Exponential Stability of Linear Systems with Multiple Time-varying Delays

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Abstract Based on the established new-type exponential stability theorem for general retarded dynamical systems and two preliminary lemmas, less conservative stability conditions for linear systems with multiple time-varying delays are established by using the new stability analysis approach of Lyapunov function. Unlike some results in the literature, none of the established results depends on the derivatives of the time-varying delays. Therefore, the results are suitable to the cases with very fast time-varying delays. An example is provided to show that the stability conditions obtained are better than the ones obtained directly based on the standard Razumikhin-type condition in the literature.

Key words Exponential stability, stability theorem, dynamical systems with time-varying delays

1 Introduction

For linear time-delay systems with time-varying delays, the main time-domain stability analysis approaches are the approach of Lyapunov functionals[1~25] and the approach of Lyapunov functions[1, 4, 12, 17, 21, 26~36]. For linear time-delay systems with time-varying delays, the approach of Lyapunov functionals needs generally to limit the bounds of the derivatives of time-varying delays (See, for example, [2, 3, 5~11, 13, 14, 16, 18~25]), so that it is not suitable for the systems with very fast time-varying delays[29~32]. On the other hand, the approach of Lyapunov functions with Razumikhin technique[27] can be used to deal easily with the case of time-varying delays without the limitations on the derivatives of delays. Unfortunately, the existing Razumikhin technique[1, 4, 12, 17, 21, 26, 27] also results in conservative stability conditions. Recently, new-type stability theorems together with new stability analysis techniques based on a Lyapunov function approach have been developed in [28~36]. This paper is a continuation to these previous researches and focuses on the application of the new method for linear systems with multiple time-varying delays.

The organization of this paper is as follows. In Section 2, the new-type exponential stability theorem for general retarded dynamical systems and two lemmas is established as preliminary results. Then, in Section 3, as the application of the established preliminary results, less conservative stability conditions for linear systems with multiple time-varying delays are established by using the new stability analysis approach of Lyapunov function. Unlike the existing results in the literature, where the derivatives of all time-varying delays are limited to be less than one, the established results do not depend on the derivative of time-varying delays. In addition, some remarks and figures are given in Sections 2 and 3 to explain the obtained results. Finally, the paper is concluded in Section 4.

2 Preliminary results

The notations used in this paper are as follows. $\mathbf{R}^n$ is the real vector space of dimension n; $\mathbf{R}^{n\times n}$ is the real matrix space of dimension $n\times n$; $\mathbf{R}_+$ denotes the set of nonnegative real numbers; $J=[r,\infty)$ with $r\in\mathbf{R}$; $\mathbf{C}^n$ is the complex vector space of dimension n; $\mathbf{C}^{n\times n}$ is the complex matrix space of dimension $n\times n$; C_n denotes the Banach space of continuous functions mapping $[-\tau,0]$ into $\mathbf{R}^n$, where $\tau>0$ is a constant; $\boldsymbol{y}_t(\theta)\in\mathbf{R}^n$ denotes $\boldsymbol{y}(t+\theta)\in\mathbf{R}^n$ for $t\in\mathbf{R}$ and $\theta\in\mathbf{R}$ so that $\boldsymbol{y}(t)=\boldsymbol{y}_t(0)$; $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$; $|\cdot|$ denotes the absolute value; $\|\boldsymbol{\phi}\|_\tau=\sup_{-\tau\le\theta\le0}\|\boldsymbol{\phi}(\theta)\|$ with $\boldsymbol{\phi}(\theta)\in\mathbf{R}^n$ for given $\boldsymbol{\phi}\in C_n$; $A^{\rm T}$ is the transpose of $A\in\mathbf{R}^{n\times n}$; A^* is the conjugate transpose of $A\in\mathbf{C}^{n\times n}$, *i.e.*, $\bar{A}^{\rm T}$; $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalue, respectively; $\|A\|=\max_i\{\lambda_i^{1/2}(A^*A)\}$ for $A\in\mathbf{C}^{n\times n}$; $A>0$ (or <0) denotes a positive definite (or negative definite) matrix; $A\le B$ means that $A-B\le0$ is negative semi-definite; and finally, $j^2=-1$.

Consider a retarded dynamical system described by a general retarded functional differential equation

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{f}(t,\boldsymbol{x}_t) \quad (1)$$

where "$\cdot$" denotes the right-hand derivative, $\boldsymbol{f}:J\times C_n\longrightarrow\mathbf{R}^n$ takes $J\times$(bounded sets of C_n) into bounded sets of $\mathbf{R}^n$, and $\boldsymbol{f}(t,\boldsymbol{\phi})$ is continuous and Lipschitzian in $\boldsymbol{\phi}\in C_n$, so that for an initial function $\boldsymbol{\phi}=x_{t_0}\in C_n$ at $t=t_0\in J$, system (1) has a unique solution, $\boldsymbol{x}(t_0,x_{t_0})(t)$, on $[t_0-\tau,\infty)$. Suppose that $\boldsymbol{f}(t,0)=0$ for all $t\in\mathbf{R}$, so that $x_e=0$ is an equilibrium of system (1). As far as local results are concerned, we always suppose that $\boldsymbol{f}:J\times C_n^\rho\longrightarrow\mathbf{R}^n$, where $C_n^\rho=\{\boldsymbol{\phi}\in C_n|\|\boldsymbol{\phi}\|_\tau<\rho,\rho>0\}$. For simplicity, we also denote the value of the solution $x(t_0,x_{t_0})(t)\in\mathbf{R}^n$ by $x(t)$ and the solution segment $\boldsymbol{x}(t+\theta)=\boldsymbol{x}_t(\theta)\in\mathbf{R}^n$ for all $\theta\in[-\tau,0]$ by $\boldsymbol{x}_t\in C_n$ at $t\ge t_0$.

Definition 1. Let $\alpha\in\mathbf{R}_+$ be a nonnegative constant number. The equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (1) is called exponentially stable with respect to the constant decay degree $\alpha\in\mathbf{R}_+$ if there exists a constant $\Gamma\ge1$ such that for any $\varepsilon>0$, there exists a $0<\delta<\varepsilon/\Gamma$ satisfying $\delta<\rho$ such that along the solution of system (1) through any $(t_0,\boldsymbol{x}_{t_0})\in J\times C_n^\rho$ satisfying $\|\boldsymbol{x}_{t_0}\|_\tau\le\delta$, we have $\|\boldsymbol{x}(t)\|\le\Gamma\|\boldsymbol{x}_{t_0}\|_\tau\exp\{-\alpha(t-t_0)\}$ for all $t\ge t_0\in J$. The equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (1) is said to be globally exponentially stable with respect to the constant decay degree $\alpha\in\mathbf{R}_+$ if there exists a constant $\Gamma\ge1$ such that for any $\delta>0$, along the solution of system (1) through any $(t_0,\boldsymbol{x}_{t_0})\in J\times C_n$ satisfying $\|\boldsymbol{x}_{t_0}\|_\tau\le\delta$, we have $\|\boldsymbol{x}(t)\|\le\Gamma\|\boldsymbol{x}_{t_0}\|_\tau\exp\{-\alpha(t-t_0)\}$ for all $t\ge t_0\in J$.

Remark 1. It is obvious that the exponential stability of the equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (1) defined by Definition 1 for a known or unknown constant $\tau>0$ implies that the uniform asymptotical stability (case $\alpha>0$) of the equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (1) and at least the uniform

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stability (e.g. $\alpha = 0$) of the equilibrium $\boldsymbol{x}_e = \mathbf{0}$ of system (1) for the same fixed $\tau > 0$.

Theorem 1. Let $\alpha \in \mathbf{R}_+$ be a nonnegative constant number, $P > 0 \in \mathbf{R}^{n\times n}$ is a positive definite constant matrix, and $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}$ with $\boldsymbol{x} \in \mathbf{R}^n$. Assume that there exits a positive constant scalar $\Pi \geq 1$ such that

$$\begin{aligned}&\sup_{s\in\mathbf{R}}\{2\boldsymbol{\phi}^{\mathrm{T}}(0)Pf(s,\boldsymbol{\phi})\} \leq \\ &\Pi\sup_{\theta\in[-\tau,0]}\{\boldsymbol{\phi}^{\mathrm{T}}(\theta)P\boldsymbol{\phi}(\theta)\}, \forall\boldsymbol{\phi}\in C_n\end{aligned} \tag{2}$$

where $\boldsymbol{f} : J\times C_n \to \mathbf{R}^n$ is defined as in (1). The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ of system (1) is exponentially stable with respect to the constant decay degree $\alpha \in \mathbf{R}_+$ if there exists a constant $\Gamma \geq 1$ such that for any $\varepsilon > 0$, there exists a $0 < \delta \leq \varepsilon/\Gamma$ satisfying $\delta < \rho$ such that along the solution of system (1) through any $(t_0, \boldsymbol{x}_{t_0}) \in J \times C_n^\rho$ satisfying $\|\boldsymbol{x}_{t_0}\|_\tau \leq \delta$, we have

$$\dot{V}(\boldsymbol{x}_t(0)) \leq -2\alpha V(\boldsymbol{x}_t(0)) \tag{3}$$

whenever $\boldsymbol{x}_t \in S(L_t(\theta))$, where

$$S(L_t(\theta)) = \left\{\boldsymbol{y}_t\in C_n \left| \begin{array}{c} L_t(\theta) = L(t+\theta) = \Pi V(\tilde{\boldsymbol{y}}_{t_0}(0))\mathrm{e}^{-2\alpha(t+\theta-t_0)} \\ V(\tilde{\boldsymbol{y}}_{t_0}(0)) = \lambda_{\max}(P)\|\boldsymbol{x}_{t_0}\|_\tau^2 \\ L_t(0) = L(t) = V(\boldsymbol{y}_t(0)) \\ V(\boldsymbol{y}_t(\theta)) = V(\boldsymbol{y}_t(0))\cos^2(\omega\theta)\mathrm{e}^{-2\alpha\theta} \\ \theta\in[-\tau,0], \omega\in\mathbf{R}, t\geq t_0\in J \end{array}\right.\right\} \tag{4}$$

Where $\tilde{\boldsymbol{y}}_{t_0}(0) \in \mathbf{R}^n$ is a real vector satisfying $V(\tilde{\boldsymbol{y}}_{t_0}(0)) = \lambda_{\max}(P)\|\boldsymbol{x}_{t_0}\|_\tau^2$ at $t_0 \in J$.

Proof. Noting that along the solution of system (1) through any $(t_0, \boldsymbol{x}_{t_0}) \in J \times C_n^\rho$, we have $\dot{V}(\boldsymbol{x}_t(0)) = 2\boldsymbol{x}_t^{\mathrm{T}}(0)Pf(t,\boldsymbol{x}_t)$ for $t \geq t_0 \in J$. By $\lambda_{\min}(P)\|\boldsymbol{x}\|^2 \leq V(\boldsymbol{x}) \leq \lambda_{\max}(P)\|\boldsymbol{x}\|^2$, $\Pi \geq 1$ in (2), and $V(\tilde{\boldsymbol{y}}_{t_0}(0)) = \lambda_{\max}(P)\|\boldsymbol{x}_{t_0}\|_\tau^2$ in (4), we have

$$\begin{aligned}\lambda_{\min}(P)\|\boldsymbol{x}_{t_0}(0)\|^2 &\leq V(\boldsymbol{x}_{t_0}(0)) \leq \lambda_{\max}(P)\|\boldsymbol{x}_{t_0}\|_\tau^2 \\ &\leq \Pi V(\tilde{\boldsymbol{y}}_{t_0}(0)) = \Pi\lambda_{\max}(P)\|\boldsymbol{x}_{t_0}\|_\tau^2\end{aligned} \tag{5}$$

so that

$$\begin{aligned}\|\boldsymbol{x}_{t_0}(0)\| &\leq \sqrt{\Pi\lambda_{\max}(P)/\lambda_{\min}(P)}\|\boldsymbol{x}_{t_0}\|_\tau \\ &\leq \sqrt{\Pi\lambda_{\max}(P)/\lambda_{\min}(P)}\delta\end{aligned} \tag{6}$$

for any $\|\boldsymbol{x}_{t_0}\|_\tau \leq \delta$. According to Definition 1, (5), and (6), let $\Gamma = \sqrt{\Pi\lambda_{\max}(P)/\lambda_{\min}(P)}$. Then for any $\varepsilon > 0$, there is a $0 < \delta \leq \varepsilon/\Gamma$ satisfying $\delta < \rho$ such that along the solution of system (1) through any $(t_0, \boldsymbol{x}_{t_0}) \in J\times C_n^\rho$ satisfying $\|\boldsymbol{x}_{t_0}\|_\tau \leq \delta$, whenever $\boldsymbol{x}_s \in S(L_s(\theta))$ at $s \geq t_0 \in J$, we have $V(\boldsymbol{x}_s(0)) = \Pi V(\tilde{\boldsymbol{y}}_{t_0}(0))\exp\{-2\alpha(s-t_0)\}$. As shown in Fig. 1, condition (3) implies that at this moment

$$\dot{V}(\boldsymbol{x}_s(0)) \leq -2\alpha V(\boldsymbol{x}_s(0)), \boldsymbol{x}_s \in S(L_s(\theta)) \tag{7}$$

Now, let us prove the sufficiency of condition (3) by contradiction. Assume that the condition (7) holds, but there is a sufficiently small $\delta s > 0$ such that $\dot{V}(\boldsymbol{x}_t(0)) > -2\alpha V(\boldsymbol{x}_t(0))$ for all $t \in (s, s+\delta s]$ so that $\dot{V}(\boldsymbol{x}_t(0)) \geq -2\alpha V(\boldsymbol{x}_t(0))$ for all $t \in (s-\delta s, s]$, $\dot{V}(\boldsymbol{x}_t(0)) = -2\alpha V(\boldsymbol{x}_t(0))$ at $t = s$, and $\dot{V}(\boldsymbol{x}_t(0)) > -2\alpha V(\boldsymbol{x}_t(0))$ for all $t \in (s, s+\delta s]$. According to the continuous dependence property of the solution of system (1) on its initials, the above assumption implies that there must be a new initial $\tilde{\boldsymbol{x}}_{t_0} \in C_n$ satisfying $\|\tilde{\boldsymbol{x}}_{t_0}\|_\tau = \|\boldsymbol{x}_{t_0}\|_\tau$ with the corresponding solution $\tilde{\boldsymbol{x}}_t(0) \in \mathbf{R}^n$ of system (1) through $(t_0, \tilde{\boldsymbol{x}}_{t_0}) \in J \times C_n^\rho$ such that $V(\tilde{\boldsymbol{x}}_{\tilde{s}}(0)) = \Pi V(\tilde{\boldsymbol{y}}_{t_0}(0))\exp\{-2\alpha(\tilde{s}-t_0)\}$ when $\tilde{\boldsymbol{x}}_{\tilde{s}} \in S(L_{\tilde{s}}(\theta))$ at some moment $\tilde{s} \in [s-\delta s, s+\delta s]$ but $\dot{V}(\tilde{\boldsymbol{x}}_{\tilde{s}}(0)) > -2\alpha V(\tilde{\boldsymbol{x}}_{\tilde{s}}(0))$ (See one case shown in Fig. 2). This contradicts condition (3), *i.e.*, $\dot{V}(\tilde{\boldsymbol{x}}_t(0)) \leq -2\alpha V(\tilde{\boldsymbol{x}}_t(0))$ whenever $\tilde{\boldsymbol{x}}_t \in S(L_t(\theta))$ on $t \geq t_0 \in J$. Therefore, condition (3) guarantees that along the solution of system (1) through any $(t_0, \boldsymbol{x}_{t_0}) \in J \times C_n^\rho$ satisfying $\|\boldsymbol{x}_{t_0}\|_\tau \leq \delta$, we have

$$V(\boldsymbol{x}_t(0)) \leq \Pi V(\tilde{\boldsymbol{y}}_{t_0}(0))\mathrm{e}^{-2\alpha(t-t_0)}, \forall t \geq t_0 \in J \tag{8}$$

that is,

$$\|\boldsymbol{x}_t(0)\| \leq \Gamma\|\boldsymbol{x}_{t_0}\|_\tau\mathrm{e}^{-\alpha(t-t_0)} \leq \Gamma\delta\mathrm{e}^{-\alpha(t-t_0)}, \forall t \geq t_0 \in J \tag{9}$$

where $\Gamma = \sqrt{\Pi\lambda_{\max}(P)/\lambda_{\min}(P)} \geq 1$. By Definition 1 and (9), the proof is completed. □

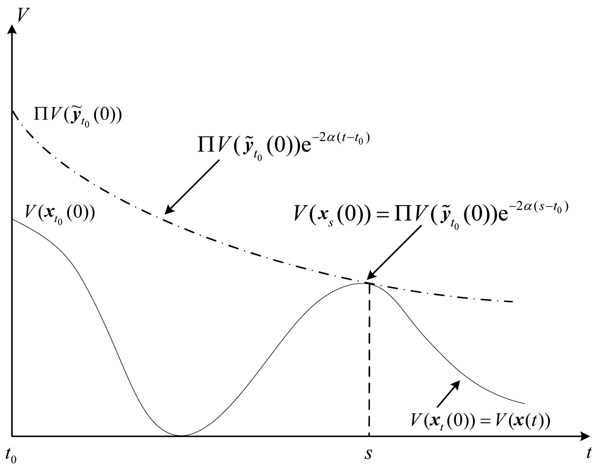

Fig. 1 The case of the sign-definite-to-sign-indefinite mode

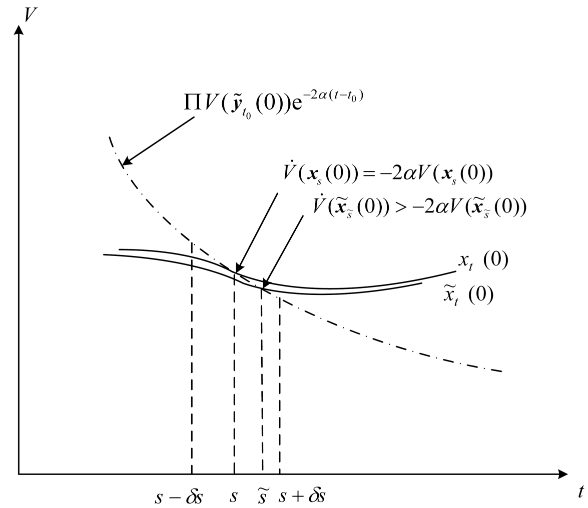

Fig. 2 A contradiction case to condition (3)

Remark 2. As we know, the classical Lyapunov stability theorems provided us the stability analysis methods in the so-called sign-definite-to-sign-definite mode, *i.e.* from a sign positive definite function $V>0$ to a sign negative definite derivative $\dot{V}<0$ of the function along the solution of system. For the case of $\dot{V}\leq 0$, the concept of Lyapunov function is extended by the invariant set theorems, which work in a sign-definite-to-sign-semidefinite mode, *i.e.*, from $V>0$ to $\dot{V}\leq 0$. In fact, it is the sign-definite-to-sign-definite mode that causes the difficulty of choosing Lyapunov functions for complex systems. As we have seen from the proof of Theorem 1, Theorem 1 works in a sign-definite-to-sign-indefinite mode, for example, from $V(\boldsymbol{x}_t(0))>0$ to $\dot{V}(\boldsymbol{x}_t(0))\leq -2\alpha V(\boldsymbol{x}_t(0))\leq 0$ ($\|\boldsymbol{x}_t(0)\|=\|\boldsymbol{x}(t)\|\neq 0$) whenever $\boldsymbol{x}_t\in S(L_t(\theta))$, and $\dot{V}(\boldsymbol{x}_t(0))>0$ is allowable if $\boldsymbol{x}_t\notin S(L_t(\theta))$ (See Fig. 1). Obviously, the concept of Lyapunov function is further extended here.

In the next section, we need the following lemmas, whose proofs are given in Appendix.

Lemma 1. Let $P>0\in \mathbf{R}^{n\times n}$ be a positive definite matrix, $D_k\in \mathbf{C}^{n\times n}$ for $k=0,1,\cdots,m$ be complex matrices, and $U(K)=\{\boldsymbol{u}\in \mathbf{C}^n|\boldsymbol{u}^*P\boldsymbol{u}=K>0\}$ with a constant $K>0$. Then for any given $1+m$ vectors: $\boldsymbol{u}\in U(K)$ and $\boldsymbol{v}_k\in U(K)$ for $k=1,2,\cdots,m$, and there are m real scalars $\psi_k\in[0,2\pi]$ for $k=1,2,\cdots,m$, and m nonnegative numbers $\beta_k\geq 0$ for $k=1,2,\cdots,m$ such that

$$\boldsymbol{u}^*(PD_0+D_0^*P)\boldsymbol{u}+\sum_{k=1}^{m}\boldsymbol{u}^*PD_k\boldsymbol{v}_k+\sum_{k=1}^{m}\boldsymbol{v}_k^*D_k^*P\boldsymbol{u}=$$
$$\boldsymbol{u}^*[P(D_0+\sum_{k=1}^{m}\mathrm{e}^{\beta_k+j\psi_k}D_k)+(D_0+\sum_{k=1}^{m}\mathrm{e}^{\beta_k+j\psi_k}D_k)^*P]\boldsymbol{u} \tag{10}$$

Lemma 2. Let $P>0\in \mathbf{R}^{n\times n}$ be a positive definite matrix, $T_k\in \mathbf{C}^{n\times n}$ for $k=0,1,\cdots,m$ be Hermitian matrices, $\gamma\in\mathbf{R}$, and $\beta_{kM}>0$ for $k=1,2,\cdots,m$ be real numbers. Then,

$$T_0+\sum_{k=1}^{m}\beta_kT_k\leq -2\gamma P,\quad \forall\beta_k\in[-\beta_{kM},\beta_{kM}] \tag{11}$$

if and only if

$$T_0+\left[\sum_{k=1}^{m}\pm\beta_{kM}T_k\right]_l\leq -2\gamma P,\quad \forall l=1,2,\cdots,2^m \tag{12}$$

where $[\sum_{k=1}^{m}\pm\beta_{kM}T_k]_l$ for $l=1,2,\cdots,2^m$ denotes all 2^m cases of alternating sign.

3 Application results

Consider the following linear system with multiple time-varying delays

$$\begin{cases}\dot{\boldsymbol{x}}(t)=A_0\boldsymbol{x}(t)+\sum\limits_{k=1}^{m}A_k\boldsymbol{x}(t-\tau_k(t)),\quad t\geq t_0\in J\\ \boldsymbol{x}_{t_0}(\theta)=\boldsymbol{x}(t_0+\theta)=\boldsymbol{\phi}(\theta),\quad \theta\in[-\tau,0]\end{cases} \tag{13}$$

where $\boldsymbol{\phi}\in C_n$, $\boldsymbol{x}\in\mathbf{R}^n$, and $A_k\in\mathbf{R}^{n\times n}(k=0,1,\cdots,m)$ are constant matrices, $\tau_k(t)\leq\tau_{kM}\leq\tau<\infty(k=1,2,\cdots,m)$ for $t\in\mathbf{R}$ are time-varying and bounded delays, and $\tau_{kM}>0$ and $\tau>0$ are known or unknown.

Proposition 1. Let $\alpha\in\mathbf{R}_+$ be a nonnegative constant number, $X(K)=\{\boldsymbol{x}\in\mathbf{R}^n|\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}=K>0\}$ with a positive definite matrix $P>0\in\mathbf{R}^{n\times n}$, and a constant $K>0$. Assume that $\tau_{kM}>0$ such that $\tau_k(t)\leq\tau_{kM}$ for all $k=1,2,\cdots,m$ are known and there is a nonnegative $\gamma\in\mathbf{R}_+$ such that

$$\begin{aligned}&2\boldsymbol{x}^{\mathrm{T}}PA_0\boldsymbol{x}+\sum_{k=1}^{m}2\boldsymbol{x}^{\mathrm{T}}PA_k\boldsymbol{y}_k\leq\\&2\boldsymbol{x}^{\mathrm{T}}PA_0\boldsymbol{x}+\sum_{k=1}^{m}\mathrm{e}^{\gamma\tau_{kM}}|\boldsymbol{x}^{\mathrm{T}}(PA_k+A_k^{\mathrm{T}}P)\boldsymbol{x}|\\&\forall\boldsymbol{x},\boldsymbol{y}_k\in X(K)\end{aligned} \tag{14}$$

Then, the equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (13) is exponentially stable with respect to the constant decay degree $\alpha\in\mathbf{R}_+$ if

$$PA_0+A_0^{\mathrm{T}}P+\left[\sum_{k=1}^{m}\pm\mathrm{e}^{(\alpha+\gamma)\tau_{kM}}(PA_k+A_k^{\mathrm{T}}P)\right]_l\leq-2\alpha P\quad \forall l=1,2,\cdots,2^m \tag{15}$$

where $[\sum_{k=1}^{m}\pm\mathrm{e}^{(\alpha+\gamma)\tau_{kM}}(PA_k+A_k^{\mathrm{T}}P)]_l$ for $l=1,2,\cdots,2^m$ denotes all 2^m cases of alternating sign.

Proof. Let $V(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}$, where $P>0\in\mathbf{R}^{n\times n}$ satisfying (14). According to Theorem 1, along the solution of system (13), whenever $\boldsymbol{x}_s\in S(L_s(\theta))$ on $s\geq t_0\in J$, we have $V(\boldsymbol{x}_s(0))=\Pi V(\tilde{\boldsymbol{y}}_{t_0}(0))\exp\{-2\alpha(s-t_0)\}$ with $\alpha\in\mathbf{R}_+$ and by the method developed in [30 ∼ 34], we obtain

$$\begin{aligned}\dot{V}(\boldsymbol{x}_s(0))=&2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\sum_{k=1}^{m}2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_k\boldsymbol{x}_s(-\tau_k(s))=\\&2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\\&\sum_{k=1}^{m}2\mathrm{e}^{\alpha\tau_k(s)}|\cos(\xi_k\tau_k(s))|\boldsymbol{x}_s^{\mathrm{T}}(0)PA_k\boldsymbol{x}_s(0_k)=\\&2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\\&\sum_{k=1}^{m}2\mathrm{e}^{\alpha\tau_{kM}}|\cos(\psi_k\tau_{kM})|\boldsymbol{x}_s^{\mathrm{T}}(0)PA_k\boldsymbol{x}_s(0_k)\end{aligned} \tag{16}$$

where $\xi_k,\psi_k\in\mathbf{R}$ and $\tau_k(s)\in[-\tau_{kM},0]$ for $k=1,2,\cdots,m$ are the instant values at the moment $s\geq t_0\in J$, and

$$\begin{cases}V(\boldsymbol{x}_s(0_k))&=V(\boldsymbol{x}_s(0))\\ \boldsymbol{x}_s(-\tau_k(s))&=\boldsymbol{x}_s(0_k)|\cos(\xi_k\tau_k(s))|\mathrm{e}^{\alpha\tau_k(s)}=\\&\quad\boldsymbol{x}_s(0_k)|\cos(\psi_k\tau_{kM})|\mathrm{e}^{\alpha\tau_{kM}}\\ V(\boldsymbol{x}_s(-\tau_k(s)))&=V(\boldsymbol{x}_s(0))\cos^2(\xi_k\tau_k(s))\mathrm{e}^{2\alpha\tau_k(s)}=\\&\quad V(\boldsymbol{x}_s(0))\cos^2(\psi_k\tau_{kM})\mathrm{e}^{2\alpha\tau_{kM}}\end{cases} \tag{17}$$

(See Fig. 3). Whenever $\boldsymbol{x}_t\in S(L_t(\theta))$ at $t\geq t_0\in J$, by

Lemmas 1, 2, and (16), condition (15) implies that

$$\begin{aligned}\dot{V}(\boldsymbol{x}_t(0)) =& 2\boldsymbol{x}_t^{\mathrm{T}}(0)PA_0\boldsymbol{x}_t(0)+\\ &\sum_{k=1}^{m} 2\mathrm{e}^{\alpha\tau_{kM}}|\cos(\psi_k\tau_k(t))|\boldsymbol{x}_t^{\mathrm{T}}(0)PA_k\boldsymbol{x}_t(0_k)\leq\\ &\boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0+A_0^{\mathrm{T}}P)\boldsymbol{x}_t(0)+\sum_{k=1}^{m}\mathrm{e}^{(\alpha+\gamma)\tau_{kM}}\times\\ &|\boldsymbol{x}_t^{\mathrm{T}}(0)(PA_k\mathrm{e}^{j\omega_k\tau_{kM}}+\mathrm{e}^{-j\omega_k\tau_{kM}}A_k^{\mathrm{T}}P)\boldsymbol{x}_t(0)|=\\ &\boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0+A_0^{\mathrm{T}}P)\boldsymbol{x}_t(0)+\sum_{k=1}^{m}\mathrm{e}^{(\alpha+\gamma)\tau_{kM}}\times\\ &|\cos(\omega_k\tau_{kM})\boldsymbol{x}_t^{\mathrm{T}}(0)(PA_k+A_k^{\mathrm{T}}P)\boldsymbol{x}_t(0)|\leq\\ &\max_{l=1,2,\cdots,2^m}\{\boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0+A_0^{\mathrm{T}}P+\\ &\left[\sum_{k=1}^{m}\pm\mathrm{e}^{(\alpha+\gamma)\tau_{kM}}(PA_k+A_k^{\mathrm{T}}P)\right]_l)\boldsymbol{x}_t(0)\}\leq\\ &-2\alpha V(\boldsymbol{x}_t(0))\end{aligned}\tag{18}$$

where $\omega_k\in\mathbf{R}$ for $k=1,2,\cdots,m$, that is, $\dot{V}(\boldsymbol{x}_t(0))\leq-2\alpha V(\boldsymbol{x}_t(0))$ whenever $\boldsymbol{x}_t\in S(L_t(\theta))$ on $t\geq t_0\in J$. By Theorem 1, the proof is completed. □

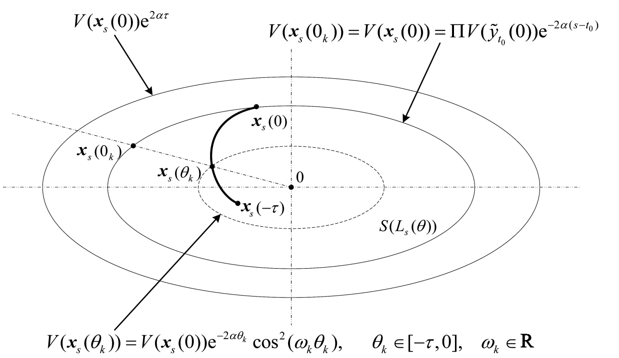

Fig. 3 A relationship between $\boldsymbol{x}_t(\theta_k)$ and $\boldsymbol{x}_t(0_k)$ for $\theta_k\in[-\tau,0]$

Proposition 2. Let $\alpha\in\mathbf{R}_+$ be a nonnegative constant number and $\tau_{kM}>0$ such that $\tau_k(t)\leq\tau_{kM}$ for all $k=1,2,\cdots,m$ are known. Then, the equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (13) is exponentially stable with respect to the constant decay degree $\alpha\in\mathbf{R}_+$ if there is a positive number $\rho>0$, a positive definite matrix $P>0\in\mathbf{R}^{n\times n}$, and m positive definite matrices $S_k>0$ for $k=1,2,\cdots,m$ such that

$$\begin{bmatrix} M & PA_1 & \cdots & PA_m\\ A_1^{\mathrm{T}}P & S_1 & & 0\\ \vdots & & \ddots & \\ A_m^{\mathrm{T}}P & 0 & & S_m\end{bmatrix}\geq 0$$

$$\rho P-S_k\geq 0,\forall k=1,2,\cdots,m\tag{19}$$

where $M=-PA_0-A_0^{\mathrm{T}}P-\rho P\sum_{k=1}^{m}\mathrm{e}^{2\alpha\tau_{kM}}-2\alpha P$.

Proof. By the standard Schur complement, condition (19) is equivalent to

$$PA_0+A_0^{\mathrm{T}}P+\rho P\sum_{k=1}^{m}\mathrm{e}^{2\alpha\tau_{kM}}+\sum_{k=1}^{m}PA_kS_k^{-1}A_kP\leq-2\alpha P\tag{20}$$

Let $V(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}$, where $P>0\in\mathbf{R}^{n\times n}$ satisfying (19). Let us start with the derivation from (16) in the proof of Proposition 1. Note that for any given $S_k>0$ for $k=1,2,\cdots,m$, there exits a positive number $\rho>0$ such that $\rho P-S_k\geq 0$ for all $k=1,2,\cdots,m$ and $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}\leq\boldsymbol{x}^{\mathrm{T}}S^{-1}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}S\boldsymbol{y}$ holds for any vectors $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$ and any positive definite matrix $S>0$. By this remark and condition (19) with (20), we further obtain, whenever $\boldsymbol{x}_s\in S(L_s(\theta))$ on $s\geq t_0\in J$,

$$\begin{aligned}\dot{V}(\boldsymbol{x}_s(0))=&2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\sum_{k=1}^{m}2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_k\boldsymbol{x}_s(-\tau_k(s))=\\ &2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\\ &\sum_{k=1}^{m}2\mathrm{e}^{\alpha\tau_{kM}}|\cos(\psi_k\tau_k(s))|\boldsymbol{x}_s^{\mathrm{T}}(0)PA_k\boldsymbol{x}_s(0_k)|\leq\\ &2\boldsymbol{x}_s^{\mathrm{T}}(0)PA_0\boldsymbol{x}_s(0)+\\ &\sum_{k=1}^{m}\boldsymbol{x}_s^{\mathrm{T}}(0)PA_kS_k^{-1}A_k^{\mathrm{T}}P\boldsymbol{x}_s(0)+\\ &\sum_{k=1}^{m}\mathrm{e}^{\alpha\tau_{kM}}\boldsymbol{x}_s^{\mathrm{T}}(0_k)S_k\boldsymbol{x}_s(0_k)+\\ &\sum_{k=1}^{m}\mathrm{e}^{2\alpha\tau_{kM}}(\rho\boldsymbol{x}_s^{\mathrm{T}}(0_k)P\boldsymbol{x}_s(0_k)-\boldsymbol{x}_s^{\mathrm{T}}(0_k)S_k\boldsymbol{x}_s(0_k))=\\ &\boldsymbol{x}_s^{\mathrm{T}}(0)(PA_0+A_0^{\mathrm{T}}P+\\ &\rho P\sum_{k=1}^{m}\mathrm{e}^{2\alpha\tau_{kM}}+\sum_{k=1}^{m}PA_kS_k^{-1}A_k^{\mathrm{T}}P)\boldsymbol{x}_s(0)\leq\\ &-2\alpha V(\boldsymbol{x}_s(0))\end{aligned}\tag{21}$$

By Theorem 1, the proof is completed. □

Proposition 3. The equilibrium $\boldsymbol{x}_e=\mathbf{0}$ of system (13) is asymptotical stable independent of delays if there is a positive number $\rho>0$, a positive definite matrix $P>0\in\mathbf{R}^{n\times n}$ and m positive definite matrices $S_k>0$ for $k=1,2,\cdots,m$ such that

$$\begin{bmatrix} -PA_0-A_0^{\mathrm{T}}P-\rho mP & PA_1 & \cdots & PA_m\\ A_1^{\mathrm{T}}P & S_1 & & 0\\ \vdots & & \ddots & \\ A_m^{\mathrm{T}}P & 0 & & S_m\end{bmatrix}>0,$$

$$\rho P-S_k\geq 0,\forall k=1,2,\cdots,m\tag{22}$$

Proof. Note that $\tau_k(t)\leq\tau<\infty$ $(k=1,2,\cdots,m)$ for a known or unknown $\tau>0$. Condition (22) implies that there is a sufficiently small positive number $\alpha=\alpha(\tau)>0$

such that

$$\begin{bmatrix} M & PA_1 & \cdots & PA_m \\ A_1^{\mathrm{T}}P & S_1 & & 0 \\ \vdots & & \ddots & \\ A_m^{\mathrm{T}}P & 0 & & S_m \end{bmatrix} \geq 0,$$

$$\rho P - S_k \geq 0, \forall k = 1, 2, \cdots, m \tag{23}$$

where $M = -PA_0 - A_0^{\mathrm{T}}P - \rho m P \mathrm{e}^{2\alpha\tau} - 2\alpha P$. By the proof of Proposition 2 and Definition 1, we prove this proposition immediately. □

Remark 3. It is easy to see that the relationship between (15) and (19) is

$$PA_0 + A_0^{\mathrm{T}} + \left[\sum_{k=1}^{m} \pm \mathrm{e}^{(\alpha+\gamma)\tau_{kM}}(PA_k + A_k^{\mathrm{T}}P)\right]_l \leq$$
$$PA_0 + A_0^{\mathrm{T}}P + \rho P \sum_{k=1}^{m} \mathrm{e}^{2\alpha\tau_{kM}} + \sum_{k=1}^{m} PA_k S_k^{-1} A_k^{\mathrm{T}} P$$
$$\forall l = 1, 2, \cdots, 2^m \tag{24}$$

Therefore, condition (15) is less conservative than (19).

Remark 4. Condition (22) is said to be independent of delays since there is no any delay information included within itself.

Remark 5. Unlike the results in [2, 3, 5∼11, 13, 14, 16, 18∼25], where the derivatives of all time-varying delays are limited to be less than one, none of the results depends on the derivative of time-varying delays. Therefore, the established results here are suitable for the cases with very fast time-varying and bounded delays.

Example 1. Consider system (13). By directly using the standard Razumikhin-type condition: $\boldsymbol{x}_t^{\mathrm{T}}(\theta)P\boldsymbol{x}_t(\theta) \leq q\boldsymbol{x}_t^{\mathrm{T}}(0)P\boldsymbol{x}_t(0)$ for all $\theta \in [-\tau, 0]$ with a constant number $q = \mathrm{e}^{2\alpha\tau} > 1$, (21) can be derived as

$$\begin{aligned}\dot{V}(\boldsymbol{x}_t(0)) =& 2\boldsymbol{x}_t^{\mathrm{T}}(0)PA_0\boldsymbol{x}_t(0) + \sum_{k=1}^{m} 2\boldsymbol{x}_t^{\mathrm{T}}(0)PA_k\boldsymbol{x}_t(-\tau_k(t)) \leq \\ & \boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0 + A_0^{\mathrm{T}}P)\boldsymbol{x}_t(0) + \sum_{k=1}^{m} \boldsymbol{x}_t^{\mathrm{T}}(0)PA_k \times \\ & P^{-1}A_k^{\mathrm{T}}P\boldsymbol{x}_t(0) + \sum_{k=1}^{m} \boldsymbol{x}_t^{\mathrm{T}}(-\tau_k(t))P\boldsymbol{x}_t(-\tau_k(t)) \leq \\ & \boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0 + A_0^{\mathrm{T}}P)\boldsymbol{x}_t(0) + \sum_{k=1}^{m} \boldsymbol{x}_t^{\mathrm{T}}(0)PA_k \times \\ & P^{-1}A_k^{\mathrm{T}}P\boldsymbol{x}_t(0) + qm\boldsymbol{x}_t^{\mathrm{T}}(0)P\boldsymbol{x}_t(0) = \\ & \boldsymbol{x}_t^{\mathrm{T}}(0)(PA_0 + A_0^{\mathrm{T}}P + qmP + \\ & \sum_{k=1}^{m} PA_kP^{-1}A_k^{\mathrm{T}}P)\boldsymbol{x}_t(0) \leq \\ & -2\alpha V(\boldsymbol{x}_t(0))\end{aligned} \tag{25}$$

where $PA_0 + A_0^{\mathrm{T}}P + mP + \sum_{k=1}^{m} PA_kP^{-1}A_k^{\mathrm{T}}P < 0$ is equivalent to

$$\begin{bmatrix} -PA_0 - A_0^{\mathrm{T}}P - mP & PA_1 & \cdots & PA_m \\ A_1^{\mathrm{T}}P & P & & 0 \\ \vdots & & \ddots & \\ A_m^{\mathrm{T}}P & 0 & & P \end{bmatrix} > 0 \tag{26}$$

By comparing (22) with (26), it is easy to see that m free positive definite matrices, $S_k > 0$, satisfying $\rho P - S_k \geq 0$ for all $k = 1, 2, \cdots, m$ can be introduced into condition (22). This shows that the proposed technique, here, can result in less conservative stability conditions than the standard Razumikhin-type technique.

4 Conclusion

Based on some preliminary results, less conservative stability conditions for linear systems with multiple time-varying delays are established by using a new stability analysis approach of Lyapunov function. Unlike some results in the literature, none of the established results depends on the derivatives of the time-varying delays. Therefore, the results are suitable to the cases with very fast time-varying delays. An example has been provided to show that the obtained stability conditions are better than the ones obtained directly based on the standard Razumikhin-type condition. Obviously, the proposed method and techniques can be used for stabilization and control synthesis of various systems with time-varying delays.

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Appendix

Proof of Lemma 1. Let $\max_{\boldsymbol{w}\in U(K)}\{\boldsymbol{w}^*D_1^*PD_1\boldsymbol{w}\}=\gamma_1^2K$ with a constant $\gamma_1>0$. According to Cauchy-Schwarz inequality, for any given $\boldsymbol{u}\in U(K)$ and $\boldsymbol{v}_1\in U(K)$, we have $|\boldsymbol{u}^*PD_1\boldsymbol{v}_1|\le(\boldsymbol{u}^*P\boldsymbol{u}\boldsymbol{v}_1^*D_1^*PD_1\boldsymbol{v}_1)^{1/2}\le\gamma_1K$ so that there are two scalars $\xi_1,\psi_1\in[0,2\pi]$ and one nonnegative number $\beta_1\ge0$ such that

$$\begin{aligned}&\boldsymbol{u}^*PD_1\boldsymbol{v}_1+\boldsymbol{v}_1^*D_1^*P\boldsymbol{u}=\boldsymbol{u}^*P\boldsymbol{u}\gamma_1\mathrm{e}^{j\xi_1}+\mathrm{e}^{-j\xi_1}\gamma_1\boldsymbol{u}^*P\boldsymbol{u}=\\&\boldsymbol{u}^*PD_1\boldsymbol{u}\mathrm{e}^{\beta_1+j\psi_1}+\mathrm{e}^{\beta_1-j\psi_1}\boldsymbol{u}^*D_1^*P\boldsymbol{u}\end{aligned}\tag{A1}$$

Therefore, it is easy to derive that for any given $1+m$ vectors: $\boldsymbol{u}\in U(K)$ and $\boldsymbol{v}_k\in U(K)$ for $k=1,2,\cdots,m$, there are m scalars $\psi_k\in[0,2\pi]$ for $k=1,2,\cdots,m$ and m nonnegative numbers $\beta_k\ge0$ for $k=1,2,\cdots,m$ such that

$$\begin{aligned}&\boldsymbol{u}^*(PD_0+D_0^*P)\boldsymbol{u}+\sum_{k=1}^m\boldsymbol{u}^*PD_k\boldsymbol{v}_k+\sum_{k=1}^m\boldsymbol{v}_k^*D_k^*P\boldsymbol{u}\\
&=\boldsymbol{u}^*\left[P(D_0+D_1\mathrm{e}^{\beta_1+j\psi_1})+(D_0+D_1\mathrm{e}^{\beta_1+j\psi_1})^*P\right]\boldsymbol{u}+\\
&\sum_{k=2}^m\boldsymbol{u}^*PD_k\boldsymbol{v}_k+\sum_{k=2}^m\boldsymbol{v}_k^*D_k^*P\boldsymbol{u}=\\
&\boldsymbol{u}^*\left[P(D_0+\sum_{k=1}^2D_k\mathrm{e}^{\beta_k+j\psi_k})+(D_0+\right.\\
&\left.\sum_{k=1}^2D_k\mathrm{e}^{\beta_k+j\psi_k})^*P\right]\boldsymbol{u}+\sum_{k=3}^m\boldsymbol{u}^*PD_k\boldsymbol{v}_k+\sum_{k=3}^m\boldsymbol{v}_k^*D_k^*P\boldsymbol{u}=\\
&\vdots\\
&\boldsymbol{u}^*\left[P(D_0+\sum_{k=1}^{m-1}D_k\mathrm{e}^{\beta_k+j\psi_k})+(D_0+\right.\\
&\left.\sum_{k=1}^{m-1}D_k\mathrm{e}^{\beta_k+j\psi_k})^*P\right]\boldsymbol{u}+\boldsymbol{u}^*PD_m\boldsymbol{v}_m+\boldsymbol{v}_m^*D_m^*P\boldsymbol{u}=\\
&\boldsymbol{u}^*\left[P(D_0+\sum_{k=1}^{m}D_k\mathrm{e}^{\beta_k+j\psi_k})+(D_0+\right.\\
&\left.\sum_{k=1}^{m}D_k\mathrm{e}^{\beta_k+j\psi_k})^*P\right]\boldsymbol{u}\end{aligned}\tag{A2}$$

□

Proof of Lemma 2. The necessity is obvious. Let us prove the sufficiency. Since $T_k\in\mathbf{C}^{n\times n}$ for $k=0,1,\cdots,m$ are all Hermitian matrices, it is easy to see that for any $\boldsymbol{u}\in\mathbf{C}^n$, $\boldsymbol{u}^*T_0\boldsymbol{u}\in\mathbf{R}$ and $\beta_k\boldsymbol{u}^*T_k\boldsymbol{u}\in\mathbf{R}$ for $k=1,2,\cdots,m$ are all real numbers. Therefore, for any $\boldsymbol{u}\in\mathbf{C}^n$, (11) guarantees that

$$\begin{aligned}&\boldsymbol{u}^*T_0\boldsymbol{u}+\sum_{k=1}^m\beta_k\boldsymbol{u}^*T_k\boldsymbol{u}\le\boldsymbol{u}^*T_0\boldsymbol{u}+\sum_{k=1}^m\beta_{kM}|\boldsymbol{u}^*T_k\boldsymbol{u}|=\\
&\max_{l=1,2,\cdots,2^m}\left\{\boldsymbol{u}^*(T_0+\left[\sum_{k=1}^m\pm\beta_{kM}T_k\right]_l)\boldsymbol{u}\right\}\le-2\gamma\boldsymbol{u}^*P\boldsymbol{u}\end{aligned}\tag{A3}$$

for all $\beta_k\in[-\beta_{kM},\beta_{kM}]$, $k=1,2,\cdots,m$ so that (12) implies (11). The sufficiency is proved. □