NOTES AND CORRESPONDENCE

A New Statistical Distribution for the Surface Elevation of Weakly Nonlinear Water Waves

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ABSTRACT

A new statistical distribution for the surface elevation of weakly nonlinear water waves is derived using the Pearson System of distributions. The new distribution avoids some problems associated with previously proposed distributions. Namely, its probability density function is positive everywhere, unlike prior results obtained with Gram–Charlier series. Furthermore, it is derived without requiring the assumptions that the wavefield is unidirectional and narrow band, as made in some earlier studies. The distribution obtained is a form of the beta distribution and depends only on two parameters, the variance and the skewness of the sea surface elevation. The new distribution is compared to wave data, measured on a reservoir, and found to give a reasonable fit.

1. Introduction

For many years Gaussian theory has been used successfully to describe the statistics of waves on the ocean surface. However, being based on the assumption that water waves are a linear phenomenon, it fails to account adequately for nonlinear wave effects such as the skewness of the sea surface elevation distribution (which is a consequence of the peakier crests and flatter troughs of nonlinear waves—Srokosz 1990). To date, two different approaches have been tried to describe these nonlinear effects, and both assume that the effects of nonlinearity are weak.

The first approach is that of Longuet-Higgins (1963), who used a Gram–Charlier series—essentially a modification of the Gaussian theory—to describe the effects of nonlinearity. He determined the moments and cumulants of the distribution by using a weakly nonlinear dynamical theory for the waves. The drawback of this approach is that the resulting probability density function (pdf) for the surface elevation is negative over part of its range (strictly it should be positive everywhere). The bulk of the pdf is positive and therefore this does provide a model for statistics of nonlinear waves.

To overcome the problem of negative values for the pdf Huang et al. (1983) specialized the Longuet-Higgins (1963) approach to the case of a narrowband unidirectional sea. By using an expansion in powers of the significant slope (assumed small) they were able to derive a pdf for the surface elevation that is positive everywhere. Unfortunately, their results are only applicable to the narrowband unidirectional case and cannot be generalized to the broadband multidirectional wave field case, which is more commonly found in nature. Tayfun (1986) noted some difficulties with Huang et al.'s (1983) approach and questioned the validity of their results. Tayfun also derived some results for the narrowband unidirectional case but did not give an explicit expression for the pdf of the surface elevation.

Here a third approach to the derivation of the statistical distribution of the surface elevation for nonlinear water waves will be presented. In common with the previous approaches, the basis of this approach is the assumption of weak nonlinearity, but it is aimed at avoiding their limitations. From Longuet-Higgins' (1963) weakly nonlinear dynamical theory it is known that, to leading order in wave steepness (a measure of nonlinearity), the distribution of surface elevation has nonzero skewness and zero kurtosis. The questions this paper addresses are: which, if any, statistical distributions have these properties? If a distribution with these properties exists, does it provide a possible model for the statistics of the surface elevation of nonlinear water waves?

2. The Pearson System

To answer the first question stated above, the Pearson System of distributions will be used. Originally due to Pearson (1895), it is fully described by Johnson and

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Kotz (1970a, chapter 12). For every member of the system the pdf, $p(x)$, of the distribution satisfies a differential equation of the form

$$\frac{1}{p}\frac{dp}{dx} = \frac{-(a + x)}{(c_0 + c_1x + c_2x^2)}, \tag{1}$$

where

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{2}$$

and

$$p(x) \geq 0. \tag{3}$$

We will attempt to find a distribution belonging to this system that has nonzero skewness and zero kurtosis. Subsequently we will test the distribution to see if it provides a reasonable fit to wave data. From Eq. (1) we obtain, for $r = 1, 2, \ldots,$

$$x^r(c_0 + c_1x + c_2x^2)\frac{dp}{dx} = -(a + x)x^rp. \tag{4}$$

On integrating Eq. (4) from $-\infty$ to ∞ and assuming that $x^rp(x) \to 0$ as $x \to \pm\infty$, which is a reasonable assumption for the behavior of a pdf, we obtain

$$-rc_0\mu'_{r-1} + [a - (r + 1)c_1]\mu'_r + [1 - (r + 2)c_2]\mu'_{r+1} = 0, \tag{5}$$

where

$$\mu'_r = \int_{-\infty}^{\infty} x^rp(x)\, dx \tag{6}$$

are the moments of the distribution about the origin. From Eqs. (2) and (6) $\mu'_0 = 1$.

To simplify the problem further, and without loss of generality, we can chose $\mu'_1 = 0$ so that the origin corresponds to the mean sea level and also scale the problem so that $\mu'_2 = 1$. Thus, x represents the sea surface elevation relative to the mean level, measured in units of the standard deviation. Now as $\mu'_1 = 0$, the moments about the mean μ_r are equal to the moments about the origin μ'_r, so that Eq. (5) can be solved for a, c_0, c_1, c_2 by setting $r = 1, 2, 3, 4$. Thus,

$$c_0 = (4\beta_2 - 3\beta_1)(10\beta_2 - 12\beta_1 - 18)^{-1} \tag{7a}$$

$$c_1 = a = \sqrt{\beta_1}(\beta_2 + 3)(10\beta_2 - 12\beta_1 - 18)^{-1} \tag{7b}$$

$$c_2 = (2\beta_2 - 3\beta_1 - 6)(10\beta_2 - 12\beta_1 - 18)^{-1}, \tag{7c}$$

where

$$\beta_1 = \mu_3^2/\mu_2^3 = \lambda_3^2 \tag{8a}$$

$$\beta_2 = \mu_4/\mu_2^2 = \lambda_4 + 3. \tag{8b}$$

Here λ_3 is the skewness and λ_4 is the kurtosis. Note that for $r \geq 4$ Eq. (5) allows the higher moments to be expressed in terms of μ_0 to μ_4.

We will now consider two special cases from the above system:

Case (a) $\lambda_3 = \lambda_4 = 0$

This assumption implies that $a = c_1 = c_2 = 0$ and $c_0 = 1$, so from Eq. (1)

$$\frac{d}{dx}(\log p) = -x;$$

hence

$$p(x) = (2\pi)^{-1/2}\exp(-x^2/2)$$

on using Eq. (2). Thus, we recover the Gaussian distribution for the surface elevation when no nonlinear effects are present ($\lambda_3 = \lambda_4 = 0$).

Case (b) $\lambda_3 \neq 0$, $\lambda_4 = 0$

This choice is made on the basis of Longuet-Higgins (1963) weakly nonlinear dynamical theory and gives the following results from Eq. (7):

$$c_0 = \frac{1}{4}(4 - \lambda_3^2)(1 - \lambda_3^2)^{-1} \tag{9a}$$

$$c_1 = a = \frac{1}{2}\lambda_3(1 - \lambda_3^2)^{-1} \tag{9b}$$

$$c_2 = -\frac{1}{4}\lambda_3^2(1 - \lambda_3^2)^{-1}. \tag{9c}$$

Note that these results are singular for $\lambda_3 = 1$, but the data examined below have λ_3 in the range 0 to 0.504, so this should cause no problems for the theory.

Johnson and Kotz (1970a) show that the form of the distribution depends on the roots of the equation [see Eq. (1) above]

$$c_2x^2 + c_1x + c_0 = 0, \tag{10}$$

which, on the use of Eq. (9), are

$$a_1 = \lambda_3^{-1}[1 - \sqrt{5 - \lambda_3^2}] \tag{11a}$$

$$a_2 = \lambda_3^{-1}[1 + \sqrt{5 - \lambda_3^2}]. \tag{11b}$$

For $0 < \lambda_3 < 1$, both roots are real and satisfy

$$a_1 < -1, \qquad a_2 > 3; \tag{12}$$

that is, there is one positive and one negative root, with $a_2 > a_1$. Hence, from Johnson and Kotz (1970a, chapter 12) we obtain

$$p(x) = K(x - a_1)^{m_1}(a_2 - x)^{m_2}, \tag{13}$$

where

$$m_1 = \frac{(a + a_1)}{c_2(a_2 - a_1)} \tag{14a}$$

$$m_2 = \frac{-(a + a_2)}{c_2(a_2 - a_1)}. \tag{14b}$$

Note that x is restricted so that

TABLE 1. Parameters of the beta distribution [Eq. (15)] as a function of the skewness λ_3. Note that $\lambda_3 = 0$ is the special case of the Gaussian distribution, $\lambda_c = 0.826490$ is a critical value at which the distribution changes form (see text), and the values for $\lambda_3 = 1$ are derived in the limit $\lambda_3 \to 1$ as the form of the distribution becomes singular [Eq. (9)].

λ_3	a_1	a_2	p	q
0.0	$-\infty$	∞	—	—
0.1	−12.34	32.34	109.92	288.09
0.2	−6.14	16.14	27.00	71.00
0.3	−4.05	10.72	11.64	30.80
0.4	−3.00	8.00	6.27	16.73
0.5	−2.36	6.36	3.79	10.21
0.6	−1.92	5.26	2.44	6.67
0.7	−1.61	4.46	1.63	4.53
0.8	−1.36	3.86	1.11	3.14
$\lambda_c = 0.826490$	−1.30	3.72	1.00	2.86
0.9	−1.16	3.39	0.75	2.19
1.0	−1.00	3.00	0.50	1.50

$$a_1 \le x \le a_2 \tag{15}$$

and that for x outside this range $p(x)$ is identically zero. In Eq. (13) K is a normalizing factor to be determined using Eq. (2).

Having found a distribution that satisfies the requirement that $\lambda_3 \neq 0$, $\lambda_4 = 0$, we will next investigate its properties.

3. The distribution of surface elevation

Equation (13) represents the pdf of a beta distribution (Johnson and Kotz 1970b, chapter 24) and may be written in normalized form as

$$p(x) = \begin{cases} \dfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \dfrac{(x-a_1)^{p-1}(a_2-x)^{q-1}}{(a_2-a_1)^{p+q-1}} \\ \quad \text{for } a_1 \le x < a_2 \\ \\ 0 \quad \text{for } x < a_1 \text{ and } x > a_2, \end{cases} \tag{16}$$

where

$$p = 1 + m_1, \qquad q = 1 + m_2; \tag{17}$$

and m_1, m_2 are given by Eq. (14); a_1, a_2 are given by Eq. (11); and $\Gamma(\cdot)$ is the gamma function.

The primary point to note about the beta distribution is that, unlike the Gaussian distribution, which sets no limits on the values of surface elevation, here the surface elevation is restricted to a range of values that depend on the skewness λ_3 [see Eqs. (11) and (15)]. From Johnson and Kotz (1970b) the beta distribution exists for $p > 0$, $q > 0$, but the form of the distribution changes if p or q or both p and $q \le 1$. For a ‘‘hump-shaped’’ distribution typical of the surface elevation of water waves (see, e.g., Huang and Long 1980, or Srokosz 1990) we require $p > 1$ and $q > 1$, which implies, from Eq. (17) that $m_1 > 0$ and $m_2 > 0$. It is easily shown, from Eqs. (9), (11), and (14), that $m_2 > 0$ for $0 < \lambda_3 < 1$, so $q > 1$. However, m_1 is not positive for the whole range $0 < \lambda_3 < 1$. Setting $m_1 = 0$ in Eq. (14) and solving for λ_3 gives a value $\lambda_3 = \lambda_c = 0.826490$ for which $p = 1$. Therefore, we might expect the beta distribution to provide a useful model for the distribution of surface elevation for $0 < \lambda_3 < \lambda_c$. The implications of this are explored in Section 4.

Here we tabulate the values of the parameters of the distribution to show their dependence on λ_3: $\lambda_3 = 0$ represents the results for the Gaussian distribution, while those for $\lambda_3 > 0$ are the results for the beta distribution. As the skewness increases it can be seen that the maximum and minimum possible values for the surface elevation decrease in magnitude. Note that nonlinear wave theory shows that there is a maximum steepness for individual waves, beyond which they break, whereas linear theory imposes no such limit (Lamb 1932; Srokosz 1990). The relationship between this deterministic result and the statistical limit on the surface elevation found here is not clear. The statistical theory sets no restriction on the wavelengths of the waves, and therefore no restriction on their steepness, and neither (formally) does the weakly nonlinear dynamical theory of Longuet-Higgins (1963), from which the statistical results have been developed.

In discussing the limitations on the values of surface elevation set by the beta distribution it is important to remember that the results for a_1 and a_2 given in Table 1 are normalized in terms of the standard deviation $\sqrt{\mu_2}$ of the elevation. Therefore, they do not represent absolute limits on the surface elevation, as larger values of $\sqrt{\mu_2}$ will lead to larger values of the elevation. Thus, the distribution of surface elevation is determined by the two parameters μ_2 and λ_3, where in Eq. (16)

$$x = \zeta/\sqrt{\mu_2}, \tag{18}$$

and ζ is the dimensional value of the surface elevation (in appropriate units).

Having derived this new distribution for the surface elevation of water waves, which by construction is positive everywhere and is not restricted to narrowband unidirectional waves, we proceed to test its validity against wave data.

TABLE 2. Results from the analysis of the five reservoir datasets. The wind speed on each occasion is given in brackets below the dataset number, in meters per second. Gauge refers to different gauges in the capacitance wire array. N is the number of degrees of freedom for the X^2 test applied to the beta distribution; G–C is an abbreviation for Gram–Charlier.

Dataset (U m s^{-1})	Gauge	λ_3	λ_4	$\frac{\zeta_{max}}{\sqrt{\mu_2}}$	$\frac{\zeta_{min}}{\sqrt{\mu_2}}$	χ^2_N (beta)	N	χ^2_{32} (G–C)	$\sqrt{\mu_2}$ (cm)
20	1	0.433	0.179	4.20	−2.90	9.13	27	9.06	7.09
(12.4)	3	0.453	0.013	3.54	−2.45	9.15	27	9.25	7.19
	6	0.504	0.306	4.05	−2.51	9.18	25	9.19	6.94
23	1	0.286	0.057	4.70	−2.69	8.99	32	9.12	5.05
(10.8)	3	0.272	−0.153	3.59	−2.95	9.09	32	9.21	5.36
	6	0.325	−0.059	4.64	−2.76	9.06	31	9.09	5.18
45	3	0.244	0.040	4.36	−3.26	8.98	32	9.05	3.14
(7.0)	5	0.290	−0.111	3.88	−3.40	9.04	32	9.15	3.19
	7	0.286	−0.092	4.80	−3.06	8.97	32	9.12	3.38
08	1	0.149	−0.136	3.99	−4.84	9.22	32	9.26	4.00
(7.1)	3	0.169	−0.196	3.99	−3.13	9.12	32	9.16	3.27
	8	0.209	−0.273	3.57	−3.21	9.09	32	9.18	3.70
05	1	0.081	−0.239	3.35	−3.89	9.33	32	9.35	1.94
(3.7)	3	0.056	0.015	4.82	−4.25	9.36	32	9.37	2.06
	8	0.190	−0.504	3.15	−2.83	9.32	32	9.40	2.21

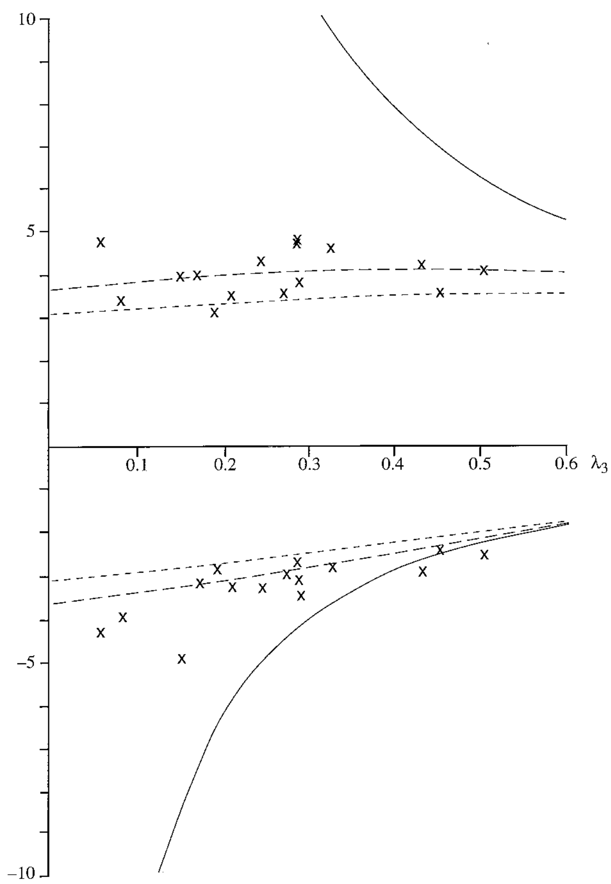

FIG. 1. Plot of the absolute limits (solid) on the beta distribution as a function of the skewness λ_3. Also shown are the 0.1 percentile (short dash) and the 0.0125 percentile (long dash), together with the maximum and minimum surface elevations ($\times$) obtained from the data (see Table 2). The units on the vertical axis are in terms of the standard deviation $\sqrt{\mu_2}$.

4. Comparison with data

As there is some question as to whether surface following wave measuring buoys correctly measure nonlinear wave effects (see, e.g., James 1986) we shall not use such data in this comparison. Instead data recorded on a local reservoir, with an array of capacitance wire wave gauges, will be used. These data were analyzed for other purposes in an earlier paper (Ewing et al. 1987) and are fully described there, so that description will not be repeated here. It is sufficient to note that there are five datasets (numbered as in Ewing et al. 1987), and for each one data from three of the capacitance wires in the array are analyzed to check the results for consistency. The degree of nonlinearity, as measured by the value of skewness λ_3, varies with wind speed (see Table 2 and Ewing et al. 1987). For these datasets the maximum value of the skewness λ_3 is less than the critical value λ_c, for which the form of the distribution changes (see previous section), so this will cause no problems here.

Table 2 gives the results obtained from the analysis of the 15 records, which are also plotted in Fig. 1. Each record consists of approximately 8000 values of surface elevation, recorded at a digitization rate of 8 Hz. Histograms of the surface elevation were obtained from the data, and the beta distribution for each case was obtained using the measured value of the skewness λ_3. Figures 2 and 3 give example plots of the results. The chi-squared test was applied to each set of data and no significant difference was found between the data and the beta distribution. The results of this test are also given in Table 2, together with those obtained by applying the chi-squared test to Longuet-Higgins (1963) Gram–Charlier series results for the elevation

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\}\left[1 + \frac{\lambda_3}{6}(x^3 - 3x)\right], \quad (19)$$

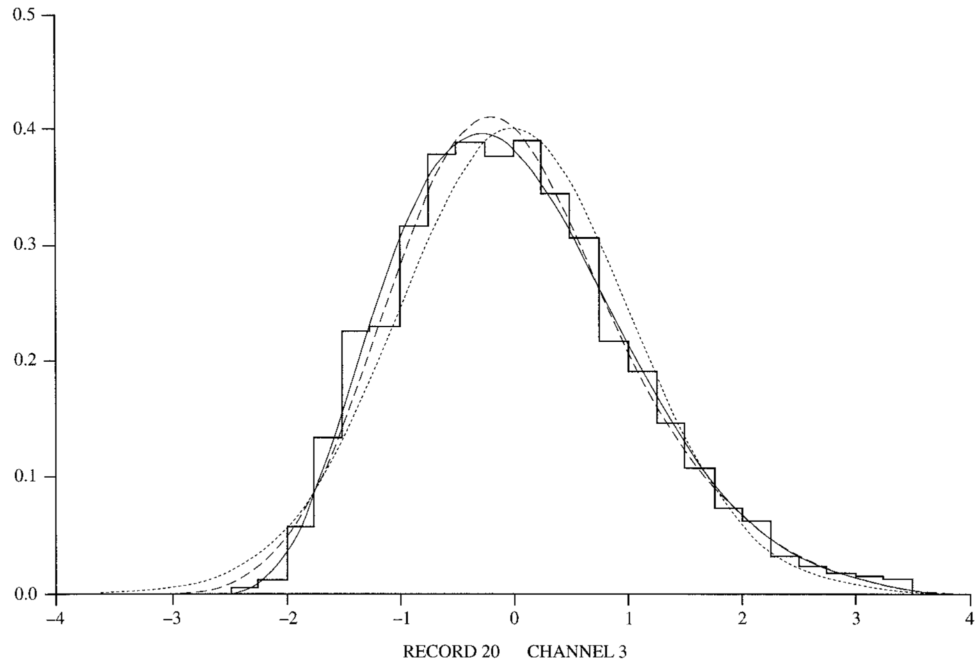

FIG. 2. Histogram of surface elevations (in standard deviation units $\sqrt{\mu_2}$), together with the theoretical pdfs, beta: solid, Gram–Charlier: long dash, Gaussian: short dash, for record 20, gauge 3, with $\lambda_3 = 0.453$ (see Table 1).

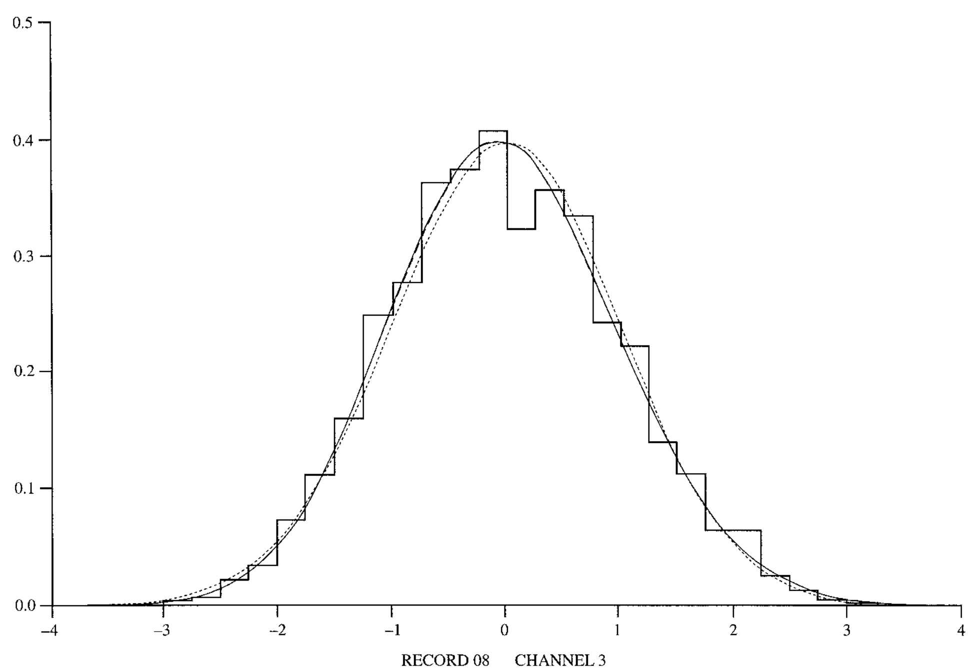

FIG. 3. Histogram of surface elevations (in standard deviation units $\sqrt{\mu_2}$), together with the theoretical pdfs, beta: solid, Gram–Charlier: long dash, Gaussian: short dash, for record 08, gauge 3, with $\lambda_3 = 0.169$ (see Table 1). Note that in this case the beta and Gram–Charlier results are indistinguishable.

where x is given by Eq. (18). In this case too, no significant difference was found. For both cases the test was set up to use 32 degrees of freedom, the number of histogram bins, except when the finite limits of the beta distribution required the reduction of that number for computational purposes. For those tests with 32 degrees of freedom (11 out of the 15 cases) the chi-squared values are consistently lower for the beta distribution, as compared to those for the Gram–Charlier series. This suggests that the beta distribution is possibly a better fit to the data.

Returning to Fig. 1 we note that these results suggest that the beta distribution may represent the distribution of surface elevation reasonably well. The maximum and minimum values of surface elevation obtained from the data generally lie close to the 0.1 (1/1000) percentile and the 0.0125 (1/8000) percentile, which might have been expected on the basis of analyzing $\sim$8000 values of surface elevation and assuming that they are not necessarily independent. Two of the (minimum) values lie just outside the bounds of the beta distribution, but this might be explained on the basis of measurement errors. Alternatively, because for these points the value of the skewness is large, the weakly nonlinear assumption $\lambda_3 \neq 0$, $\lambda_4 = 0$ used to derive the beta distribution may not be valid. Values of the kurtosis were calculated from the data (see Table 2), but the estimates are unreliable as the lengths of the data records are relatively short, so it is difficult to assess the validity of assuming $\lambda_4 = 0$. The difficulty of obtaining accurate estimates of higher-order moments from data can be seen from the work of Huang and Long (1980). They used considerably longer records ($\sim$1.3 million surface elevation values), obtained in laboratory conditions, to estimate higher-order moments. However, despite these reservations, this initial comparison of the new distribution with data gives reasonable agreement.

Given the reservations expressed by Tayfun (1986) about Huang et al.'s (1983) results for the narrowband unidirectional case, no attempt has been made to fit their distribution to the data.

5. Discussion and conclusions

In this paper it has been shown how it is possible to derive a new statistical distribution for the surface elevation of water waves when nonlinear effects are important. The fundamental assumption underlying the results derived is that the distribution has nonzero skewness ($\lambda_3 \neq 0$) and zero kurtosis ($\lambda_4 = 0$). The distribution obtained, the beta distribution, has a pdf that is everywhere positive, unlike the Gram–Charlier series obtained by Longuet-Higgins (1963). In addition, unlike the results of Huang et al. (1983) and Tayfun (1986), the one given here is not restricted to the case of narrowband unidirectional waves. Tests against wave data show that the new distribution provides a reasonable representation of the data.

Theoretically we have shown that the beta distribution is only useful for the range of skewness values $0 < \lambda_3 < \lambda_c$. The data that have been examined in this paper lie well below the upper bound λ_c. However, some of the data of Huang and Long (1980), measured in the laboratory, have values of skewness that exceed λ_c. Does this invalidate the use of this new distribution? Since the assumption on which the distribution is derived is that of weak nonlinearity (following Longuet-Higgins 1963) it would be surprising if it could represent distributions with extreme values of the skewness λ_3. Furthermore, it is well known that laboratory results are not necessarily representative of those found in the open ocean. Therefore, a qualified ''no'' may be given in answer to the question. The qualification being that the limitation of the distribution to the range $0 < \lambda_3 < \lambda_c$ must be borne in mind when it is used in practice.

The interesting difference between the new distribution and previously proposed ones, including the Gaussian one for the linear case, is that for nonzero skewness ($\lambda_3 \neq 0$) there are bounds on the maximum and minimum possible values of the normalized surface elevation. This result might best be regarded as a curiosity that remains to be explained, as its interpretation in terms of the underlying physics of the waves is problematical. The implications of this result for applications, such as the prediction of extreme wave heights, also remains to be explored (but that is left for another paper).

As pointed out by Johnson and Kotz (1970a), it is possible to derive systems of distributions that depend on more than four parameters [the Pearson System has four, see Eq. (1)]. However, their complexity increases with increasing numbers of parameters, as does the difficulty of estimating the values of the parameters by fitting to data (essentially it is necessary to estimate the values of higher-order moments of the distribution from data, which can be problematical—see the previous section). This means that the result obtained here from the Pearson System can be regarded as only a partial ''answer'' to the problem of obtaining a statistical distribution that describes the surface elevation of nonlinear water waves. Furthermore, the generalization of the approach to obtain, for example, the joint distribution of surface elevation and slopes is not obvious. [Longuet-Higgins (1963) and Srokosz (1986a) carry out this generalization using the Gram–Charlier approach for one- and two-dimensional elevation and slope distributions, respectively.]

The final point to note in discussing the result obtained here is that it does not account for strongly nonlinear effects, such as wave breaking. Alternative theoretical models have been proposed to describe the statistics of breaking waves (see, e.g., Srokosz 1986b), but these have been developed in an ad hoc heuristic manner (see the discussion in Srokosz 1990). In this paper an attempt has been made to carry out a systematic derivation of the statistical distribution of the surface elevation in the weakly nonlinear case ($\lambda_3 \neq 0$, $\lambda_4 = 0$). It remains a challenging problem to try to develop an appropriate statistical model that will be applicable more broadly.

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REFERENCES

Ewing, J. A., M. S. Longuet-Higgins, and M. A. Srokosz, 1987: Measurements of vertical acceleration in wind waves. *J. Phys. Oceanogr.,* **17,** 3–11.

Huang, N. E., and S. R. Long, 1980: An experimental study of the surface elevation probability distribution and the statistics of wind-generated waves. *J. Fluid Mech.,* **101,** 179–200.

——, ——, C. C. Tung, Y. Yuan, and L. F. Bliven, 1983: A non-Gaussian statistical model for surface elevation of nonlinear random wave fields. *J. Geophys. Res.,* **88,** 7597–7606.

James, I. D., 1986: A note on the theoretical comparison of wave staffs and waverider buoys in steep gravity waves. *Ocean Eng.,* **13,** 209–214.

Johnson, N. L., and S. Kotz, 1970a: *Distributions in Statistics: Continuous Univariate Distributions.* Vol. 1. John Wiley & Sons, 300 pp.

——, and ——, 1970b: *Distributions in Statistics: Continuous Univariate Distributions.* Vol. 2. John Wiley & Sons, 306 pp.

Lamb, H., 1932: *Hydrodynamics.* 6th ed. Cambridge University Press, 738 pp.

Longuet-Higgins, M. S., 1963: The effect of nonlinearities on statistical distributions in the theory of sea waves. *J. Fluid Mech.,* **17,** 459–480.

Pearson, K., 1895: Contributions to the mathematical theory of evolution. II Skew variations in homogeneous material. *Philos. Trans. Roy. Soc. London, Ser. A,* **186,** 343–414.

Srokosz, M. A., 1986a: On the joint distribution of surface elevation and slope for a non-linear random sea, with an application to radar altimetry. *J. Geophys. Res.,* **91,** 995–1006.

——, 1986b: On the probability of breaking in deep water. *J. Phys. Oceanogr.,* **16,** 382–385.

——, 1990: Wave statistics, *Surface Waves and Fluxes,* Vol. I, *Current Theory,* G. L. Geernaert and W. J. Plant Eds., Kluwer Academic, 285–232.

Tayfun, M. A., 1986: On narrow-band representation of ocean waves. I. Theory. *J. Geophys. Res.,* **91,** 7743–7759.