Jacobi Approximation and Its Applications *

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Abstract: Jacobi approximation in certain Hilbert spaces is investigated. It is applied to spectral methods for singular problems, unbounded domains, axisymmetric domains and exterior problems. Numerical results show the efficiency of this approach.

Key words: Jacobi approximation; spectral method; applications

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1 Introduction

The spectral method has high accuracy, and so often provides good numerical solutions of differential equations. But this merit might be destroyed by some facts, such as instability of nonlinear computations, discontinuities of data, unboundness of domains and singularities of solutions. Further the singularities of solutions could be caused by several factors, for instance, degeneration of coefficients, unboundness of data and corners of domains. Some techniques have been proposed to overcome these difficulties. KREISS and OLIGER[1], GOTTLIEB and TURKEL[2], KUO[3], VANDEVEN[4], TADMOR[5] and GUO[6] proposed various filterings to weaken the instability in nonlinear computations. CAI, GOTTLIEB and SHU[7,8] provided certain essentially nonoscillatory approximations and one-side filters for fitting discontinuous data. In particular, GOTTLIEB, SHU, SOLOMONOFF and VANDEVEN[9], and GOTTLIEB and SHU[10~13] recovered the spectral accuracy by using Gegenbauer approxi-

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mation. FUNARO and KAVIAN[14], MADAY, PERNAUD-THOMAS and VANDEVEN[15], GUO[16], GUO and SHEN[17], GUO and XU[18] and XU and GUO[19] developed some spectral methods in unbounded domains. But so far, there is no work concerning spectral method for singular problems. In fact, the Jacobi approximation can be applied to such problems, see GUO[20~23], GUO and WANG[24], and WANG and GUO[25]. The key points are fitting singular solutions by Jacobi polynomials, comparing numerical solutions with some unusual orthogonal projections of exact solutions, and measuring the errors in certain Hilbert spaces in which the exact solutions are. This new method could be used in several fields, such as spectral methods for singular problems, unbounded domains, axisymmetric domains and exterior problems. This paper is for the Jacobi approximation and its applications. We first introduce some results on the Jacobi approximation in the next section. They play important roles in the analysis of the Jacobi spectral methods. Then we discuss its various applications in Sections 3~6. In the final section, we present some numerical results showing the efficiency of this new approach.

2 Main Approximation Results

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function in the usual sense. Let

$$L_\chi^2(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_\chi < \infty\}$$

where

$$\|v\|_\chi = \left(\int_\Lambda |v(x)|^2 \chi(x)\, dx\right)^{\frac{1}{2}}.$$

Let $(u, v)_\chi$ be the corresponding inner product. Further, let $\partial_x v(x) = \frac{\partial}{\partial x} v(x)$, and for any nonnegative integer m, define

$$H_\chi^m(\Lambda) = \{v \mid \partial_x^k v \in L_\chi^2(\Lambda),\ 0 \leqslant k \leqslant m\},$$

equipped with the semi-norm $|v|_{m,\chi}$ and the norm $\|v\|_{m,\chi}$ as usual. For any real $r > 0$, we define the space $H_\chi^r(\Lambda)$ by space interpolation as in Adams[15]. Let $\mathcal{D}(\Lambda)$ be the set of all infinitely differentiable functions with compact supports in Λ, and $H_{0,\chi}^r(\Lambda)$ be its closure in $H_\chi^r(\Lambda)$. If $\chi(x) \equiv 1$, then we donote $H_\chi^r(\Lambda)$, $H_{0,\chi}^r(\Lambda)$, $|v|_{r,\chi}$, $\|v\|_{r,\chi}$, $\|v\|_\chi$ and $(u,v)_\chi$ by $H^r(\Lambda)$, $H_0^r(\Lambda)$, $|v|_r$, $\|v\|_r$, $\|v\|$ and (u,v), respectively.

The Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$ are defined by

$$(1-x)^\alpha(1+x)^\beta J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l l!}\partial_x^l\left((1-x)^{l+\alpha}(1+x)^{l+\beta}\right),\ l = 0,1,\cdots.$$

Let $\alpha, \beta > -1$ and

$$\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta.$$

The set $\{J_l^{(\alpha,\beta)}(x)\}$ is the $L_{\chi^{(\alpha,\beta)}}^2(\Lambda)$-orthogonal system. For any $v \in L_{\chi^{(\alpha,\beta)}}^2(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x)$$

where $\hat{v}_l^{(\alpha,\beta)}$ are the Jacobi coefficients, $l \geqslant 0$.

Now let N be any positive integer, and $\mathscr{P}_N$ be the set of all algebraic polynomials of degree at most N. ${}_0\mathscr{P}_N = \{v \mid v \in \mathscr{P}_N,\ v(-1) = 0\}$ and $\mathscr{P}_N^0 = \{v \mid v \in \mathscr{P}_N,\ v(-1) = v(1) = 0\}$. Denote by c a generic positive constant independent of any function and N.

We consider various orthogonal projections. The $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$- orthogonal projection $P_{N,\alpha,\beta}$: $L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathscr{P}_N$ is a mapping such that for any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathscr{P}_N.$$

For technical reasons, we introduce another Hilbert space. For any interger $r \geqslant 0$, set

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},A} < \infty\}$$

where

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} = \begin{cases} \left(\sum_{k=0}^{m-1} \|(1-x^2)^{m-k}\partial_x^{2m-k}v\|^2_{\chi^{(\alpha,\beta)}} + \|v\|^2_{m,\chi^{(\alpha,\beta)}}\right)^{\frac{1}{2}}, & \text{for } r = 2m, \\ \left(\sum_{k=0}^{m} \|(1-x^2)^{m+\frac{1}{2}-k}\partial_x^{2m+1-k}v\|^2_{\chi^{(\alpha,\beta)}} + \|v\|^2_{m,\chi^{(\alpha,\beta)}}\right)^{\frac{1}{2}}, & \text{for } r = 2m+1. \end{cases}$$

For any real $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation. Next, for any non-negative integer μ,

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda) = \{v \mid \partial_x^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},A}(\Lambda)\},$$

$$H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda) = \{v \mid v \in H^r_{\chi^{(\alpha,\beta)},*,k}(\Lambda),\ 0 \leqslant k \leqslant \mu\}$$

with the following norms

$$\|v\|_{r,\chi^{(\alpha,\beta)},*,\mu} = \|\partial_x^\mu v\|_{r-\mu,\chi^{(\alpha,\beta)},A},$$

$$\|v\|_{r,\chi^{(\alpha,\beta)},**,\mu} = \left(\sum_{k=0}^{\mu} \|v\|^2_{r,\chi^{(\alpha,\beta)},*,k}\right)^{\frac{1}{2}}.$$

For any real $\mu > 0$, we define the spaces $H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda)$ and $H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda)$ by space interpolation.

Theorem 1　If $\alpha + r > 1$ or $\beta + r > 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda)$, $r \geqslant 1$ and $\mu \leqslant r$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\sigma(\mu,r)}\|v\|_{r,\chi^{(\alpha,\beta)},**,\mu}$$

where

$$\sigma(\mu,r) = \begin{cases} 2\mu - r, & \text{for } \mu \geqslant 0, \\ \mu - r, & \text{for } \mu < 0. \end{cases}$$

In particular, for any $\alpha = \beta > -1$, the above result is valid with

$$\sigma(\mu,r) = \begin{cases} 2\mu - r - \frac{1}{2}, & \text{for } \mu > 1, \\ \frac{3}{2}\mu - r, & \text{for } 0 \leqslant \mu \leqslant 1, \\ \mu - r, & \text{for } \mu < 0. \end{cases}$$

As is well known, we usually consider the $H_\chi^{r(\alpha,\beta)}(\Lambda)$- orthogonal projection in numerical analysis of differential equations. But in many practical problems, the coefficients of derivatives of different orders may degenerate in different ways. Also by certain suitable variable transformations, differential equations in unbounded domains might be changed to some singular problems in bounded domains. In these cases, it is not possible to compare the approximate solutions with the exact solutions in the Sobolev spaces. For instance, we consider the problem

$$-\partial_x((1-x^2)\partial_x U) + U = f, \qquad x \in \Lambda.$$

Then U is not in $H^1(\Lambda)$. So the numerical solution could not tend to U in $H^1(\Lambda)$. Indeed, it is not necessary. Howerer, it might be carried out in certain Hilbert spaces.

Now, let $\alpha,\beta,\gamma,\delta > -1$, and introduce the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \leqslant \mu \leqslant 1$. For $\mu = 0$, $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \text{ is measurable and } \| v \|_{1,\alpha,\beta,\gamma,\delta} < \infty\}$$

where

$$\| v \|_{1,\alpha,\beta,\gamma,\delta} = (| v |^2_{1,\chi^{(\alpha,\beta)}} + \| v \|^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}.$$

For $0 < \mu < 1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation.

Let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}} \qquad \forall u, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

It is an inner product of the space $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$. The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta}: H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathscr{P}_N$ is a mapping such that for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \forall \phi \in \mathscr{P}_N.$$

Theorem 2 If

$$\alpha \leqslant \gamma + 2, \qquad \beta \leqslant \delta + 2, \tag{1}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},*,1}(\Lambda)$ with $r \geqslant 1$,

$$\| P^1_{N,\alpha,\beta,\gamma,\delta}v - v \|_{1,\alpha,\beta,\gamma,\delta} \leqslant c N^{1-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

If, in addition,

$$\alpha \leqslant \gamma + 1, \qquad \beta \leqslant \delta + 1, \tag{2}$$

then for all $0 \leqslant \mu \leqslant 1$,

$$\| P^1_{N,\alpha,\beta,\gamma,\delta}v - v \|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant c N^{\mu-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

In some practical problems arising in fluid dynamics, biology and other fields, the unknown functions vanish at one of the extreme points, say $x = -1$. So we need other orthogonal projections. Let

$${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0\}.$$

The orthogonal projection ${}_0P^1_N : {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathscr{P}_N$ is a mapping such that for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}({}_0P^1_N v - v, \phi) = 0 \qquad \forall \phi \in {}_0\mathscr{P}_N.$$

Theorem 3 If $\alpha \leqslant \gamma + 2$, $\beta \leqslant 0$ and $\delta \geqslant 0$, then for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap$

$H^r_{\chi^{(\alpha,\beta)},*,1}(\Lambda)$ with $r \geqslant 1$,

$$\| {}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v \|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

If, in addition, (2) holds, then for all $0 \leqslant \mu \leqslant 1$,

$$\| {}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v \|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

When we study the movements of fluid flows in bounded domains with fixed non-slip walls, the populations of budworms in bounded forests with lethal boundary conditions, and some problems on other topics, we meet homogenous boundary conditions. In those cases, we have to consider another projection. Let

$$H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) = \{ v \mid v \in H_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = v(1) = 0 \}.$$

The orthogonal projection $P^{1,0}_{N,\alpha,\beta,\gamma,\delta}$: $H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \phi \in \mathcal{P}^0_N.$$

Theorem 4　If $\gamma \leqslant \alpha \leqslant \gamma + 1$, $\delta \leqslant \beta \leqslant \delta + 1$ and $\gamma, \delta < 1$, then for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*,2}(\Lambda)$ with $r \geqslant 2$,

$$\| P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v \|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,2}.$$

If, in addition, $\alpha = \gamma$, $\beta = \delta$ and $\alpha, \beta > 0$, then for all $0 \leqslant \mu \leqslant 1$,

$$\| P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v \|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r} \| v \|_{r,\chi^{(\alpha,\beta)},*,2}.$$

The theorems in this section play important roles in numerical analysis of Jacobi spectral method. The proofs of these theorems can be found in Guo[22, 23].

Remark 1　Guo and Wang[24], and Wang and Guo[25] also considered Jacobi interpolation. Their results laid the mathematical foundation of Jacobi pseudospectral method, which is easier to be performed in actual calculation.

Remark 2　All results can be generalized to multiple-dimensional spaces, see Guo and Wang[26].

3　Singular Problems

As an example, we consider the following problem

$$-\partial_x(k(x)\partial_x U(x)) + b(x)U(x) = f(x), \qquad x \in \Lambda, \tag{3}$$

where $k(x) \geqslant 0$, $b(x) \geqslant 0$ and $f(x)$ are given functions. Assume that $k(x)$ and $b(x)$ only degenerate as $|x| \to 1$. Suppose that

$$k(x) \sim \chi^{(\alpha,\beta)}(x), \qquad b(x) \sim \chi^{(\gamma,\delta)}(x),$$

and for certain positive constants c_1 and c_2,

$$\chi^{(\alpha,\beta)}(x) \leqslant k(x) \leqslant c_1\chi^{(\alpha,\beta)}(x),$$
$$\chi^{(\gamma,\delta)}(x) \leqslant b(x) \leqslant c_2\chi^{(\gamma,\delta)}(x).$$

We look for the solution of (3) such that at least $k(x)\partial_x U(x) \to 0$ as $|x| \to 1$. A weak formulation of this problem is to find $U \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ such that

$$(\partial_x U, \partial_x v)_k + (bU, v) = (f, v), \qquad \forall\ v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{4}$$

If $f \in (H^1_{\alpha,\beta,\gamma,\delta}(\Lambda))'$, then (4) has a unique solution.

Let $u_N \in \mathscr{P}_N$ be the approximation to U, satisfying

$$a_{\alpha,\beta,\gamma,\delta}(u_N, \phi) + ((k - \chi^{(\alpha,\beta)})\partial_x u_N, \partial_x \phi) + ((b - \chi^{(\gamma,\delta)})u_N, \phi) = (f, \phi), \ \forall \phi \in \mathscr{P}_N. \tag{5}$$

Theorem 5 Let (1) hold. If $U \in H_{\chi^{(\alpha,\beta)},*,1}(\Lambda)$ with $r \geqslant 1$, then

$$\| P^1_{N,\alpha,\beta,\gamma,\delta}U - u_N \|_{1,\alpha,\beta,\gamma,\delta} \leqslant c N^{1-r} \| U \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

If, in addition, (2) holds and $k(x) \equiv \chi^{(\alpha,\beta)}(x)$, then for all $0 \leqslant \mu \leqslant 1$,

$$\| U - u_N \|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant c N^{\mu-r} \| U \|_{r,\chi^{(\alpha,\beta)},*,1}.$$

Remark 3 If $k(x)$ degenerates at several distinct points, then we divide the interval into several subintervals. Their extreme points coincide with these distinct points. Further we use different Jacobi approximations in different subintervals.

Remark 4 We can use Jacobi approximation or Jacobi interpolation to solve boundary value problems with singular source terms, and initial-boundary value problems with singular source terms and singular initial states, see Guo and Wang[26]. Also, we can use these methods for inhomogeneous boundary value problems defined on domains with corners.

4 Unbounded Domains

As an example, we consider the logistic equation governing the population of budworms in an unbounded forest, say $\tilde{\Lambda} = \{y \mid 0 < y < \infty\}$. Suppose that the boundary condition at $y = 0$ is lethal, and the population $V(y, t)$ grows infinitely as $y \to \infty$, but at least $e^{-2y}\partial_y V(y, t) \to 0$. This problem is of the form

$$\begin{cases} \partial_t V(y,t) - \partial_y^2 V(y,t) = V(y,t)(1 - V(y,t)), & y \in \tilde{\Lambda}, 0 < t \leqslant T, \\ V(0,t) = \lim\limits_{y\to\infty} e^{-2y}\partial_y V(y,t) = 0 & 0 \leqslant t \leqslant T, \\ V(y,0) = V_0(y), & y \in \tilde{\Lambda}. \end{cases} \tag{6}$$

Now we make the variable transformation (see Guo[21]),

$$y(x) = -2\ln(1 - x) + 2\ln 2.$$

Let $U(x, t) = V(y(x), t)$ and $U_0(x) = V_0(y(x))$. Then (6) becomes

$$\begin{cases} \partial_t U(x,t) - \frac{1}{4}(1-x)\partial_x((1-x)\partial_x U(x,t)) = U(x,t)(1-U(x,t)), x \in \Lambda, 0 < t \leqslant T, \\ U(-1,t) = \lim\limits_{x\to 1}(1-x)^2\partial_x U(x,t) = 0, & 0 \leqslant t \leqslant T, \\ U(x,0) = U_0(x). \end{cases} \tag{7}$$

A weak formulation of (6) is to find $U \in L^\infty(0, T; L^2(\Lambda)) \cap L^2(0, T; {}_0H^1_{\chi^{(2,0)}}(\Lambda))$ such that

$$\begin{cases}(\partial_t U(x,t),v)+\frac{1}{4}a_{2,0,0,0}(U(t),v)+\frac{1}{4}(U(t),\partial_x v)_{\chi^{(1,0)}}=\\ (\frac{3}{2}U(t)-U^2(t)), \qquad \forall v\in {}_0H^1_{2,0,0,0}(\Lambda),\ 0<t\leqslant T,\\ U(0)=U_0.\end{cases} \tag{8}$$

If $U_0\in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$, then (8) has a unique solution.

Let $u_N(t)$ be the approximation to $U(t)$. The Jacobi spectral scheme for (8) is to find $u_N\in {}_0\mathscr{P}_N$ such that

$$(\partial_t u_N(t),\phi)+\frac{1}{4}a_{2,0,0,0}(u_N(t),\phi)+\frac{1}{4}(u_N(t),\partial_x\phi)_{\chi^{(1,0)}}=$$
$$(\frac{3}{2}u_N(t)-u_N^2(t),\phi),\quad \forall \phi\in {}_0\mathscr{P}_N,\ 0<t\leqslant T. \tag{9}$$

In addition, $u_N(0)=u_{N,0}={}_0P^1_{N,2,0,0,0}U_0$.

Let

$$E(v,t)=\|v(t)\|^2+\int_0^t\|v(s)\|^2_{1,2,0,0,0}\mathrm{d}s.$$

Theorem 6 If for $r\geqslant 0$ and $d>1$,

$$U\in L^2(0,T;H^{r+1}_{\chi^{(2,0)},*,1}(\Lambda))\cap L^\infty(0,T;H^d(\Lambda)\cap {}_0H^1_{\chi^{(2,0)},*,1}(\Lambda)\cap H^{1+d}_{\chi^{(2,0)},*,1}(\Lambda)),$$

then for all $0\leqslant t\leqslant T$,

$$E(U-u_N,t)\leqslant bN^{-2r}$$

where b is a positive constant depending only on the norms of U in the mentioned spaces.

Remark 5 By the variable transformation (see Guo[20])

$$y=\ln\frac{1+x}{1-x},$$

the problems on the whole line become certain singular problems on a finite intervals. Then we can also use the Jacobi spectral method for their numerical slutions.

Remark 6 In multiple-dimensional space, we can use different variable transformations and choose some suitable Jacobi approximations with different parameters α and β in different directions. So we can solve problems in different unbounded domains, such as the whole space, the half space, some infinte straps and so on, see Wang and Guo[27].

5 Mixed Approximation

For instance, we consider the Laplace equation on an axisymmetric domain Ω, i.e.,

$$\begin{cases}-\Delta U=f, & \text{in } \Omega,\\ U=0, & \text{on } \partial\Omega.\end{cases} \tag{10}$$

By using the cylindrical coordinates (r,θ,z), we get that

$$-\partial_r(r\,\partial_r U)-\frac{1}{r}\partial_\theta{}^2 U-r\partial_z{}^2U=rf,\qquad \text{in } \Omega. \tag{11}$$

We can resolve (11) numerically, by using the Jacobi approximation in the r-direction, the Fourier approximation in the θ- direction and the Jacobi approximation with $\alpha=\beta=0$ (the Legendre approximation) in the z-direction.

Azaiez, Bernardi, Dauge and Maday[28] first used such method by using some combinations of Legendre polynomials. Indeed, it is equivalent to the Jacobi approximation with $\alpha=1$ and $\beta=0$, as studied in Guo[21]. We also refer to Wang and Guo[27].

6 Exterior Problems

We can use the Jacobi approximation to solve some exterior problems. For instance, we consider the twodimensional problem

$$\begin{cases} -\Delta U = f, & r>1, \\ U=0, & r=1. \end{cases} \tag{12}$$

Here the obstacle is a unit disc. We resolve (12) by a variable transformation and rational approximation induced by the Jacobi approximation. If the obstacle is not a disc, we can resolve it by using the above method combined with domain-decomposition and certain specific finite element method with high accuracy.

7 Numerical Results

We now present some numerical results. We consider problem (8) and take the following test function

$$V(y,t)=\frac{c(e^{by}-1)}{ce^{by}+e^{dt}},$$

where $b=0.5$, $c=1.0$ and $d=0.01$. Clearly, $V(y,t)\to 1$ as $y\to\infty$. In this case, the corresponding test function solution

$$U(x,t)=\frac{4^b c-c(1-x)^{2b}}{4^b c-e^{dt}(1-x)^{2b}}.$$

We use (9) to solve (8) with the related non-zero source term. In actual computation, we advance in time by using the Runge-Kutta method of fourth order with mesh size τ. Let $u_N(x,t)$ be the numerical solution. For description of numerical errors, let $\zeta_{N,j}^{(0,0)}$ be the Legendre interpolation nodes, and $\omega_{N,j}^{(0,0)}$ be the corresponding Legendre weights. The errors $E_1(u_N,t)$ and $E_2(u_N,t)$ are defined by

$$E_1(u_N,t)=\Big(\sum_{j=0}^{N}(U(\zeta_{N,j}^{(0,0)},t)-u_N(\zeta_{N,j}^{(0,0)},t))^2\omega_{N,j}^{(0,0)}\Big)^{\frac{1}{2}}$$

and

$$E_2(u_N,t)=\left(\sum_{j=0}^{N}\frac{(U(\zeta_{N,j}^{(0,0)},t)-u_N(\zeta_{N,j}^{(0,0)},t))^2}{|U(\zeta_{N,j}^{(0,0)},t)|^2}\omega_{N,j}^{(0,0)}\right)^{\frac{1}{2}}$$

The numerical errors of scheme (9) at $t = 0.5$ are presented in Table 1, which show the convergence of scheme (9).

Table 1 The errors $E_1(u_N, 0.5)$ and $E_2(u_N, 0.5)$

	$E_1(u_N, 0.5)$	$E_2(u_N, 0.5)$
$N = 8,\quad \tau = 0.01$	6.903E−07	2.157E−06
$N = 16,\quad \tau = 0.001$	7.324E−13	2.135E−12
$N = 24,\quad \tau = 0.0001$	1.351E−15	3.434E−15

In Table 2, we list the numerical errors of scheme (9) with $\tau = 0.0001$ and $N = 16$. It indicates that the stability of the calculation.

Table 2 The errors $E_1(u_N, t)$ and $E_2(u_N, t)$

	$E_1(u_N, t)$	$E_2(u_N, t)$
0.5	7.694E−13	2.148E−12
1.0	7.570E−13	2.266E−12
1.5	7.761E−13	2.395E−12
2.0	8.094E−13	2.532E−12

We next take the test function

$$U(x,t) = (1-x)^{\gamma}\sin\left(\frac{1}{2}(1+x)(1+t)\right), \qquad \gamma = -10^{-4}.$$

Clearly, $|U(x,t)| \to \infty$ as $x \to 1$, and $U \notin L^2(0,T;H^1(\Lambda))$. But $U \in L^2(0,T;H^1_{2,0,0,0}(\Lambda))$. We use scheme (9) to solve (8) with the corresponding source term. The numerical errors at $t = 0.5$ are presented in Table 3. The numerical errors of scheme (9) with $\tau = 0.0001$ and $N = 16$ are listed in Table 4. The numerical results show the convergence of scheme (9) and the stability of the calculation again. But the errors do not decay as fast as in the first example, since the exact solution in the second example has certain singularity as $x \to 1$. It coincides with the theoretical analysis.

Table 3 The errors $E_1(u_N, 0.5)$ and $E_2(u_N, 0.5)$

	$E_1(u_N, 0.5)$	$E_2(u_N, 0.5)$
$N = 8,\quad \tau = 0.01$	1.046E−6	2.374E−6
$N = 16,\quad \tau = 0.001$	4.477E−7	7.413E−7
$N = 24,\quad \tau = 0.0001$	3.245E−7	4.230E−7

Table 4 The errors $E_1(u_N, t)$ and $E_2(u_N, t)$.

t	$E_1(u_N, t)$	$E_2(u_N, t)$
0.5	4.477E-7	7.413E-7
1.0	6.273E-7	7.443E-7
1.5	1.131E-6	1.770E-6
2.0	3.285E-6	2.014E-6

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Jacobi 逼近及其应用

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摘要:研究一些 Hilbert 空间中的 Jacobi 逼近。它被应用于奇异问题,无界区域,轴对称区域和外部问题的谱方法。数值结果显示了这一方法的有效性。

关键词:Jacobi 逼近;谱方法;应用