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A Short Proof of Ramanujan's ${}_1\psi_1$ Formula

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Abstract In this paper, we give a short proof of the celebrated Ramanujan's ${}_1\psi_1$ formula.

Keywords q-binomial theorem; Ramanujan's ${}_1\psi_1$ formula; bilateral q-series.

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Among the classical hierarchy of basic hypergeometric summation formulas, one of fundamental identities is Ramanujan's ${}_1\psi_1$ sum[2, II. 29], which reads

Theorem 1 *For $|q|<1$ and $|b/a|<|x|<1$,*

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k=\frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(x;q)_\infty(b/ax;q)_\infty(b;q)_\infty(q/a;q)_\infty},\tag{1}$$

where the q-shifted factorial $(a;q)_n$ is defined as

$$(a;q)_n=\begin{cases}\prod_{k=0}^{n-1}(1-aq^k), & n>0;\\ 1, & n=0;\\ \frac{(-1/a)^{-n}q^{\binom{-n+1}{2}}}{(q/a;q)_{-n}}, & n<0.\end{cases}$$

In particular, $(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$.

As is well known to us, Ramanujan's ${}_1\psi_1$ sum is usually considered as a bilateral generalization of the following q-binomial theorem[2, II. 3]. We require it in our proof.

Lemma 1 *For $|x|<1, |q|<1$, and a parameter a,*

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k=\frac{(ax;q)_\infty}{(x;q)_\infty}.\tag{2}$$

Since Ramanujan's ${}_1\psi_1$ sum was first brought before the mathematical public by Hardy[3] in 1940 and first proved by Hahn[4] and Jackson[5] respectively, to find any possible elegant and simple proof of this identity has still been a charming problem in the theory of q-series. As of today, many such proofs have been published, which can be divided, in the authors' point

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of view, into two kinds of methods: analytic and combinatorial. For analytic proofs, we refer the reader to Andrews[6], Andrews and Askey[7], Askey[8], Adiga[9], Fine[11, Eq. (18.3)], Ismail[12], Mimachi[13], and Chan[14], Schlosser[15]. A good survey of various proofs of this formula built on some classical summation formulae in q-series can be found in Chu and Wang's recent paper[1]; For combinatorial proofs, we refer to Corteel and Lovejoy[16], Corteel[17], Joichi and Stanton[18], and Yee[19]. We especially recommend Kadell[20] for its probabilistic proof.

In what follows, we give another analytic proof of this identity, as the main result of this short paper. Since there is no evidence that it has been previously known in the literature, so we think it is new.

Proof Write that

$$F(a,x)=\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k.$$

At first, it is easy to verify directly that

$$(1-x)F(a,x)=\frac{b-a}{q-a}F(aq^{-1},qx)$$

or equivalently

$$F(a,x)=\frac{1-b/a}{(1-x)(1-q/a)}F(aq^{-1},qx).$$

Iterate this relation $n\geq 0$ times to arrive at

$$F(a,x)=\frac{(b/a;q)_n}{(x;q)_n(q/a;q)_n}F(aq^{-n},q^nx). \tag{3}$$

Let n tend to ∞ on both sides of this relation. By Tannery's theorem[10], we get

$$\lim_{n\to\infty}F(aq^{-n},q^nx)=\sum_{k=-\infty}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(ax)^k}{(b;q)_k}.$$

Observe that the above bilateral power series is absolutely convergent in the domain $|b/a|<|x|<\infty$ and its limitation is analytic in variable x. Define

$$H(x)=\sum_{k=-\infty}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}x^k}{(b;q)_k}.$$

Then from (3) it follows immediately that for $x:|b/a|<|x|<1$,

$$F(a,x)=\frac{(b/a;q)_\infty}{(x;q)_\infty(q/a;q)_\infty}H(ax). \tag{4}$$

Finally, set $a=1$ in (4) to get

$$H(x)=\frac{(x;q)_\infty(q;q)_\infty}{(b;q)_\infty}F(1,x). \tag{5}$$

By the q-binomial theorem, we have

$$F(1,x)=\sum_{k=0}^{\infty}\frac{(q/b;q)_k}{(q;q)_k}(b/x)^k=\frac{(q/x;q)_\infty}{(b/x;q)_\infty}. \tag{6}$$

As a consequence, we get

$$F(a,x) = \frac{(b/a;q)_\infty}{(x;q)_\infty (q/a;q)_\infty} H(ax)$$
$$= \frac{(b/a;q)_\infty}{(x;q)_\infty (q/a;q)_\infty} \frac{(ax;q)_\infty (q;q)_\infty}{(b;q)_\infty} \frac{(q/ax;q)_\infty}{(b/ax;q)_\infty}.$$

Remark In the above argument, we omit, for the sake of brevity, the parameter(s) b or a and b of the functions $F(a,x)$ and $H(x)$, although both of them, certainly, depend on b or a and b. Also, the q-binomial theorem can also be proved with the same argument, whereas it is regarded as a known fact in this paper, as described by Lemma 1.

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