Indefinite Linear Quadratic Optimal Control Problem for Singular Linear Discrete-time System: Krein Space Method

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Abstract The finite time horizon indefinite linear quadratic(LQ) optimal control problem for singular linear discrete time-varying systems is discussed. Indefinite LQ optimal control problem for singular systems can be transformed to that for standard state-space systems under a reasonable assumption. It is shown that the indefinite LQ optimal control problem is dual to that of projection for backward stochastic systems. Thus, the optimal LQ controller can be obtained by computing the gain matrices of Kalman filter. Necessary and sufficient conditions guaranteeing a unique solution for the indefinite LQ problem are given. An explicit solution for the problem is obtained in terms of the solution of Riccati difference equations.

Key words Singular linear discrete-time system, LQ optimal control, finite time horizon, Riccati difference equation, Kalman filter

1 Introduction

Singular systems have comprehensive practical background such as power systems[1,2], social economic systems[3], circuit systems[4], and so on. Great progress has been made in the theory and application since 1970s[5,6]. The linear quadratic optimal control problem is important in control theory and has been widely used in practice. The treatment of the LQ problem for linear systems has been well studied since the early 1960s. A detailed discussion of this topic can be found in [7] and [8]. For linear singular systems, there have also been a lot of excellent results about the LQ problem[9∼11].

A linear quadratic control problem is indefinite when the cost weighting matrices for the state and the control are allowed to be indefinite. To our best knowledge, reference [12] is the first article, which studied a nonstandard quadratic regulator on a finite time horizon, with indefinite weighting matrix for the control. Up to now, indefinite LQ theory has been extensively developed and has found interesting applications in finance, especially in dynamic macro economics theory[13]. Many applications for this problem can be seen in [14], [15], and the references therein.

A lot of literature about indefinite LQ control problem can be found. The finite horizon nonstandard LQ problem for an abstract dynamic system, which models a large class of hyperbolic-like partial differential equations, was studied in [16]. It provides necessary and sufficient conditions for finiteness of the value function corresponding to the control problem. Reference [17] considered the unique control problem that minimizes a general quadratic cost functional for a discrete-time system. Necessary and sufficient conditions were derived for the problem under conditions that a solution exists for an arbitrary finite planning horizon. The discrete-time infinite-horizon linear quadratic problem with indefinite cost criterion was considered in [18]. Necessary and sufficient conditions for the existences of optimal controls were given based on a geometric characterization of the set of all Hermitian solutions of the discrete-time algebraic Riccati equation. Reference [19] dealt with the free-endpoint regular linear quadratic problem with indefinite cost functional in the context of linear quadratic optimal control. Necessary and sufficient conditions were established for the existence of optimal controls.

Krein space is a kind of indefinite-metric space, which is different from Hilbert space in certain basic ways; in particular, it can contain nonzero vectors of zero length (such vectors are called neutral), as well as subspaces that contain nonzero vectors orthogonal to all vectors in the subspace (such subspaces are called degenerate, and such vectors are called isotropic)[20]. It has been proved that the Krein space linear estimation theory is efficient for H_∞ filtering[21], LQR optimal control[22], and Risk-sensitive filtering[23]. A complete solution to the LQR problem for systems with multiple input delays, which is dual to that of fixed-lag smoothing for a backward system without involving delays, was presented in [22]. Thus, the optimal controller can be obtained by computing the gain matrices of the standard fixed-lag smoothing.

In this article, the finite time horizon indefinite LQ optimal control problem for discrete time-varying linear singular systems is investigated. The rest of the article is organized as follows. An indefinite LQ optimal control problem for singular systems is given in Section 2. It is shown that the indefinite LQ problem for singular systems can be transformed to that for standard state-space systems under a reasonable assumption and the indefinite LQ problem is dual to that of projection for a backward stochastic system in Section 3. A necessary and sufficient condition guaranteeing that there is a unique solution for the indefinite LQ problem is given, and it is shown that the optimal LQ controller can be obtained by computing the gain matrices of Kalman filter in Section 4. An example is given to show the validity of the proposed method in Section 5, and some conclusions are drawn in Section 6.

2 Problem statement

Consider the optimal control problem of linear singular discrete-time systems

$$E\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{u}(k), \quad E\boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

with quadratic cost

$$J(\boldsymbol{u}(k), \boldsymbol{x}(k)) = \boldsymbol{x}^{\mathrm{T}}(N+1)E^{\mathrm{T}}S_0E\boldsymbol{x}(N+1) + \sum_{k=0}^{N}\left(\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{u}(k)\end{bmatrix}^{\mathrm{T}} Q(k)\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{u}(k)\end{bmatrix}\right) \tag{2}$$

where $\boldsymbol{x}(k) \in \mathbf{R}^n$ is the state, $\boldsymbol{u}(k) \in \mathbf{R}^r$ is the system input, $A(k) \in \mathbf{R}^{n\times n}, B(k) \in \mathbf{R}^{n\times r}$, $E \in \mathbf{R}^{n\times n}$ is a sin-

Received March 31, 2006; in revised form September 25, 2006
Supported by National Natural Science Foundation of P. R. China (50477042), Ph. D. Programs Foundation of Ministry of Education of P. R. China (20040422052), and the Natural Science Foundation of Shandong Province (Z2004G04)
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DOI: 10.1360/aas-007-0635

gular matrix with rank $E = p < n$. S_0 and $Q(k) \triangleq \begin{bmatrix} Q_{11}(k) & Q_{12}(k) \\ Q_{12}^{\mathrm{T}}(k) & Q_{22}(k) \end{bmatrix} \in \mathbf{R}^{(n+r)\times(n+r)}$ are symmetric matrices.

We denote the above quadratic cost optimal control problem about (1) and (2) as problem P, and define the set of admissible control-trajectory pairs of problem P as

$$\mathcal{J} = \{(\boldsymbol{u}(k), \boldsymbol{x}(k)) | (\boldsymbol{u}(k), \boldsymbol{x}(k)) \text{ satisfies } (1)\} \tag{3}$$

Solving the problem P is to find $(\boldsymbol{u}^*(k), \boldsymbol{x}^*(k)) \in \mathcal{J}$ such that

$$J(\boldsymbol{u}^*(k), \boldsymbol{x}^*(k)) = \min_{(\boldsymbol{u}(k), \boldsymbol{x}(k)) \in \mathcal{J}} J(\boldsymbol{u}(k), \boldsymbol{x}(k)) \tag{4}$$

Remark 1. Problem P, which can be named indefinite LQ problem, is still an open problem. It is a singular LQ problem if $Q_{22} \geq 0$ and a regular problem if $Q_{22} > 0$, which have been discussed in many articles.

3 Preliminaries

It is well known that there exists the following restricted system equivalence(r.s.e.) transformation for system (1):

$$\begin{aligned} &MEN = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \quad MA(k)N = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix} \\ &MB(k) = \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix}, \quad \boldsymbol{x}(k) = N \begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \widetilde{\boldsymbol{x}}(k) \end{bmatrix} \end{aligned} \tag{5}$$

where $M, N \in \mathbf{R}^{n\times n}$ are nonsingular matrices, $A_{11}(k) \in \mathbf{R}^{p\times p}$, $A_{12}(k) \in \mathbf{R}^{p\times(n-p)}$, $A_{21}(k) \in \mathbf{R}^{(n-p)\times p}$, $A_{22}(k) \in \mathbf{R}^{(n-p)\times(n-p)}$, $B_1(k) \in \mathbf{R}^{p\times r}$, $B_2(k) \in \mathbf{R}^{(n-p)\times r}$, $\bar{\boldsymbol{x}}(k) \in \mathbf{R}^p$, and $\widetilde{\boldsymbol{x}}(k) \in \mathbf{R}^{(n-p)}$. This transformation changes system (1) with the quadratic cost (2) into the system

$$\begin{cases} \bar{\boldsymbol{x}}(k+1) = A_{11}(k)\bar{\boldsymbol{x}}(k) + A_{12}(k)\widetilde{\boldsymbol{x}}(k) + B_1(k)\boldsymbol{u}(k) \\ \boldsymbol{0} = A_{21}(k)\bar{\boldsymbol{x}}(k) + A_{22}(k)\widetilde{\boldsymbol{x}}(k) + B_2(k)\boldsymbol{u}(k) \\ \bar{\boldsymbol{x}}(0) = [I_p \ \ 0]M\boldsymbol{x_0} \end{cases} \tag{6}$$

with cost

$$\begin{aligned} J_1(\boldsymbol{u}(k), \bar{\boldsymbol{x}}(k), \widetilde{\boldsymbol{x}}(k)) = \bar{\boldsymbol{x}}^{\mathrm{T}}(N+1)S_1\bar{\boldsymbol{x}}(N+1) + \\ \sum_{k=0}^{N}\left(\begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \widetilde{\boldsymbol{x}}(k) \\ \boldsymbol{u}(k) \end{bmatrix}^{\mathrm{T}} Q_1(k) \begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \widetilde{\boldsymbol{x}}(k) \\ \boldsymbol{u}(k) \end{bmatrix} \right) \end{aligned} \tag{7}$$

where

$$\begin{aligned} Q_1(k) &= \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} Q(k) \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \\ S_1 &\triangleq \begin{bmatrix} I_p & 0 \end{bmatrix} M^{-\mathrm{T}} S_0 M^{-1} \begin{bmatrix} I_p \\ 0 \end{bmatrix} \end{aligned}$$

We denote P_1 as the optimal control problem of system (6) with cost (7), and define the set of admissible control-trajectory pairs of problem P_1 as

$$\begin{aligned} \mathcal{J}_1 = \{(\boldsymbol{u}(k), \bar{\boldsymbol{x}}(k), \widetilde{\boldsymbol{x}}(k)) | (\boldsymbol{u}(k), \bar{\boldsymbol{x}}(k), \widetilde{\boldsymbol{x}}(k)) \\ \text{satisfies } (6)\} \end{aligned} \tag{8}$$

It is evident that problem P is equivalent to problem P_1 according to the definition of an equivalent relation of the two optimal control problems[10]. Thus, we can solve P_1 as substitute for solving P. Next, we will solve problem P_1 instead of problem P.

Assumption 1. $[E \ \ A(k) \ \ B(k)]$ is causality controllable[24], *i.e.*,

$$\operatorname{rank} \begin{bmatrix} 0 & E & 0 \\ E & A(k) & B(k) \end{bmatrix} = n + \operatorname{rank} E \ , \ k = 0, 1, \cdots, N$$

Lemma 1[24]. System (1) is causality controllable if and only if the matrix $[A_{22}(k) \ B_2(k)]$ is of full row rank, $k = 0, 1, \cdots, N$.

Under Assumption 1, the following equation

$$\begin{bmatrix} A_{22}(k) & B_2(k) \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{x}}(k) \\ \boldsymbol{u}(k) \end{bmatrix} = -A_{21}(k)\bar{\boldsymbol{x}}(k)$$

has solutions, and its general solution can be expressed as

$$\begin{bmatrix} \widetilde{\boldsymbol{x}}(k) \\ \boldsymbol{u}(k) \end{bmatrix} = -\bar{A}^+(k)A_{21}(k)\bar{\boldsymbol{x}}(k) + \widetilde{P}(k)\boldsymbol{v}(k) \tag{9}$$

where

$$\bar{A}^+(k) \triangleq \begin{bmatrix} A_{22}^{\mathrm{T}}(k) \\ B_2^{\mathrm{T}}(k) \end{bmatrix} \left([A_{22}(k) \ \ B_2(k)] \begin{bmatrix} A_{22}^{\mathrm{T}}(k) \\ B_2^{\mathrm{T}}(k) \end{bmatrix}\right)^{-1} \tag{10}$$

is the Moore Penrose inverse of $[A_{22}(k) \ \ B_2(k)]$. $\widetilde{P}(k) \in R^{(n-p+r)\times r}$ is of full column rank, and satisfies

$$[A_{22}(k) \ \ B_2(k)]\widetilde{P}(k) = 0 \tag{11}$$

According to expression on (9), there exists the following transformation

$$\begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \widetilde{\boldsymbol{x}}(k) \\ \boldsymbol{u}(k) \end{bmatrix} = T(k) \begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \boldsymbol{v}(k) \end{bmatrix} \tag{12}$$

where $T(k) = \begin{bmatrix} I_p & 0 \\ -\bar{A}^+(k)A_{21}(k) & \widetilde{P}(k) \end{bmatrix}$. Substituting (9) into the first equation of (6), we obtain

$$\bar{\boldsymbol{x}}(k+1) = \widetilde{A}_{11}(k)\bar{\boldsymbol{x}}(k) + \widetilde{B}_1(k)\boldsymbol{v}(k), \ \ \bar{\boldsymbol{x}}(0) = [I_p \ \ 0]M\boldsymbol{x_0} \tag{13}$$

where

$$\begin{aligned} \widetilde{A}_{11}(k) &= A_{11}(k) - [A_{12}(k) \ \ B_1(k)]\bar{A}^+(k)A_{21}(k) \\ \widetilde{B}_1(k) &= [A_{12}(k) \ \ B_1(k)]\widetilde{P}(k) \end{aligned} \tag{14}$$

Under transformation (12), cost (7) is changed identically into

$$\begin{aligned} J_2(\boldsymbol{v}(k), \bar{\boldsymbol{x}}(k)) = \bar{\boldsymbol{x}}^{\mathrm{T}}(N+1)S_1\bar{\boldsymbol{x}}(N+1) + \\ \sum_{k=0}^{N}\left(\begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \boldsymbol{v}(k) \end{bmatrix}^{\mathrm{T}} \bar{Q}(k) \begin{bmatrix} \bar{\boldsymbol{x}}(k) \\ \boldsymbol{v}(k) \end{bmatrix} \right) \end{aligned} \tag{15}$$

where

$$\bar{Q}(k) \triangleq T(k)^{\mathrm{T}}Q_1(k)T(k) \tag{16}$$

Denote P_2 as the optimal control problem of system (13) with cost (15), and define the set of admissible control-trajectory pairs of problem P_2 as

$$\mathcal{J}_2 = \{(\boldsymbol{v}(k), \bar{\boldsymbol{x}}(k)) | (\boldsymbol{v}(k), \bar{\boldsymbol{x}}(k)) \text{ satisfies } (13)\} \tag{17}$$

Investigating the relation between P_1 and P_2, we have

Lemma 2. If matrix $[A_{22}(k) \ \ B_2(k)], k = 0, 1, \cdots, N$ is of full row rank, then transformation (12) is a bijection of J_2 onto J_1 and the problem P_1 is equivalent to problem P_2.

Proof. See [25].

It is clear that P_2 is an optimal control problem of standard state-space discrete-time linear system. Lemma 2 shows that if the discrete-time singular system satisfies Assumption 1, then its optimal control problem with the indefinite linear-quadratic cost can be transformed into an equivalent indefinite linear-quadratic optimal control problem for standard state-space discrete-time linear system. We will show that the indefinite LQ problem is dual to projection problem for a stochastic backward system. Thus, the optimal LQ controller can be obtained by computing the gain matrix of the stochastic backward system.

Let us begin by defining $\boldsymbol{v} \triangleq col\{\boldsymbol{v}(0),\cdots,\boldsymbol{v}(N)\}$, $\bar{\boldsymbol{x}} \triangleq col\{\bar{\boldsymbol{x}}(0),\cdots,\bar{\boldsymbol{x}}(N)\}$ and $\Phi_{i,j} \triangleq \widetilde{A}_{11}(i)\widetilde{A}_{11}(i-1)\cdots\widetilde{A}_{11}(j), i\geq j$, so that we may use the state equation (13) to write

$$\bar{\boldsymbol{x}}(N+1)=\Phi_{N,0}\bar{\boldsymbol{x}}(0)+\mathcal{C}\boldsymbol{v},\quad \bar{\boldsymbol{x}}=\mathcal{O}\bar{\boldsymbol{x}}(0)+\Gamma\boldsymbol{v} \tag{18}$$

where

$$\begin{aligned}&\mathcal{C}\triangleq(\Phi_{N,1}\widetilde{B}_1(0),\cdots,\Phi_{N,N}\widetilde{B}_1(N-1),\widetilde{B}_1(N))\\&\mathcal{O}\triangleq\begin{bmatrix} I & \Phi_{0,0}^{\mathrm{T}} & \cdots & \Phi_{N-1,0}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\&\Gamma\triangleq\begin{bmatrix} 0 & & & \\ \widetilde{B}_1(0) & 0 & & \\ \Phi_{1,1}\widetilde{B}_1(0) & \widetilde{B}_1(1) & 0 & \\ \cdots & \cdots & \cdots & \\ \Phi_{N-1,1}\widetilde{B}_1(0) & \Phi_{N-1,2}\widetilde{B}_1(1) & \cdots & 0\end{bmatrix}\end{aligned} \tag{19}$$

We denote

$$\bar{I}\triangleq\left(\begin{array}{cccc|cccc} I & & & & 0 & & & \\ 0 & & & & I & & & \\ & I & & & & 0 & & \\ & 0 & & & & I & & \\ & & \ddots & & & & \ddots & \\ & & & I & & & & 0\\ & & & 0 & & & & I\end{array}\right)\in\mathbf{R}^{2N\times 2N} \tag{20}$$

where the $(2i-1)$th row vector of $\bar{I}$ is $(\overbrace{0,\cdots,0}^{(i-1)},I,\overbrace{0,\cdots,0}^{(2N-i)})$ and the $(2i)$th row vector of $\bar{I}$ is $(\underbrace{0,\cdots,0}_{(N+i-1)},I,\underbrace{0,\cdots,0}_{(N-i)})$, $i=1,\cdots,N$, and

$$\begin{aligned}\bar{Q}&\triangleq\bar{I}^{\mathrm{T}}\mathrm{diag}\{\bar{Q}(0),\bar{Q}(1),\cdots,\bar{Q}(N)\}\bar{I}\\&\triangleq\begin{pmatrix}\bar{Q}_{11} & \bar{Q}_{12}\\ \bar{Q}_{12}^{\mathrm{T}} & \bar{Q}_{22}\end{pmatrix}\end{aligned} \tag{21}$$

where $\bar{Q}_{ij}=\mathrm{diag}\{\bar{Q}_{ij}(0),\bar{Q}_{ij}(1),\cdots,\bar{Q}_{ij}(N)\}$. Then, J_2 can be written as follows.

$$\begin{aligned}J_2&=(\Phi_{N,0}\bar{\boldsymbol{x}}(0)+\mathcal{C}\boldsymbol{v})^{\mathrm{T}}S_1(\Phi_{N,0}\bar{\boldsymbol{x}}(0)+\mathcal{C}\boldsymbol{v})+\\&\quad\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathcal{O} & \Gamma\\ 0 & I\end{bmatrix}^{\mathrm{T}}\bar{Q}\begin{bmatrix}\mathcal{O} & \Gamma\\ 0 & I\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix}\\&=\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Phi_{N,0}^{\mathrm{T}}S_1\Phi_{N,0}+\mathcal{O}^{\mathrm{T}}\bar{Q}_{11}\mathcal{O} & \Phi_{N,0}^{\mathrm{T}}S_1\mathcal{C}+\mathcal{O}^{\mathrm{T}}\bar{Q}_{11}\Gamma+\mathcal{O}^{\mathrm{T}}\bar{Q}_{12}\\ * & \mathcal{C}^{\mathrm{T}}S_1\mathcal{C}+\Gamma^{\mathrm{T}}\bar{Q}_{11}\Gamma+\Gamma^{\mathrm{T}}\bar{Q}_{12}+\bar{Q}_{12}^{\mathrm{T}}\Gamma+\bar{Q}_{22}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix}\end{aligned} \tag{22}$$

Introduce the following backward stochastic system associated with (13) as follows.

$$\begin{cases}\boldsymbol{\xi}(k)=\widetilde{A}_{11}^{\mathrm{T}}(k)\boldsymbol{\xi}(k+1)+\boldsymbol{\omega}(k)\\ \boldsymbol{\zeta}(k)=\widetilde{B}_1^{\mathrm{T}}(k)\boldsymbol{\xi}(k+1)+\boldsymbol{\theta}(k)\end{cases} \tag{23}$$

where $\boldsymbol{\xi}(N+1)$, $\boldsymbol{\omega}(k)$ and $\boldsymbol{\theta}(k)$ are Krein space variables and white noises with zero means and covariances

$$\begin{aligned}&\left\langle\begin{pmatrix}\boldsymbol{\xi}(N+1)\\ \boldsymbol{\omega}(k)\\ \boldsymbol{\theta}(k)\end{pmatrix},\begin{pmatrix}\boldsymbol{\xi}(N+1)\\ \boldsymbol{\omega}(j)\\ \boldsymbol{\theta}(j)\end{pmatrix}\right\rangle\\&=\begin{pmatrix}S_1 & 0 & 0\\ 0 & \bar{Q}_{11}(k)\delta_{kj} & \bar{Q}_{12}(k)\delta_{kj}\\ 0 & \bar{Q}_{12}^{\mathrm{T}}(k)\delta_{kj} & \bar{Q}_{22}(k)\delta_{kj}\end{pmatrix}\\&\quad k,j=0,1\cdots,N\end{aligned} \tag{24}$$

where $\delta_{kj}=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}$ is the Kronecker Delta function.

Denoting $\boldsymbol{\omega}\triangleq\begin{pmatrix}\boldsymbol{\omega}(0)\\ \vdots\\ \boldsymbol{\omega}(N)\end{pmatrix}$, $\boldsymbol{\theta}\triangleq\begin{pmatrix}\boldsymbol{\theta}(0)\\ \vdots\\ \boldsymbol{\theta}(N)\end{pmatrix}$, $\boldsymbol{\zeta}\triangleq\begin{pmatrix}\boldsymbol{\zeta}(0)\\ \vdots\\ \boldsymbol{\zeta}(N)\end{pmatrix}$, we can obtain the following lemma with the above Krein space model.

Lemma 3. Under the Krein space model (23), J_2 has the following quadratic form

$$J_2=\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R_{\boldsymbol{\xi}_0} & R_{\boldsymbol{\xi}_0\boldsymbol{\zeta}}\\ R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0} & R_{\boldsymbol{\zeta}}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{v}\end{bmatrix} \tag{25}$$

where

$$\begin{bmatrix}R_{\boldsymbol{\xi}_0} & R_{\boldsymbol{\xi}_0\boldsymbol{\zeta}}\\ R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0} & R_{\boldsymbol{\zeta}}\end{bmatrix}\triangleq\left\langle\begin{bmatrix}\boldsymbol{\xi}(0)\\ \boldsymbol{\zeta}\end{bmatrix},\begin{bmatrix}\boldsymbol{\xi}(0)\\ \boldsymbol{\zeta}\end{bmatrix}\right\rangle \tag{26}$$

Proof. According to (23) and (19), $\boldsymbol{\xi}(0)$ and $\boldsymbol{\zeta}$ can be written as follows:

$$\boldsymbol{\xi}(0)=\Phi_{N,0}^{\mathrm{T}}\boldsymbol{\xi}(N+1)+\mathcal{O}^{\mathrm{T}}\boldsymbol{\omega},\quad \boldsymbol{\zeta}=\mathcal{C}^{\mathrm{T}}\boldsymbol{\xi}(N+1)+\Gamma^{\mathrm{T}}\boldsymbol{\omega}+\boldsymbol{\theta}$$

Therefore, we can easily obtain (25) by using (24). □

In view of Lemma 3, if $R_{\boldsymbol{\zeta}}$ is invertible, the minimizing solution of J_2 with respect to control inputs $v(k)$ is now readily given by[20]

$$\hat{\boldsymbol{v}}=-R_{\boldsymbol{\zeta}}^{-1}R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0}\bar{\boldsymbol{x}}(0)$$

and thus

$$J_2=\bar{\boldsymbol{x}}^{\mathrm{T}}(0)P_0\bar{\boldsymbol{x}}(0)+(\boldsymbol{v}-\hat{\boldsymbol{v}})^{\mathrm{T}}R_{\boldsymbol{\zeta}}(\boldsymbol{v}-\hat{\boldsymbol{v}})$$

where $P_0=<\bar{\boldsymbol{x}}(0)-\hat{\bar{\boldsymbol{x}}}(0),\ \bar{\boldsymbol{x}}(0)-\hat{\bar{\boldsymbol{x}}}(0)>$, and $\hat{\bar{\boldsymbol{x}}}(0)$ is the projection of $\bar{\boldsymbol{x}}(0)$ onto the linear space $\mathcal{L}\{\boldsymbol{\zeta}(0),\cdots,\boldsymbol{\zeta}(N)\}$.

4 Main result

In this section, we shall give an explicit solution to the LQ problem for system (1). In view of the results of the previous section, the gain matrix $-R_{\boldsymbol{\zeta}}^{-1}R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0}$ of the controller is the negative transpose of the gain matrix in estimating the random variable $\boldsymbol{\xi}(0)$ from $\boldsymbol{\zeta}$. Therefore, to give a solution to the optimal LQ problem, we shall calculate the gain matrix $R_{\boldsymbol{\xi}_0\boldsymbol{\zeta}}R_{\boldsymbol{\zeta}}^{-1}$.

By applying the Kalman filtering formulation for the backward system (23), the filtering estimate is calculated directly as[20]

$$\hat{\boldsymbol{\xi}}(k)=\widetilde{A}_{11}^{\mathrm{T}}(k)\hat{\boldsymbol{\xi}}(k+1)+K_p(k)\boldsymbol{\varepsilon}(k),\quad \hat{\boldsymbol{\xi}}(N+1)=\mathbf{0} \tag{27}$$

where

$$\boldsymbol{\varepsilon}(k)=\boldsymbol{\zeta}(k)-\widetilde{B}_1^{\mathrm{T}}(k)\hat{\boldsymbol{\xi}}(k+1)$$

is the innovations,

$$K_p(k) = [\widetilde{A}_{11}^{\mathrm{T}}(k)P(k+1)\widetilde{B}_1(k) + \bar{Q}_{12}(k)]Q_\varepsilon^{-1}(k) \tag{28}$$

and

$$Q_\varepsilon(k) = \widetilde{B}_1^{\mathrm{T}}(k)P(k+1)\widetilde{B}_1(k) + \bar{Q}_{22}(k)$$

are the Kalman gain and innovations variance, respectively, and $P(k)$ satisfies the backward-time Riccati recursion:

$$\begin{aligned} P(k) = \widetilde{A}_{11}^{\mathrm{T}}(k)P(k+1)\widetilde{A}_{11}(k) + \bar{Q}_{11}(k) - \\ K_p(k)Q_\varepsilon(k)K_p^{\mathrm{T}}(k) \end{aligned} \tag{29}$$

with $P(N+1) = S_1$.

Now (27) may be rewritten as

$$\hat{\boldsymbol{\xi}}(k) = F_p(k)\hat{\boldsymbol{\xi}}(k+1) + K_p(k)\boldsymbol{\zeta}(k) \tag{30}$$

where $F_p(k) = \widetilde{A}_{11}^{\mathrm{T}}(k) - K_p(k)\widetilde{B}_1^{\mathrm{T}}(k)$.

The recursion (30) can be solved to yield

$$\hat{\boldsymbol{\xi}}(0) = [K_p(0),\ F_p(0)K_p(1), \cdots, F_p(0, N-1)K_p(N)] \times \begin{pmatrix} \boldsymbol{\zeta}(0) \\ \vdots \\ \boldsymbol{\zeta}(N) \end{pmatrix} \tag{31}$$

where $F_p(i,j) = F_p(i)F_p(i+1)\cdots F_p(j), i < j$.

Thus, the solution to the LQ problem is given by the negative of the transpose of the above solution.

$$\begin{pmatrix} \hat{\boldsymbol{v}}(0) \\ \hat{\boldsymbol{v}}(1) \\ \vdots \\ \hat{\boldsymbol{v}}(N) \end{pmatrix} = - \begin{pmatrix} K_p^{\mathrm{T}}(0) \\ K_p^{\mathrm{T}}(1)F_p^{\mathrm{T}}(0) \\ \vdots \\ K_p^{\mathrm{T}}(N)F_p^{\mathrm{T}}(0, N-1) \end{pmatrix} \bar{\boldsymbol{x}}(0)$$

But this implies

$$\hat{\boldsymbol{v}}(0) = -K_p^{\mathrm{T}}(0)\bar{\boldsymbol{x}}(0)$$

and

$$\hat{\boldsymbol{v}}(1) = -K_p^{\mathrm{T}}(1)F_p^{\mathrm{T}}(0)\boldsymbol{x}_{10} = -K_p^{\mathrm{T}}(1)\bar{\boldsymbol{x}}(1)$$

and continuing in a similar fashion

$$\hat{\boldsymbol{v}}(i) = -K_p^{\mathrm{T}}(i)\bar{\boldsymbol{x}}(i), \quad i = 0, 1, \cdots, N \tag{32}$$

Equations (28), (29), and (32) constitute the solution to the indefinite LQ control problem. Of course, the condition for a minimum over $\{v(i)\}$ will be discussed. Next, we will give necessary and sufficient conditions that there exists a unique solution for a minimum over $\{v(i)\}$.

Lemma 4[20]. $\hat{\boldsymbol{v}} = -R_{\boldsymbol{\zeta}}^{-1}R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0}\bar{\boldsymbol{x}}(0)$ is a unique minimum if and only if $R_{\boldsymbol{\zeta}} > 0$, and $R_{\boldsymbol{\zeta}} > 0$ is equivalent to $Q_\varepsilon(k) > 0,\ k = 0, 1, \cdots, N$.

Proof. The proof of the first part involves the (lower-upper) triangular factorization of $\begin{bmatrix} R_{\boldsymbol{\xi}_0} & R_{\boldsymbol{\xi}_0\boldsymbol{\zeta}} \\ R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0} & R_{\boldsymbol{\zeta}} \end{bmatrix}$ and is straightforward. The equivalence of $R_{\boldsymbol{\zeta}} > 0$ and $Q_\varepsilon(k) > 0,\ k = 0, 1, \cdots, N$ can be obtained by Schur complement. □

Therefore, the Krein space Kalman filter corresponding to $R_{\boldsymbol{\zeta}}$ allows us to recursively check the above condition *via* the innovations Gramian.

Remark 2. $R_{\boldsymbol{\zeta}} \geq 0$ or $Q_\varepsilon(k) \geq 0,\ k = 0, 1, \cdots, N$ just need to be satisfied if $\hat{v}$ has not to be a unique minimum.

Remark 3. The control input $\hat{\boldsymbol{v}} = -R_{\boldsymbol{\zeta}}^{-1}R_{\boldsymbol{\zeta}\boldsymbol{\xi}_0}\bar{\boldsymbol{x}}(0)$ is referred to as the optimal open-loop control because it only depends on the initial state, $\bar{\boldsymbol{x}}(0)$. The control input $\hat{\boldsymbol{v}}(i) = -K_p^{\mathrm{T}}(i)\bar{\boldsymbol{x}}(i)$, on the other hand, is referred to as the optimal closed-loop control because it depends only on the current state, $\bar{\boldsymbol{x}}(i)$. Note that at any given time, i, the optimal closed-loop control input, $\hat{\boldsymbol{v}}(i) = -K_p^{\mathrm{T}}(i)\bar{\boldsymbol{x}}(i)$, coincides with the optimal open-loop input if and only if all previous choices of the control input were also optimal.

We can now summarize the results obtained so far in the following theorem.

Theorem 1. Under Assumption 1, the optimal control problem P has a unique solution if and only if $Q_\varepsilon(i) > 0, i = 0, 1, \cdots, N$. The optimal control can be synthesized as the linear state feedback such that the optimal closed-loop system is casual, i.e. its state trajectory is causally and uniquely determined by the initial value. Thus, problem P has a unique solution determined by the initial value. The optimal LQ controller can be shown as $\boldsymbol{u}(i) = K(i)\boldsymbol{x}(i)$, where

$$K(i) = [0 \quad I_r] \begin{bmatrix} I_p & 0 \\ -\bar{A}^{+}(i)A_{21}(i) & \widetilde{P}(i) \end{bmatrix} \begin{bmatrix} I_p \\ -K_p^{\mathrm{T}}(i) \end{bmatrix} (I_p \quad 0)N^{-1}$$

Proof. It is easily shown by (12) and (32). □

Remark 4. It is obvious that the method discussed in the article can be used to deal with the indefinite LQ optimal control problem for normal linear systems, which are viewed as special cases of system (1) when E is nonsingular.

5 Example

Consider the optimal control problem of linear singular discrete-time system (1) with quadratic cost (2), where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$x_0 = 1, N = 7$$

According to the method above, we can compute the gain matrices of controller as follows.

$$\begin{gathered} K(0) = 3.5000, \quad K(1) = 3.5002, \quad K(2) = 3.5007 \\ K(3) = 3.5029, \quad K(4) = 3.5116, \quad K(5) = 3.5455 \\ K(6) = 3.6667, \quad K(7) = 4 \end{gathered}$$

and

$$J(u^*(k), x^*(k)) = \bar{x}^{\mathrm{T}}(0)P_0\bar{x}(0) = 3$$

6 Conclusion

In this article, the indefinite LQ optimal control problem for a class of discrete-time linear singular systems has been discussed. It has been pointed out that this problem is equivalent to an indefinite LQ problem for standard state-space discrete-time linear systems under Assumption 1. The indefinite LQ problem for standard state-space systems is dual to that of projection for a backward stochastic system. Thus, the optimal LQ controller can be obtained by computing the gain matrices of linear estimation. Necessary and sufficient conditions guaranteeing a unique solution for the indefinite LQ problem are given.

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