dense in c_0, i.e. $\overline{R(\Delta_v - \alpha I)} \neq c_0$ and hence $\alpha \in C_2\sigma(\Delta_v, c_0)$. □

Theorem 3.10. *Let (v_k) be a strictly decreasing sequence of positive real numbers and $\alpha \in \sigma_r(\Delta_v, c_0)$. Then $\alpha \in C_2\sigma(\Delta_v, c_0)$.*

Proof. We have,

$$\sigma_r(\Delta_v, l_1) = \left\{\alpha \in \mathbf{C} : \left|1 - \frac{\alpha}{L}\right| < 1\right\} \setminus \{v_0, v_1, v_2, \cdots\}.$$

Since $\alpha \neq v_k$, for all k, therefore the operator $(\Delta_v - \alpha I)$ is a triangle. Hence it has an inverse. For $v_k \neq \alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$, the inverse of the operator $(\Delta_v - \alpha I)$ is discontinuous by statements (3.6) and (3.8). Thus, $(\Delta_v - \alpha I)$ is injective and $(\Delta_v - \alpha I)^{-1}$ is discontinuous.

Again by Theorem 3.4, $(\Delta_v - \alpha I)^*$ is not one to one for $\alpha \in \mathbf{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$. But Lemma 2.5 yields the fact that the range of the operator $(\Delta_v - \alpha I)$ is not dense in c_0, i.e. $\overline{R(\Delta_v - \alpha I)} \neq c_0$ and hence $\alpha \in C_2\sigma(\Delta_v, c_0)$. □

Theorem 3.11. *Let (v_k) be a constant sequence, say $v_k = L$ and $\alpha \in \sigma_c(\Delta_v, c_0)$. Then $\alpha \in B_2\sigma(\Delta_v, c_0)$.*

Proof. For $\alpha \in \mathbf{C}$ with $\left|1-\frac{\alpha}{L}\right| = 1$, the operator $(\Delta_v - \alpha I)$ is a triangle and has an inverse. The inverse of the operator $(\Delta_v - \alpha I)$ is discontinuous by statement (3.7). Therefore, $(\Delta_v - \alpha I)$ has an unbounded inverse. Again by Theorem 3.4, $(\Delta_v - \alpha I)^*$ is one to one for $\alpha \in \mathbf{C}$ with $\left|1-\frac{\alpha}{L}\right| = 1$. By Lemma 2.5, the range of the operator $(\Delta_v - \alpha I)$ is dense in c_0, i.e. $\overline{R(\Delta_v - \alpha I)} = c_0$.

Now, we have to show that the operator $(\Delta_v - \alpha I)$ is not surjective. It is sufficient to show that there is no sequence $x = (x_n)$ in c_0 such that $(\Delta_v - \alpha I)x = y$, for some $y \in c_0$. Clearly, $y = (1, 0, 0, \cdots) \in c_0$. We have,

$$(\Delta_v - \alpha I)x = y \;\Rightarrow\; x_n = \frac{L^n}{(L-\alpha)^{n+1}}, \text{ for each n} \geq 0.$$

therefore $|x_n| = \frac{1}{L}$, for each n ≥ 0 because $\left|1-\frac{\alpha}{L}\right| = 1$. Consequently $\lim_{n\to\infty} |x_n| = \dfrac{1}{L} \neq 0$. This shows that, $x \notin c_0$ and hence the operator $(\Delta_v - \alpha I)$ is not surjective. □

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