Stability of Niiler's Solution of the General Circulation Problem

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ABSTRACT

Niiler's model of the general wind-driven ocean circulation is reformulated in a quasigeostrophic framework and solved analytically to first order of the perturbative expansion, both for interior and boundary layer regions. On this basis, the nonlinear unconditional stability of the solution is proven by means of the Lyapunov direct method.

1. Introduction

The basic picture of the basin-scale ocean circulation is founded on the motion of a single-layered fluid in a bounded region on a beta plane. This is the starting point of an hierarchy of models arranged with increasing mechanical and thermodynamical complexity, in which the continuous evolution of numerical techniques hardly allows one to isolate the most refined model at the moment. In this paper we focus on the bottom of this hierarchy of models in order to locate preliminarily in a suitable framework the subject of our investigation, that is, Niiler's (1966) model.

The prototype of (oversimplified) inertial circulation is the Fofonoff (1954) mode that is able to reproduce qualitatively both the formation region of westward intensified ocean currents and the inertial stream crossing the basin from coast to coast. However, the inability to model westward intensification led to consideration of the Fofonoff mode as purely abstract, until its review in terms of equilibrium statistical mechanics (Carnevale and Frederiksen 1987; Griffa and Salmon 1989). The key step in the evolution is represented by the models of wind-driven circulation with bottom friction only, formulated first by Stommel (1948) in a historical linear version and then by Veronis (1966) in a full nonlinear context. The investigation of Veronis has been later reconsidered repeatedly, from different points of view. For instance, Harrison and Stalos (1982) evaluate, in the framework of the Veronis model, regional vorticity budgets over a wide range of parameter choices. Merkine et al. (1985) reexamine the visual resemblance between highly nonlinear solutions of the Veronis model and Fofonoff's solution and establish that the apparent resemblance is misleading. Zimmerman (1993) derives analytically the symmetry presented in the numerical simulations of Veronis by a perturbation expansion in the inverse Ekman number, valid in the "small beta regime." He also shows that Veronis' sequence does not lead to the inertial Fofonoff mode, the latter being originated from the limit of another sequence in the parameter space spanned by the Rossby and Ekman numbers. Niiler's model also follows this trend. It explicitly includes the Fofonoff mode as a precise asymptotic limit for vanishing forcing dissipation but, at the same time, retains the basic features of the wind-driven circulation, like the Sverdrup balance and westward intensification, even if the wind stress adopted is less realistic than that taken into account by Veronis.

Common characteristics of the above quoted models are the uniqueness of boundary conditions (no mass flux only), the absence of recirculation whatever nonlinearity may be, and the absence of multiple steady states (Ierley and Sheremet 1995).

The subsequent step in the evolution, and the last considered here, comes from the adoption of lateral diffusion of relative vorticity as a frictional parameterization. This ingredient, which is a heritage of the Navier–Stokes equations, has profound implications. For instance, additional, but largely arbitrary, boundary conditions are required besides no mass flux, and these deeply influence the overall current field. Moreover, in the regime of marked nonlinearity recirculation develops in the northwestern corner of the basin. Furthermore, multiple steady solutions are allowed. A detailed and up-to-date review of these arguments is found in Pedlosky (1996). A numerical phenomenon that can arise, especially in the presence of lateral friction and no-slip boundary conditions, is the generation of instability, in the sense of the impossibility to achieve a steady state in correspondence to given points in the parameter space of the system considered (Bryan 1963;

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Blandford 1971). Since only instability can be numerically detected (stability would require an infinite computation time!) and if instability is not observed in a system or in another dynamically close to the previous one, it is expected that stability (in a certain norm) holds. Thus, this conjecture needs to be proven. This is the case in Niiler's model, which is close in a sense to that of Veronis in the highly nonlinear regime, this last being able to converge in any case to a steady state.

It is stressed again that Niiler's model of a full-basin oceanic circulation investigates analytically the wind-driven motion of a one-layered ocean in a strongly nonlinear regime and in the presence of bottom friction only as a dissipative mechanism. The identification of two modes simultaneously present in the motion, with inertial motion predominant, leads to a particular scaling of the governing equations that exhibits a weak coupling between the potential vorticity conservation tendency and the response to the applied forcing. The meridionally modulated wind stress field is chosen so that the inertial mode coincides with that of Fofonoff, while the higher-order correction satisfies the Sverdrup balance in the interior. The overall solution is a crude truncated expansion in powers of a proper coupling parameter constituted by two terms only, the first being just the Fofonoff mode. In regards to the second term, Niiler does not give its explicit form but points out only the Sverdrup balance between this term and the wind curl.

The assumed smallness of the coupling has two important consequences:

1) The flow pattern is close to that of Fofonoff and the inertial boundary layer expands toward the interior, opposing to the east–west asymmetry typical of the Sverdrup balance with the tendency to a complete east–west invariance;
2) The southward drift of the interior is clearly depressed with respect to the value taken in the dissipative regime.

These facts show that, in spite of the different form of the adopted wind stress, the streamlines of Niiler's model are in some accordance with the numerical solutions of Veronis, in the strongly nonlinear regime obtained by using a very similar homogeneous model.

Since a steady state was actually reached in all the numerical experiments of Veronis, it is quite reasonable to conjecture that Niiler's solution is also stable. The aim of the present paper is to study this point thoroughly in order to prove the nonlinear unconditional stability of Niiler's solution with respect to a given norm by deriving a special time evolution of the norm above which is neither conserved (as in the case of the Fofonoff mode) nor tends necessarily to zero (as in the case of the Stommel–Veronis solution in the dissipative regime (Crisciani et al. 1994).

To this purpose, Niiler's model is reformulated in the quasigeostrophic framework (section 2), and then (section 3) the analytical form of the overall solution is deduced by boundary layer methods. In section 4 an upper bound of the time derivative of the perturbation norm square is explicitly evaluated, thus achieving the results we have anticipated.

2. Niiler's model revisited

Niiler's model is restated in the framework of the quasigeostrophic dynamics governed by the following dimensional (starred) vorticity equation:

$$J_*(\psi_*, \nabla_*^2\psi_*) + \beta\frac{\partial\psi_*}{\partial x_*} = \frac{1}{\rho H}\hat{z}\cdot\boldsymbol{\nabla}_* \times \boldsymbol{\tau}_* - k\nabla_*^2\psi_*, \qquad (x_*, y_*) \in D_*, \quad (1)$$

with no-mass flux boundary condition

$$\psi_* = 0 \qquad (x_*, y_*) \in \partial D_*. \quad (2)$$

The original notation is used, so

$$k = \frac{f_0}{2}E_V^{1/2},$$

where f_0 is the Coriolis parameter and E_V is the vertical Ekman number.

In that part of the ocean basin D_* far removed from the boundaries and in addition to the Sverdrup mode excited by the wind stress $\boldsymbol{\tau}_*$ and having the amplitude

$$U_S = \frac{\tau_0}{\rho H\beta L}, \quad (3)$$

where τ_0 is the typical intensity of $\boldsymbol{\tau}_*$, the dominant dynamical equilibrium permits a free zonal flow, whose amplitude is denoted by U_f. In the assumed strongly nonlinear regime,

$$U_f \gg U_S \quad (4)$$

and U_f is taken as velocity scale. Near the boundaries the typical velocity is $U_f\, L/\lambda$, where $\lambda = (U_f/\beta)^{1/2}$ is the width of the inertial region.

To find the relation between U_f and U_S, we integrate (1) over the area enclosed by the circuit C coinciding with a streamline of ψ_*. The result is

$$\frac{1}{\rho H}\oint_C \boldsymbol{\tau}_*\cdot d\mathbf{l} - k\oint_C \mathbf{u}_*\cdot\, d\mathbf{l} = 0, \quad (5)$$

where $\mathbf{u}_* = \hat{z} \times \boldsymbol{\nabla}_*\psi_*$. With the aid of (3), the first term of (5) is of the order of $U_S\beta L$, while the second one is $kU_f\, L/\lambda$, so, equating the two quantities and recalling the dependence of λ on U_f, one obtains the relation

$$\beta^{1/2}U_S = kU_f^{1/2}. \quad (6)$$

At this point, the nondimensional version of (1) for $\psi = \psi_*/(U_f\, L)$ can be deduced by substituting (3) and (6) into (1), which yields

$$\frac{U_f}{\beta L^2}J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \frac{U_S}{U_f}\hat{z}\cdot\nabla\times\boldsymbol{\tau} - \frac{U_S}{U_f}\left(\frac{U_f}{\beta L^2}\right)^{1/2}\nabla^2\psi,$$

and putting

$$\varepsilon = \left(\frac{U_f}{\beta L^2}\right)^{1/2}, \qquad \delta = \frac{U_S}{U_f}, \tag{7}$$

the final version can be written as

$$\varepsilon^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \delta(\hat{z}\cdot\nabla\times\boldsymbol{\tau} - \varepsilon\nabla^2\psi), \qquad (x, y) \in D, \tag{8}$$

with the boundary condition

$$\psi = 0, \qquad (x, y) \in \partial D, \tag{9}$$

which comes from (2).

Note that, since $\lambda \lll L$ and $\varepsilon = \lambda/L$, $\varepsilon < O(1)$. Moreover, inequality (4) implies $\delta < O(1)$.

Niiler's solution is based on the expansion of ψ in powers of δ, truncated at the first order; that is,

$$\psi \approx \psi_0 + \delta\psi_1. \tag{10}$$

Then, substitution of (10) into (8) and (9) determines to the zeroth and first order in δ the following problems for ψ_0 and ψ_1 respectively:

$$\varepsilon^2 J(\psi_0, \nabla^2\psi_0) + \frac{\partial\psi_0}{\partial x} = 0, \qquad (x, y) \in D \tag{11}$$

$$\psi_0 = 0, \qquad (x, y) \in \partial D \tag{12}$$

$$\varepsilon^2[J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0)] + \frac{\partial\psi_1}{\partial x} = \hat{z}\cdot\nabla\times\boldsymbol{\tau} - \varepsilon\nabla^2\psi_0, \qquad (x, y) \in D \tag{13}$$

$$\psi_1 = 0, \qquad (x, y) \in \partial D. \tag{14}$$

Niiler was able to prove that the choice

$$\varepsilon^2\nabla^2\psi_0 + y = \psi_0, \tag{15}$$

identically satisfies (11) and is consistent with (13) if a wind stress of the form

$$\boldsymbol{\tau} = (\tilde{\tau} + 3y - y^2)\,\mathbf{i} \tag{16}$$

is postulated in the square fluid domain

$$D = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\},$$

where $\tilde{\tau}$ is an arbitrary constant. Thus, in the presence of the forcing (16), the highly nonlinear regime is constituted by the superposition of the Fofonoff mode ψ_0, which is the solution of problem (11), (12) with the first-order contribution ψ_1. In the basin interior (I) this last field satisfies the Sverdrup balance

$$\frac{\partial\psi_{1I}}{\partial x} = \hat{z}\cdot\nabla\times\boldsymbol{\tau}, \tag{17}$$

as is apparent from (13) if the terms proportional to ε and ε^2, growing only near the boundaries, are disregarded.

Note that, from (16),

$$\hat{z}\cdot\nabla\times\boldsymbol{\tau} = 2y - 3. \tag{18}$$

The only information about the interior flow given by Niiler comes from (17), (18) and is represented by Eq. (26) of the original paper.

Finally, from (11), (13), and (18) one obtains the equation

$$J(\psi_0, \varepsilon^2\nabla^2\psi_1 - \psi_1) = 2y - 3 - \varepsilon\nabla^2\psi_0, \tag{19}$$

which will be useful in what follows.

3. On the solution of Niiler's model

The analytical expression of ψ_1 is not given in the original paper of Niiler but it must be preliminarily determined in order to investigate the stability of the basic state (10). Consider first the situation in the basin interior, where Eq. (17) holds. We introduce the transform

$$(x, y) \to (1 - x, y) \tag{20}$$

of D into itself, which is simply the mirror reflection of D with respect to the midbasin longitude, and define $\overline{\psi}(x, y) = \psi(1 - x, y)$. With reference to the identity

$$\psi_1 = \frac{1}{2}(\psi_1 + \overline{\psi_1}) + \frac{1}{2}(\psi_1 - \overline{\psi_1})$$

we put

$$\psi_1 = \psi_s + \psi_a, \tag{21}$$

where $\psi_s = (\psi_1 + \overline{\psi_1})/2$ and $\psi_a = (\psi_1 - \overline{\psi_1})/2$. Obviously, $\overline{\psi_s} = \psi_s$ and $\overline{\psi_a} = -\psi_a$. In the interior, (21) is denoted as

$$\psi_{1I} = \psi_{sI} + \psi_{aI}. \tag{22}$$

Substitution of (22) into (17) gives

$$\frac{\partial\psi_{sI}}{\partial x} + \frac{\partial\psi_{aI}}{\partial x} = 2y - 3, \tag{23}$$

and the application of transform (20) to (23) yields

$$-\frac{\partial\psi_{sI}}{\partial x} + \frac{\partial\psi_{aI}}{\partial x} = 2y - 3. \tag{24}$$

From (23) and (24) we obtain both

$$\frac{\partial\psi_{sI}}{\partial x} = 0 \tag{25}$$

and

$$\psi_{aI} = (2y - 3)x + G(y), \tag{26}$$

where G is any function of y. Equation (25) shows that ψ_{sI} is purely zonal, while the antisymmetric behavior of ψ_{aI}, that is to say $\psi_{aI}(1 - x, y) = -\psi_{aI}(x, y)$, implies, via (26), that

$$\psi_{aI} = (2y - 3)\left(x - \frac{1}{2}\right). \qquad (27)$$

If the same procedure is followed by substituting (21) into (19), we see that ψ_s satisfies the equation $J(\psi_0, \varepsilon^2\nabla^2\psi_s - \psi_s) = 0$; that is to say

$$\varepsilon^2\nabla^2\psi_s - \psi_s = P(\psi_0). \qquad (28)$$

Since in the interior $\psi_{0I} = y$ [recall (15)], Eq. (28) implies that $\psi_{sI} = -P(y)$: This equation is consistent with (25) whatever the function P may be. Therefore, in the interior the complete solution is

$$\psi_I = y + \delta\psi_{sI}(y) + \delta\psi_{aI}(x, y),$$

where the $O(\delta)$ contribution of ψ_{sI} is arbitrary due to the arbitrariness of the function $P(\psi_0)$ appearing in (28). This fact allows one to choose, on simplicity grounds, $P(\psi_0) = 0$ and hence $\psi_s = 0$. This, since the solution of the problem,

$$\varepsilon^2\nabla^2\psi_s - \psi_s = 0, \qquad (x, y) \in D$$
$$\psi_s = 0, \qquad (x, y) \in \partial D,$$

is identically vanishing.

Thus, recalling (27), we have

$$\psi_{1I} = (2y - 3)\left(x - \frac{1}{2}\right). \qquad (29)$$

The streamlines of the interior come from the equation $\psi_{0I} + \delta\psi_{1I} = p$, which can be written under the form of a function $y(x)$ as

$$y_p(x) = \frac{p + 3\delta\left(x - \frac{1}{2}\right)}{1 + 2\delta\left(x - \frac{1}{2}\right)},$$

where the parameter p ($0 < p < 1$) singles out each streamline. One immediately verifies that dy_p/dx is always positive, while d^2y_p/dx^2 is always negative in the fluid domain D. This last result should provide a more detailed flow path, represented schematically in Fig. 3 of Niiler's original paper.

Once the interior solution (29) is given, one can apply boundary layer methods to deduce the complete solution up to the first order in δ.

With reference to one (say K, $K = S, N, W, E$) of the four boundaries of D, Fofonoff's solution can be written as

$$\psi_0 = y + \phi_0^{(K)}, \qquad (30)$$

while the $O(\delta)$ terms take the form

$$\psi_1 = \psi_{1I} + \phi_1^{(K)}. \qquad (31)$$

Now, substitution of (30) and (31) into (19) leads to the equation

$$J(\phi_0^{(K)}, \varepsilon^2\nabla^2\phi_1^{(K)} - \phi_1^{(K)}) - J(\phi_0^{(K)}, \psi_{1I}) - \frac{\partial}{\partial x}(\varepsilon^2\nabla^2\phi_1^{(K)} - \phi_1^{(K)}) = -\varepsilon\nabla^2\phi_0^{(K)}. \qquad (32)$$

At this point, Eq. (32) must be solved separately for each boundary, making use of the analytical form of the correction $\phi_0^{(K)}$, which is known from Fofonoff's solution. For instance, in the southern boundary $\phi_0^{(S)} = 0$ and, in terms of the boundary layer coordinate $\eta = \varepsilon^{-1}y$, Eq. (32) takes the form

$$\frac{\partial}{\partial x}\left[\varepsilon^2\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \eta^2}\right)\phi_1^{(S)} - \phi_1^{(S)}\right] = 0.$$

The dominant balance is

$$\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \phi_1^{(S)}}{\partial x}\right) - \frac{\partial \phi_1^{(S)}}{\partial x} = 0.$$

Moreover, $\phi_1^{(S)}$ must satisfy the following boundary conditions:

$$\lim_{\eta\to+\infty} \phi_1^{(S)} = 0$$

and

$$\psi_{1I}(x, 0) + \phi_1^{(S)}(x, 0) = 0.$$

The boundary layer correction is therefore

$$\phi_1^{(S)}(x, \eta) = 3\left(x - \frac{1}{2}\right)\exp(-\eta),$$

and the complete solution, valid for the interior and the southern boundary layer, is

$$\psi_1^{(S)} = \left(x - \frac{1}{2}\right)\left[2y - 3 + 3\exp\left(-\frac{y}{\varepsilon}\right)\right]. \qquad (33)$$

The procedure for the other boundaries is analogous, even if it is slightly more involved because of the presence of the nonvanishing functions $\phi_0^{(K)}$ ($K = N, W, E$). Here the results are summarized.

Northern boundary: It is known that $\phi_0^{(N)} = -\exp[(y - 1)/\varepsilon]$. Using the approximation $\partial\psi_{1I}/\partial x \approx -1$ near $y = 1$ in the development of the second term of (32), one achieves the solution, valid for the interior and the northern boundary layer,

$$\psi_1^{(N)} = \left(x - \frac{1}{2}\right)\left[2y - 3 + \exp\left(\frac{y - 1}{\varepsilon}\right)\right]. \qquad (34)$$

Western boundary: In this boundary layer $\phi_0^{(W)} = -y\exp(-x/\varepsilon)$. As before, with the aid of the approximation $\partial\psi_{1I}/\partial y \approx -1$ near $x = 0$, the solution obtained is

$$\psi_1^{(W)} = (2y - 3)\left[x - \frac{1}{2} + \frac{1}{2}\exp\left(-\frac{x}{\varepsilon}\right)\right]. \qquad (35)$$

Eastern boundary: We simply have $\psi_1^{(E)}(x, y) = -\psi_1^{(W)}(1 - x, y)$ and hence

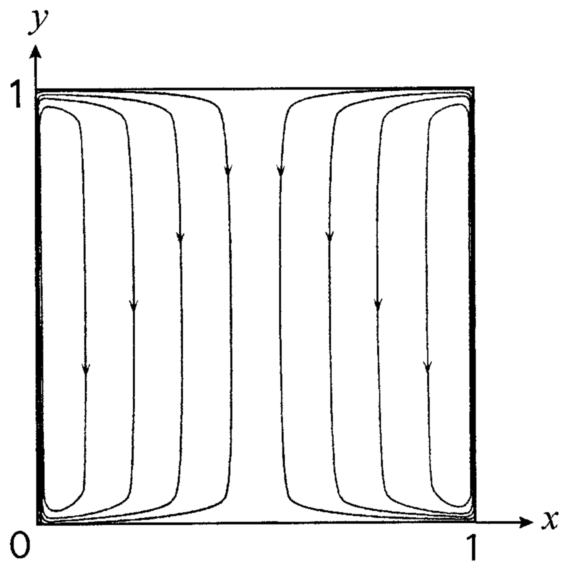

FIG. 1. Sketch of the streamlines of ψ_1 obtained from the boundary layer solutions (33), (34), (35), and (36). The double-gyre system generates a southward interior drift and two northward intensified currents along the meridional boundaries.

$$\psi_1^{(E)} = (2y - 3)\left[x - \frac{1}{2} - \frac{1}{2}\exp\left(\frac{x-1}{\varepsilon}\right)\right]. \quad (36)$$

A remarkable property of the partial solutions (33), (34), (35), and (36) is that each of them satisfies the equation

$$\varepsilon^2\nabla^2\psi_1^{(K)} - \psi_1^{(K)} = \left(\frac{1}{2} - x\right)(2y - 3) \quad K = S, N, W, E,$$

and therefore one can establish for the whole domain the extrapolation

$$\varepsilon^2\nabla^2\psi_1 - \psi_1 \approx \left(\frac{1}{2} - x\right)(2y - 3), \quad (37)$$

which approximates $\varepsilon^2\nabla^2\psi_1 - \psi_1$ near the corners of the basin where boundary layer techniques cannot be applied. Note that $\nabla^2\psi_1^{(K)}$ is proportional to ε^{-2}, so the accuracy of (37) does not depend on ε. Equation (37) will be useful in next section.

The field ψ_1 given by (33), (34), (35), and (36) represents a double gyre, generating a southward interior current together with two northward return currents close to $x = 0$ and $x = 1$. The superposition of these currents with those of Fofonoff's solution, having opposite sign in the western and eastern side, explains both the western intensification and the eastern weakening of the complete solution (10). The southward drift of the interior originates from the convergence in $x = ½$ of the zonal current along the northern boundary, while the divergence, at the same longitude, of the zonal current along the southern boundary feeds the northward return flow. The streamlines of ψ_1 are sketched in Fig. 1. Some features of the total solution $\psi_0 + \delta\psi_1$ are pointed out in Figs. 2 and 3.

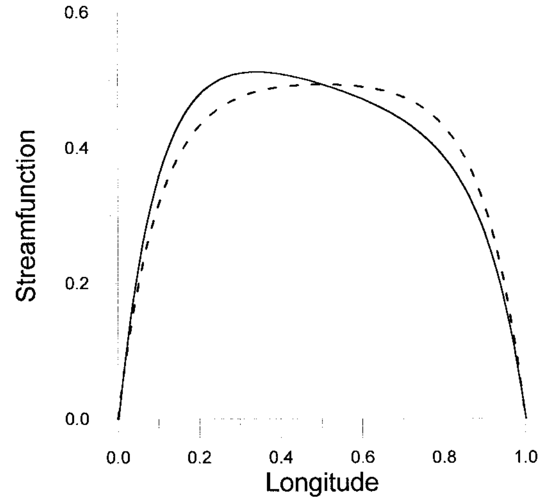

FIG. 2. Plot of the total solution $\psi = \psi_0 + \delta\psi_1$ (continuous line) superimposed to the Fofonoff solution ψ_0 (dashed line) vs longitude, both evaluated at the midbasin latitude $y = ½$. To make the difference between these solutions perceptible, the values $\varepsilon = \delta = 10^{-1}$ and $\varepsilon = 10^{-1}$, $\delta = 0$ have been used respectively. The asymmetry of ψ with respect to ψ_0 is related, for increasing values of the longitude, to the western intensified boundary current (northward), the southward drift of the interior, and the eastern weakened boundary current (southward).

4. Stability analysis of the solution

First, the following time-dependent version of Eq. (8) is postulated:

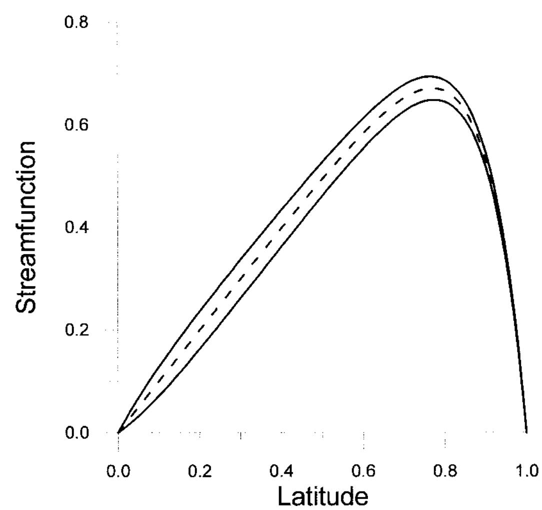

FIG. 3. Plot of the total solution (continuous lines) compared with that of Fofonoff (dashed line) vs latitude, for $\varepsilon = \delta = 10^{-1}$. The upper branch refers to $x = ⅓$, the lower one to $x = ⅔$. The different slope of these branches in the proximity of the zonal boundaries, in $y = 0$ and $y = 1$, are caused by the divergence or convergence of the interior meridional current (reported in Fig. 1) near these boundaries, which modulates the intensity of the local zonal current.

$$\frac{\partial}{\partial t}\nabla^2\psi + \varepsilon^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \delta(\hat{z}\cdot\nabla\times\boldsymbol{\tau} - \varepsilon\nabla^2\psi). \quad (38)$$

From the stability point of view, only the sign of the time derivative is relevant, so multiplication of $\partial\nabla^2\psi/\partial t$ by some scale factor does not change the conclusions. The perturbed state ψ is the superposition of the steady basic state (10) with a time-dependent perturbation, expressed as $\varphi_0 + \delta\varphi_1$, in order to keep the problem for φ_0 separate from that for φ_1. Substitution of $\psi = \psi_0 + \varphi_0 + \delta(\psi_1 + \varphi_1)$ into (38) gives, to the zeroth and first order in δ, the following equations:

$$\frac{\partial}{\partial t}\nabla^2\varphi_0 = -\varepsilon^2 J(\psi_0, \nabla^2\varphi_0) + J(\psi_0, \varphi_0) - \varepsilon^2 J(\varphi_0, \nabla^2\varphi_0) \quad (39)$$

$$\frac{\partial}{\partial t}\nabla^2\varphi_1 = -\varepsilon^2 J(\varphi_0, \nabla^2\psi_1) + \varepsilon^2 J(\nabla^2\varphi_1, \psi_0) + \varepsilon^2 J(\nabla^2\varphi_1, \varphi_0) - \varepsilon^2 J(\psi_1, \nabla^2\varphi_0) - J(\varphi_1, \psi_0) - \varepsilon^2 J(\varphi_1, \nabla^2\varphi_0) - \varepsilon\nabla^2\varphi_0. \quad (40)$$

In the deduction of (39) and (40) use has been made of (11), (13), and (15) involving only the basic-state components.

Consider now Eq. (39). Multiplication by

$$\nabla^2\varphi_0 - \frac{1}{\varepsilon^2}\varphi_0$$

and the subsequent integration on D with the aid of boundary condition (9) yields

$$\frac{1}{2}\frac{d}{dt}\int_D \left[(\nabla^2\varphi_0)^2 + \frac{1}{\varepsilon^2}|\nabla\varphi_0|^2\right] dx\, dy = 0; \quad (41)$$

that is, the conservation of the perturbation norm

$$n(\varphi_0) = \left\{\iint_D \left[(\nabla^2\varphi_0)^2 + \frac{1}{\varepsilon^2}|\nabla\varphi_0|^2\right] dx\, dy\right\}^{1/2}. \quad (42)$$

Equation (41) states the stability of Fofonoff's mode ψ_0 with respect to the norm (42). It is not a new result, as Eq. (15) with boundary conditions (12) satisfies the conditions for nonlinear stability of ψ_0 formerly established by Arnold's (1965).

It is quite natural to extend the norm (42) to the complete perturbation $\varphi_0 + \delta\varphi_1$, thus obtaining for the square norm the equation

$$n^2(\varphi_0 + \delta\varphi_1) \approx n^2(\varphi_0) + 2\delta\int_D \left(\nabla^2\varphi_0\nabla^2\varphi_1 + \frac{1}{\varepsilon^2}\nabla\varphi_0\cdot\nabla\varphi_1\right) dx\, dy, \quad (43)$$

where the $O(\delta^2)$ terms have been disregarded in accordance with the statement of Niiler's model. From (43), recalling (41), we have within this approximation

$$\frac{d}{dt}n^2(\varphi_0 + \delta\varphi_1) = 2\delta\int_D \left(\nabla^2\varphi_0\frac{\partial}{\partial t}\nabla^2\varphi_1 + \nabla^2\varphi_1\frac{\partial}{\partial t}\nabla^2\varphi + \frac{1}{\varepsilon^2}\nabla\varphi_0\cdot\frac{\partial}{\partial t}\nabla\varphi_1 + \frac{1}{\varepsilon^2}\nabla\varphi_1\cdot\frac{\partial}{\partial t}\nabla\varphi_0\right) dx\, dy, \quad (44)$$

The first two terms appearing in the integral of (44) can be evaluated directly by using (39) and (40). About the term

$$\nabla\varphi_0\cdot\frac{\partial}{\partial t}\nabla\varphi_1$$

one can use the identity

$$\varphi_0\frac{\partial}{\partial t}\nabla^2\varphi_1 = \nabla\cdot\left(\varphi_0\frac{\partial}{\partial t}\nabla\varphi_1\right) - \nabla\varphi_0\cdot\frac{\partial}{\partial t}\nabla\varphi_1$$

and therefore again (40) and the boundary condition $\varphi_0 = 0$ along ∂D. The procedure is analogous to evaluate the last integral. The final result of this cumbersome computation is

$$\frac{d}{dt}n^2(\varphi_0 + \delta\varphi_1) = 2\delta\left[\iint_D (\varepsilon^2\nabla^2\psi_1 - \psi_1)J(\varphi_0, \nabla^2\varphi_0)\, dx\, dy - \varepsilon n^2(\varphi_0)\right]. \quad (45)$$

Note that φ_1 does not appear in the rhs of (45). A short comment about (45): Equation (43) implies that, *up to first order in* δ,

$$\delta n^2(\varphi_0) = \delta n^2(\varphi_0 + \delta\varphi_1). \quad (46)$$

The square norm $n^2(\varphi_0)$ is conserved [Eq. (41)], so from (46) we have

$$\frac{d}{dt}\delta n^2(\varphi_0 + \delta\varphi_1) = 0.$$

Since we are working with truncated expansions, this last equation does not imply the conservation of $n^2(\varphi_0 + \delta\varphi_1)$; rather, that

$$\frac{d}{dt}n^2(\varphi_0 + \delta\varphi_1) = O(\delta),$$

in accordance with (45).

Equation (45) is equivalent to

$$\frac{d}{dt}n^2(\varphi_0 + \delta\varphi_1) = 2\delta \int_D \nabla^2\varphi_0 \boldsymbol{\nabla}\varphi_0 \cdot \mathbf{B}\, dx\, dy - 2\delta\varepsilon n^2(\varphi_0), \quad (47)$$

where $\mathbf{B} = \hat{z} \times \boldsymbol{\nabla}(\varepsilon^2\nabla^2\psi_1 - \psi_1)$, that is to say, recalling (37), $\mathbf{B} = (2x - 1)\mathbf{i} + (3 - 2y)\mathbf{j}$. In particular,

$$\max_D |\mathbf{B}| = 10^{1/2}. \quad (48)$$

To obtain a suitable differential inequality to prove that $n^2(\varphi_0 + \delta\varphi_1)$ remains bounded in time if it is initially (say in $t = 0$) bounded, we proceed as follows, with reference to (47):

1) Assume for the time being that a positive constant M exists such that

$$\int_D \nabla^2\varphi_0 \boldsymbol{\nabla}\varphi_0 \cdot \mathbf{B}\, dx\, dy \le Mn^2(\varphi_0);$$

2) Substitute (46) into the last term on the rhs of (47).

As a consequence of (i) and (ii), Eq. (47) yields, *up to first order in* δ,

$$\frac{d}{dt}n^2(\varphi_0 + \delta\varphi_1) \le 2\delta M n^2(\varphi_0) - 2\delta\varepsilon n^2(\varphi_0 + \delta\varphi_1). \quad (49)$$

Time integration of (49) gives

$$n^2(\varphi_0 + \delta\varphi_1; t) \le \frac{M}{\varepsilon}n^2(\varphi_0) + \exp(-2\delta\varepsilon t)\left[n^2(\varphi_0 + \delta\varphi_1; 0) - \frac{M}{\varepsilon}n^2(\varphi_0)\right]. \quad (50)$$

Inequality (50) shows that the dissipation, whose strength is $\delta\varepsilon$, controls only at first the evolution of the norm, while, asymptotically, this is bounded by the constant quantity $\varepsilon^{-1}Mn^2(\varphi_0)$.

Consider separately the two possibilities

$$n^2(\varphi_0 + \delta\varphi_1; 0) < \frac{M}{\varepsilon}n^2(\varphi_0) \quad (51)$$

and

$$n^2(\varphi_0 + \delta\varphi_1; 0) \ge \frac{M}{\varepsilon}n^2(\varphi_0). \quad (52)$$

Case (51) implies, through (50),

$$n^2(\varphi_0 + \delta\varphi_1; t) < \frac{M}{\varepsilon}n^2(\varphi_0)$$

and hence, in particular

$$n^2(\varphi_0 + \delta\varphi_1; t) + n^2(\varphi_0) < \left(1 + \frac{M}{\varepsilon}\right)[n^2(\varphi_0 + \delta\varphi_1; 0) + n^2(\varphi_0)]. \quad (53)$$

In terms of the norm

$$\|\varphi_0 + \delta\varphi_1\| = [n^2(\varphi_0 + \delta\varphi_1) + n^2(\varphi_0)]^{1/2} \quad (54)$$

operating on the space of the truncated perturbations $\varphi_0 + \delta\varphi_1$ and recalling (41), inequality (53) is equivalent to

$$\|\varphi_0 + \delta\varphi_1; t\| < \left(1 + \frac{M}{\varepsilon}\right)^{1/2} \|\varphi_0 + \delta\varphi_1; 0\|, \quad (55)$$

and thus the stability of $\psi_0 + \delta\psi_1$ in the norm (54) immediately follows.

On the other hand, case (52) corresponds to

$$n^2(\varphi_0 + \delta\varphi_1; t) \le n^2(\varphi_0 + \delta\varphi_1; 0), \quad (56)$$

so (56) itself ensures the stability of the basic state in the norm $n(\varphi_0 + \delta\varphi_1)$. However, (56) also implies

$$n^2(\varphi_0 + \delta\varphi_1; t) + n^2(\varphi_0) \le n^2(\varphi_0 + \delta\varphi_1; 0) + n^2(\varphi_0);$$

that is, by resorting again to (54),

$$\|\varphi_0 + \delta\varphi_1; t\| \le \|\varphi_0 + \delta\varphi_1; 0\|. \quad (57)$$

By means of (55) and (57) we have proven the *unconditional* stability of Niiler's solution with respect to the norm (54), in correspondence to both the possibilities (51) and (52).

Finally, M has to be estimated. Note that if $M \le \varepsilon$, then (47) would imply

$$\frac{d}{dt}n(\varphi_0 + \delta\varphi_1) \le 0,$$

and hence stability would immediately follow. However, we are only able to evaluate an upper bound of M. To this purpose, it is proven that the ratio

$$R \equiv \frac{\int_D \nabla^2\varphi_0 \boldsymbol{\nabla}\varphi_0 \cdot \mathbf{B}\, dx\, dy}{n^2(\varphi_0)} \quad (58)$$

is bounded by a constant. It is useful to define, for shortness, the following functionals:

$$K = K(\varphi_0) \equiv \int_D |\boldsymbol{\nabla}\varphi_0|^2\, dx\, dy, \quad (59)$$

and

$$V = V(\varphi_0) \equiv \int_D (\nabla^2\varphi_0)^2\, dx\, dy, \quad (60)$$

which are proportional to the kinetic energy and the potential enstrophy of the perturbation φ_0, respectively. By using the inequality

$$\nabla^2\varphi_0\boldsymbol{\nabla}\varphi_0\cdot\mathbf{B} \leq |\nabla^2\varphi_0|\,|\boldsymbol{\nabla}\varphi_0| \max_D |\mathbf{B}|$$

and then the Schwarz inequality in the numerator of (58), one can write, recalling (48), (59), and (60),

$$R \leq 10^{1/2}\frac{(KV)^{1/2}}{V + \varepsilon^{-2}K}. \qquad (61)$$

Putting $a = a(\varphi_0) \equiv (K/V)^{1/2}$, the Wirtinger inequality [see (A6) in the appendix] $K/V \leq 1/\pi^2$ ensures that $a(\varphi_0)$ is never singular, being included into the interval $0 \leq a \leq 1/\pi$. In terms of a, (61) takes the form

$$R \leq 10^{1/2}\frac{a}{1 + \varepsilon^{-2}a^2}$$

and hence

$$R \leq \frac{10^{1/2}}{2}\varepsilon$$

(a stronger upper bound would be obtained for $\varepsilon > 1/\pi$ but this inequality is physically unexpected). Therefore,

$$\int_D \nabla^2\varphi_0\boldsymbol{\nabla}\varphi_0\cdot\mathbf{B}\,dx\,dy \leq \frac{10^{1/2}}{2}\varepsilon n^2(\varphi_0);$$

that is

$$M = \frac{10^{1/2}}{2}\varepsilon.$$

This completes our proof.

5. Remarks

From (50) with

$$\frac{M}{\varepsilon} = \frac{10^{1/2}}{2},$$

one obtains $n^2(\varphi_0 + \delta\varphi_1;\ +\infty) < 1.58n^2(\varphi_0)$, while condition $n^2(\varphi_0 + \delta\varphi_1;\ +\infty) \approx n^2(\varphi_0)$ should be more close to intuition since one can imagine that bottom dissipation is able to erode wholly, in the course of an infinite time interval, the difference $n^2(\varphi_0 + \delta\varphi_1) - n^2(\varphi_0)$. This last concept is probably correct. However, we recall that the evaluation of M is based on the application of inequalities that overestimate the numerator of (51) and thus M itself.

One could wonder why (46) is not used also in the first term of the rhs of (49). The reason is that the *sign* of the difference $n^2(\varphi_0 + \delta\varphi_1) - n^2(\varphi_0)$ is not known, so inequality

$$\delta\int_D \nabla^2\varphi_0\boldsymbol{\nabla}\varphi_0\cdot\mathbf{B}\,dx\,dy \leq \delta M n^2(\varphi_0 + \delta\varphi_1)$$

could be false, even if the inequality

$$\delta\int_D \nabla^2\varphi_0\boldsymbol{\nabla}\varphi_0\cdot\mathbf{B}\,dx\,dy \leq \delta M n^2(\varphi_0)$$

is correct. On the other hand, it is just the time independence of the first term in the rhs of (49) that makes the integration of (49) very simple.

If one accounts for, in a formal way, the $O(\delta^2)$ contribution to (50), an additional term at the rhs of (50) is obtained, of the kind

$$\delta^2\exp(-2\delta\varepsilon t)\int_0^t \exp(2\delta\varepsilon\vartheta)F(\vartheta)\,d\vartheta,$$

but $F(t)$ is an $O(1)$ function that cannot be explicitly determined in the framework of Niiler's model.

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APPENDIX

The Wirtinger Inequality

Lemma: Let $\mathbf{u} = (u, v)$ be a two-dimensional current field in the domain $D = \{(x, y): 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ with no-mass flux boundary conditions:

$$u(0, y) = u(1, y) = 0 \quad \forall y\colon 0 \leq y \leq 1 \qquad (A1)$$

$$v(x, 0) = v(x, 1) = 0 \quad \forall x\colon 0 \leq x \leq 1. \qquad (A2)$$

It is proven that

$$N \equiv \int_D J(v, u)\,dx\,dy = 0, \qquad (A3)$$

where $J(v, u)$ is the Jacobian determinant with respect to x and y.

By using the identity

$$J(v, u) = \boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla}u \times \mathbf{k})$$

($\mathbf{k}$ is the unit ‘‘upward’’ vector) and the divergence theorem, the integral N takes the form

$$N = \oint_{\partial D} v\boldsymbol{\nabla}u \times \mathbf{k}\cdot\mathbf{n}\,ds$$

($\mathbf{n}$ is the unit normal to ∂D in the x–y plane and ds is the differential arclength along ∂D); that is

$$N = \oint_{\partial D} v\boldsymbol{\nabla}u\cdot d\mathbf{r}, \qquad (A4)$$

($d\mathbf{r} = \mathbf{t}\,ds$ where $\mathbf{t} = \mathbf{k} \times \mathbf{n}$ and, with reference to the Cartesian x–y plane, $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy$). Because of the boundary conditions (A2), the integral (A4) is simply

$$N = \int_0^1 \upsilon(1, y)\frac{\partial u}{\partial y}\bigg|_{x=1} dy - \int_0^1 \upsilon(0, y)\frac{\partial u}{\partial y}\bigg|_{x=0} dy;$$

that is

$$N = \int_0^1 \left\{\frac{\partial}{\partial y}[\upsilon(1, y)u(1, y)] - u(1, y)\frac{\partial \upsilon}{\partial y}\bigg|_{x=1}\right\} dy - \int_0^1 \left\{\frac{\partial}{\partial y}[\upsilon(0, y)u(0, y)] - u(0, y)\frac{\partial \upsilon}{\partial y}\bigg|_{x=0}\right\} dy.$$

Recalling boundary condition (A1), the integral (A5) reduces to a linear combination of products υu evaluated in the stagnation points (r, s) with $r = 0, 1$ and $s = 0, 1$. Therefore, $N = 0$.

The Wirtinger inequality (Straughan 1992)

Let V be the cell, in three dimensions, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z$ 1. Suppose that the function $\sigma(x, y, z)$ is such that

1) $$\frac{\partial \sigma}{\partial z} = 0 \quad \text{on } z = 0, 1$$

2) $$\int_V \sigma \, dx\, dy\, dz = 0;$$

then the Wirtinger inequality states that

$$\int_V \sigma^2 \, dx\, dy\, dz \le \frac{1}{\pi^2}\int_V \left[\left(\frac{\partial \sigma}{\partial x}\right)^2 + \left(\frac{\partial \sigma}{\partial y}\right)^2 + \left(\frac{\partial \sigma}{\partial z}\right)^2\right) dx\, dy\, dz.$$

Let $\sigma \equiv \partial\varphi_0/\partial x$. Since φ_0 does not depend on z, condition (1) is verified. Moreover, condition (2) becomes

$$\int_0^1 dy[\varphi_0(1, y) - \varphi_0(0, y)] = 0$$

and it is also verified. Therefore,

$$\int_D \left(\frac{\partial \varphi_0}{\partial x}\right)^2 dx\, dy \le \frac{1}{\pi^2}\int_D \left[\left(\frac{\partial^2 \varphi_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \varphi_0}{\partial x \partial y}\right)^2\right] dx\, dy.$$

In the same way, for $\sigma \equiv \partial\varphi_0/\partial y$, one obtains

$$\int_D \left(\frac{\partial \varphi_0}{\partial y}\right)^2 dx\, dy \le \frac{1}{\pi^2}\int_D \left[\left(\frac{\partial^2 \varphi_0}{\partial y^2}\right)^2 + \left(\frac{\partial^2 \varphi_0}{\partial x \partial y}\right)^2\right] dx\, dy.$$

Adding the last two inequalities, we have

$$\int_D |\nabla\varphi_0|^2 \, dx\, dy \le \frac{1}{\pi^2}\int_D (\nabla^2\varphi_0)^2 \, dx\, dy + \frac{2}{\pi^2}\int_D \left[\left(\frac{\partial^2 \varphi_0}{\partial x \partial y}\right)^2 - \frac{\partial^2 \varphi_0}{\partial x^2}\frac{\partial^2 \varphi_0}{\partial y^2}\right] dx\, dy.$$

Setting $u = -\partial\varphi_0/\partial y$ and $\upsilon = \partial\varphi_0/\partial x$, the last integral coincides with the integral N of the lemma. Thus, identity (A3) holds and we conclude that

$$\int_D |\nabla\varphi_0|^2 \, dx\, dy \le \frac{1}{\pi^2}\int_D (\nabla^2\varphi_0)^2 \, dx\, dy. \quad \text{(A6)}$$

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