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Graded Prime Ranks and Graded Antisimple Radicals of Rings Graded by Finite groups

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Abstract Let R be graded by a finite group G. Our main purpose is to see whether a graded ring with a certain property has a similar property when regarded as being without grading. First, we prove the incomparability: Let $P \subsetneqq Q$ be ideals of $R\#G^*$ with P being prime, then $P \cap R \subsetneqq Q \cap R$. As its application, we show that the graded prime rank of R is equal to the prime rank of R. Then, we prove that if $|G|^{-1} \in R$, then the graded antisimple radical of R is equal to the antisimple radical of R.

Key words (graded) essential modules; (graded) prime ranks; (graded) antisimple radicals

0 Introduction

Throughout the paper, all rings are associative rings with an identity, and all modules are right modules. R will denote a graded ring of type G, where G is a finite group.

For a G-graded ring R, there exists the extension ring $R\#G^*$ of R with the same identity 1, which is called the smash product. The ring is a free left and right R-module with basis $\{p_x : x \in G\}$ such that $\sum_{x \in G} p_x = 1$ is a decomposition of 1 into orthogonal idempotent. Its multiplication is given by defining $(ap_g)(bp_h) = ab_{gh^{-1}}p_h$, where $a, b \in R, g, h \in G$. The group G acts as automorphisms on $R\#G^*$ by defining $(ap_x)^y = ap_{xy}$, where $a \in R, x, y \in G$. If V is a $R\#G^*$-module, it becomes a graded R-module by defining $V_g = V \cdot p_{g^{-1}}$. If V is a graded R-module, it becomes a $R\#G^*$-module by defining $V \cdot (ap_h) = (V \cdot a)_{h^{-1}}$[1].

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1 Graded prime ranks of graded rings

Recall that a (graded) submodule N of a (graded) module M is said to be (graded) essential in M if N intersects all nonzero (graded) submodules of M non-trivially. By [1, Lemma 2.3], it is easy to obtain that N is graded essential in M as graded R-modules if and only if N is essential in M as $R\#G^*$-modules.

Theorem 1 Let $P \subsetneqq Q$ be ideals of $R\#G^*$ with P being prime. Then $P \cap R \subsetneqq Q \cap R$.

Proof It is easy to see that Q/P is essential in $R\#G^*/P$ as $R\#G^*$-modules since P is a prime ideal of $R\#G^*$, and hence it is graded essential as graded R-modules.

Since $R+P/P$ is a graded submodule of $R\#G^*/P$ as graded R-modules, it is easy to see that $0 \neq Q/P \cap R+P/P = Q\cap R+P/P \cong Q \cap R/P \cap R$. Therefore we obtain that $p \cap R \subsetneqq Q \cap R$.

Recall that for any ring S, the prime rank is the largest integer n such that S has a chain of prime ideals $P_0 \subsetneqq P_1 \subsetneqq P_2 \cdots \subsetneqq P_n$; if no such maximum exists, the prime rank of S is infinite. If R is a graded ring, the graded prime rank is defined analogously using graded prime ideals instead of prime ideals.

Theorem 2 Let R be a graded ring by a finite group G. Then the prime rank of $R\#G^*$ is equal to the graded prime rank of R.

Proof Assume that there exists a chain

$$Q_0 \subsetneqq Q_1 \subsetneqq Q_2 \subsetneqq \cdots \subsetneqq Q_n$$

of prime ideals of $R\#G^*$. By Theorem 1, we have that

$$Q_0 \cap R \subsetneqq Q_1 \cap R \subsetneqq Q_2 \cap R \subsetneqq \cdots \subsetneqq Q_n \cap R.$$

By [1, Theorems 6.2 and 6.3], each $Q_i \cap R$ is a graded prime ideal of R where $i = 0,1,\cdots,n$. Hence we obtain that the prime rank of $R\#G^*$ is at most equal to the graded prime rank of R.

For the converse, assume that there exists a chain

$$P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$$

of graded prime ideals of R. Then it is obvious that

$$P_0\#G^* \subsetneqq P_1\#G^* \subsetneqq P_2\#G^* \subsetneqq \cdots \subsetneqq P_n\#G^*$$

is a chain of ideals of $R\#G^*$. By [1, Theorem 6.2], $P_i\#G^* = \bigcap_{g\in G} Q_i^g$, where $i = 0,1,\cdots,n$; and each Q_i^g is a prime ideal of $R\#G^*$.

Now fix $i (i \neq 0)$. If Q_i is given, then, since $Q_i \supset \bigcap_{g\in G} Q_i^g = P_i\#G^* \supsetneqq P_{i-1}\#G^* = \bigcap_{g\in G} Q_{i-1}^g$, there exists some $g \in G$ such that $Q_i \supset Q_{i-1}^g$ by the primeness of Q_i. Thus $Q_i \supsetneqq Q_{i-1}^g$ since $P_i \neq P_{i-1}$. For convenience, use Q_{i-1} instead of Q_{i-1}^g. Thus by successively choosing $Q_n, Q_{n-1}, \cdots, Q_1, Q_0$, we obtain the chain of prime ideals of $R\#G^*$:

$$Q_0 \subsetneqq Q_1 \subsetneqq Q_2 \subsetneqq \cdots \subsetneqq Q_n.$$

Hence we have that the graded prime rank of R is at most equal to the prime rank of $R\#G^*$.

Therefore the two ranks must be equal.

Corollary 1 Let R be graded by a finite group G. Then the graded prime rank of R is equal to the prime rank of R and is at most equal to the prime rank of R_e.

Proof Obviously, the prime rank of $M_n(R)$ is equal to the prime rank of R.

By [1, Theorem 3.5], that is, $(R\#G^*)*G \cong M_n(R)$, and [2, Theorem 4.7], we obtain that the prime rank of R is equal to the prime rank of $R\#G^*$. Thus we have that the graded prime rank of R is equal to the prime rank of R by Theorem 2.

We know that $p_eR\#G^*p_e \cong R_e$, Thus, by [2, Theorem 4.7], there exists a ono-to-one correspondence between $\mathrm{Spec}_{p_e}(R\#G^*)$ and $\mathrm{Spec}(R_e)$, preserving inclusions. Hence the prime rank of $R\#G^*$ is at most equal to the prime rank of R_e. By Theorem 2, the graded prime rank of R is at most equal to the prime rank of R_e.

2 Graded antisimple radicals of graded rings

For any ring A, $H(A)=\bigcap\{I: I$ is a nonzero ideal of $A\}$ is the heart of A. If R is a graded ring, $H_G(R)=\bigcap\{I: I$ is a nonzero graded ideal of $R\}$ is the graded heart of R:

Let C be the class of prime rings with a nnzero heart. Obviously, if $A\in C$, then A satisfies the minimum condition for ideals. Let $\bigcap\{I: I$ is an ideal of A such that $A/I\in C\}$ be denoted by $S(A)$. If A has no image in C, then we write $S(A)=A$. The radical S decided by C is called antisimple[7]. Analogously we define a graded antisimple radical S_G, that is, for a graded ring R, $S_G(R)=\bigcap\{I: I$ is a graded ideal of R such that $R/I\in C_G\}$ where C_G is the class of graded prime rings with a nonzero graded heart.

Lemma If $R\#G^*/P\in C$, then $R\#G^*/P^g\in C$, for all $g\in G$.

Proof Obviously P is a prime ideal of $R\#G^*$. So it is easy to see that each P^g is a prime ideal of $R\#G^*$, where $g\in G$.

Assume that $R\#G^*/P$ has a nonzero heart H/P. Thus H^g/P^g is a nonzero ideal of $R\#G^*/P^g$. If W is an ideal of $R\#G^*$ such that $P^g\subset W\subsetneqq H^g$, then $P\subset W^{g^{-1}}\subsetneqq H$. Thus $W^{g^{-1}}\subset P$ and so $W\subset P^g$. Hence H^g/P^g is a nonzero heart of $R\#G^*/P^g$. Consequently, $R\#G^*/P^g\in C$.

Theorem 3 Let R be graded by a finite group G. Then

$$S_G(R)\#G^*=S(R\#G^*).$$

Proof Let P be a prime ideal of $R\#G^*$. Thus $P\cap R$ is a graded prime ideal of R. Suppose that $P\cap R=0$, that is, R is graded prime.

Assume that $R\#G^*/P\in C$. Then $R\#G^*/P^g\in C$, for all $g\in G$ by the Lemma. So $R\#G^*/P^g$ satisfies the minimum condition for ideals, where $g\in G$. Since $P\cap R=0$, $\bigcap_{g\in G}P^g=0$[1]. By [8], we obtain that $R\#G^*$ satisfies the minimum condition for ideals. Hence R satisfies the minimum condition for graded ideals.

Since R is graded prime, it has a uniquely minimal graded ideal. So $R\in C_G$.

Therefore $S_G(R)\#G^*\subset S(R\#G^*)$.

On the other hand, let Q be a graded prime ideal of R such that $H_G(R/Q) = H/Q \neq 0$, where H is a graded ideal of R. By [1, Theorem 6. 2], there exists a prime ideal P of $R\#G^*$ such that $P \cap R = Q$. We will show that $H\#G^* + P/P$ is the heart of $R\#G^*/P$ non-trivially.

Obviously, $H\#G^* + P/P$ is a nonzero ideal of $R\#G^*/P$. If L is an ideal of $R\#G^*$ such that $P \subsetneqq L$, then $Q = P \cap R \subsetneqq L \cap R$ by Theorem 1. Thus $L \cap R \supset H$ and so $L/P \supset H\#G^* + P/P$. Hence $R\#G^*/P \in C$ and so $R\#G^*/P^g \in C$, for all $g \in G$. Since $Q\#G^* = \bigcap_{g\in G} P^g \supset S(R\#G^*)$, we obtain that $S_G(R)\#G^* \supset S(R\#G^*)$.

Corollary 2 Let R be graded by a finite group G. Then

(1) $S_G(R) \subset S(R)$, and furthermore, $S_G(R) \subset S(R)_G$;

(2) $S_G(R) \cap R_e = S(R_e)$;

(3) if the grading is strong, then $S_G(R) = RS(R_e) = S(R_e)R$.

Proof (1) Since the skew group ring $A * G$ is a normalizing extension of the ring A, we have that $S(R\#G^*) * G \subset S((R\#G^*) * G)$ by [3, Proposition 3. 3 and examples 3. 6].

We see that $M_n(S_G(R)) \cong (S_G(R)\#G^*) * G = S(R\#G^*) * G \subset S((R\#G^*) * G) \cong S(M_n(R)) = M_n(S(R))$. Therefore $S_G(R) \subset S(R)$.

Since $S(R)_G$ is the largest graded ideal of R contained in $S(R)$, we obtain that $S_G(R) \subset S(R)_G$.

(2) By [1, Corollary 1. 5] and [4, Proposition 4 and Theorem 5], $S(R_e)p_e = S(R_e p_e) = p_e S(R\#G^*)p_e = p_e(S_G(R)\#G^*)p_e = S_G(R)_e p_e = (S_G(R) \cap R_e)p_e$. Thus $S(R_e) = S_G(R) \cap R_e$.

(3) Since R is strongly graded, the result follows by (2) and [6, Corollary I. 3. 8].

Theorem 4 Let R be graded by a finite group G. Then $S_G(R) = S(R)_G$.

Proof Assume that Q is a graded prime ideal of R such that $H_G(R/Q) = H/Q \neq 0$, where H is a graded ideal of R. By [1, the proof of Lemma 5. 1], there exists a prime ideal P of R which is maximal with respect to the property $P_G = Q$.

Obviously, $H + P/P$ is a nonzero ideal of R/P. If J is an ideal of R such that $R \subsetneqq J$, then $Q = P_G \subsetneqq J_G$ by the maximum of P with respect to the property $P_G = Q$. Thus $J_G \supset H$ and so $J/P \supset H + P/P$. Hence $R/P \in C$. So $P \supset S(R)$, and $Q = P_G \supset S(R)_G$. Consequently, $S_G(R) \supset S(R)_G$.

On the other hand, by Corollary 2(1), $S_G(R) \subset S(R)_G$.

Therefore $S_G(R) = S(R)_G$.

Theorem 5 Let R be graded by a finite group G and $|G|^{-1} \in R$. Then $S_G(R) = S(R)$.

Proof Assume that $R \in C_G$. By [1, Theorem 6. 3], we have that $0 = P_1 \cap P_2 \cap \cdots \cap P_k$, $k \leqslant |G|$ and that each P_i is a minimal prime ideal of R over the graded prime 0, $i = 1,2,\cdots,k$, when $|G|^{-1} \in R$. We will show that $R/P_i \in C$ for all i.

Let H be the nonzero graded heart of R. Then $H + P_i/P_i$ is a nonzero ideal of R/P_i. If I is an ideal of R such that $I \supsetneqq P_i$, then $I_G \neq 0$ by [3, Corollary 6. 6]. Thus $I_G \supset H$ and so $I/P_i \supset H + P_i/P_i$. Hence $H + P_i/P_i$ is the nonzero heart of R/P_i. Consequently, $R/P_i \in C$.

Therefore $P_i \supset S(R)$, for all i. We see that $S(R) \subset \bigcap_{i=1}^{k} P_i = 0$.

Consequently we obtain that $S_G(R) \supset S(R)$. Hence the result follows by Corollary 2(1).

Corollary 3 Let A be a ring and $|G|^{-1} \in A$. Then $S(A) * G = S(A * G)$.

Proof Since the skew group ring $A * G$ is a graded ring of type G by defining $(A * G)_g = Ag$, we have that $S(A) * G = S_G(A * G)$ by Corollary 2(3). Therefore the result follows by Theorem 5.

References

1 Cohen M, Montgomery S. Group graded rings, smash products and group actions. Trans Amer Math Soc, 1984, 282: 237～258

2 Lorenz M, Passman D S. Prime ideals in crossed products of finite groups. Israel J Math, 1978, 33: 105～114

3 Nastasescu C. Group rings of graded rings: Applications. J Pure Appl Alg, 1984, 33: 313～335

4 Nicholson W K, Watters J F. Normal radicals and normal classes of rings. J Alg, 1979, 59: 5～15

5 Stewart P N. Properties of normalizing extensions and fixed rings. Comm Alg, 1984, 12: 1067～1098

6 Nastasescu C, Oystaeyen F V. Graded ring theory. North-Holland Amsterdam, 1982

7 Szasz F A. Radicals of rings. Chichester New York Brisbane Toronto, 1981: 74

8 谢邦杰. 抽象代数学. 上海: 上海科学技术出版社, 1982: 469

分次环的分次素秩和分次反单根

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提 要 研究对于具有某种性质的G-分次环R(G是有限群),当不考虑分次时,是否具有类似的性质. 为此,首先证明了不相容性,即若$P \subsetneqq Q$是$R\#G^*$的两个理想且P是素的,则$P \cap R \subsetneqq Q \cap P$. 作为它的应用,证得分次环的分次素秩与素秩是相等的,其次,得到当$|G|^{-1} \in R$时,R的分次反单根与反单根是一致的.

关键词 (分次)本质模; (分次)素秩; (分次)反单根

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