Structural Eng./Earthquake Eng., JSCE, Vol.24, No.2, 84s-91s, 2007
(Copy of Dobokugakkai Ronbunsyuu A Vol.63 No.4, 685-692, 2007.10)

A DISCRETIZATION OF TAPERED BEAMS UP TO THE SECOND-ORDER NONLINEARITY *

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It is quite complicated for the linear differential equation of beam deflection to be expanded as to varying cross section. But, once a tapered beam element is solved, its nodal stiffness relations can be adopted into a general discrete analysis of framed structures. In this paper, the method of separation into rigid displacement and deformation, which has been developed in the geometrically nonlinear analysis, is found to have a fitness for dealing with the varying beam elements; and typical two types of tapered 2-D beams are discretized from their linear solutions into the geometric and second-order stiffness relations.

Key Words: *tapered beam, nonlinear discretization, geometric stiffness*

1. INTRODUCTION

In the preceding age of manual formulations, the beams with varying cross section were to be solved individually according to their own and external conditions: for instance, in case the moment of inertia of cross section varies in a power form of the axial coordinate, as identified by Timoshenko,[1)] the differential equation of beam-column had been carried out by means of Bessel functions; and the convoluted buckling equations of various non-uniform beams were listed in certain books such as Ref.2).

In the history of our discrete methods, the displacement method of framed structures was first established from the slope deflection equations of members, and which motivated the sequent FEM to be developed upon the interpolation of displacements. Let the following facts with the tapered beam elements be focused on: while there exist the stiffness matrices formulated through advanced interpolations,[e.g.3)] for the complexity, they have been regarded not to be derived from the formal solution of their differential equations. In particular, since the geometric stiffness matrix for a uniform beam was derived by Hartz,[4)] *nonlinear treatments have not been presented at all for the tapered ones*.

In this study, not directly on its total nodal freedom, the linear differential equations of a tapered beam are really solved for its deformation on the simple support, and to which the effects of displacement as a rigid body are adjoined. Then, to the same extent as the ordinary ones, the geometric stiffness matrix and the second-order stiffness relations are developed on the basis of the linear solution and the displacement as a rigid body.

* A main part of this paper has been presented at the JSCE Annual Conference,[8)] held in Tokyo, 2005-9.

2. LINEAR SOLUTION

Consider a planar beam element tapered such that its height h of cross section is linearly sloped along its axial coordinate x, with a constant width b (see **Fig.1**): area A and moment of inertia I of cross section in its length l are written as

$$A(x) = A_0(1+\alpha \frac{x}{l}) \ , \ I(x) = I_0(1+\alpha \frac{x}{l})^3 \quad \text{(1a,b)}$$

$$\left(h(x) = h_0(1+\alpha \frac{x}{l}) \right)$$

where suffix $(\ )_0$ indicates a quantity at $x=0$, and the varying rate lies in $\alpha > -1$.

Similarly to a uniform beam, the interpolation functions in the higher order discretizations will be set up from the linear solution. The usual expansions on the six nodal displacements are too complicated for a varying cross section, and so let our element (e) be subjected to the simple support shown in **Fig.1**. The linear solution is developed between the *deformation force* and *deformation*:

$$\boldsymbol{f}_{(e)} = \{H, M_A, M_B\} \Leftrightarrow \boldsymbol{\varepsilon}_{(e)} = \{w, \varphi_A, \varphi_B\}$$

With no external forces on the axial line, the bending moment $M(x)$ is in a linear distribution. By substitution of the above $I(x)$, the differential equation for deflection $v_G(x)$ is written as

$$\frac{EI_0}{l^2}(1+\alpha\xi)^3\frac{d^2v_G}{d\xi^2}\big(=-M(x)\big)=D+G\xi$$

where the axial coordinate has been normalized into $\xi=x/l$. When $(1+\alpha\xi)^3$ is shifted to the right-hand side, expression $(D+G\xi)/(1+\alpha\xi)^3$ with undetermined constants D and G can be rewritten into $A/(1+\alpha\xi)^3+B/(1+\alpha\xi)^2$. Thus we have

$$\frac{d^2v_G}{d\xi^2}=\frac{A}{(1+\alpha\xi)^3}+\frac{B}{(1+\alpha\xi)^2} \qquad (2)$$

The twice integration with the supporting conditions, $v_G(0)=0$ and $v_G(1)=0$, yields

$$v_G(\xi)=A\left\{\frac{1}{2\alpha^2(1+\alpha\xi)}+\frac{\xi}{2\alpha(1+\alpha)}-\frac{1}{2\alpha^2}\right\}+B\left\{-\frac{1}{\alpha^2}\log(1+\alpha\xi)+\frac{\log(1+\alpha)}{\alpha^2}\cdot\xi\right\}$$

Then, from compatibility $\theta(0)=\varphi_A$ and $\theta(1)=\varphi_B$, we have the equations to determine A and B for the deformation parameters:

$$\frac{A}{l}\left\{-\frac{1}{2(1+\alpha)}\right\}+\frac{B}{l}\left\{-\frac{1}{\alpha}+\frac{1}{\alpha^2}\log(1+\alpha)\right\}=\varphi_A$$

$$\frac{A}{l}\left\{\frac{1}{2(1+\alpha)^2}\right\}+\frac{B}{l}\left\{-\frac{1}{\alpha(1+\alpha)}+\frac{1}{\alpha^2}\log(1+\alpha)\right\}=\varphi_B$$

from which

$$A=\frac{2l}{C}\Big\{(\alpha(1+\alpha)-(1+\alpha)^2\log(1+\alpha))\varphi_A+(-\alpha(1+\alpha)^2+(1+\alpha)^2\log(1+\alpha))\varphi_B\Big\}$$

$$B=\frac{l}{C}\left\{\alpha^2\varphi_A+\alpha^2(1+\alpha)\varphi_B\right\}$$

where

$$C=-2\alpha+(2+\alpha)\log(1+\alpha) \qquad (3)$$

Eventually, the deflection $v_G(\xi)$ and slope $\theta(\xi)$ are obtained as follows:

$$v_G(\xi)=\Gamma_{vA}(\xi)\varphi_A+\Gamma_{vB}(\xi)\varphi_B$$
$$\theta(\xi)=\Gamma_{\theta A}(\xi)\varphi_A+\Gamma_{\theta B}(\xi)\varphi_B \qquad (4a,b)$$

in which $\Gamma_{\theta A}=1/l\cdot d\Gamma_{vA}/d\xi$, $\Gamma_{\theta B}=1/l\cdot d\Gamma_{vB}/d\xi$ and

$$\Gamma_{\theta A}(\xi)=\frac{1}{C}\left\{-\frac{\alpha}{1+\alpha\xi}+\left(-1+\frac{(1+\alpha)}{\alpha}\log(1+\alpha)\right)\cdot\frac{(1+\alpha)}{(1+\alpha\xi)^2}+\left(1-\frac{1}{\alpha}\log(1+\alpha)\right)\right\}$$

$$\Gamma_{\theta B}(\xi)=\frac{1}{C}\left\{-\frac{(1+\alpha)\alpha}{1+\alpha\xi}+\left(1-\frac{1}{\alpha}\log(1+\alpha)\right)\cdot\frac{(1+\alpha)^2}{(1+\alpha\xi)^2}+(1+\alpha)\left(-1+\frac{(1+\alpha)}{\alpha}\log(1+\alpha)\right)\right\} \qquad (5)$$

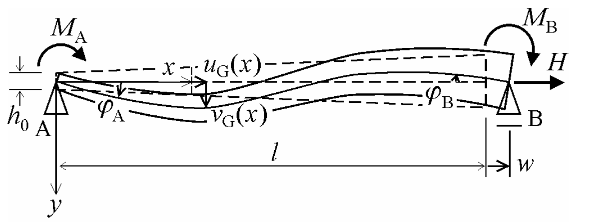

Fig.1 A tapered beam on the simple support

From curvature $\kappa(\xi)=1/l\cdot d\theta/d\xi$, the end moments corresponding to φ_A and φ_B are obtained by $M_A=-EI_0\kappa(0)$ and $M_B=EI_0(1+\alpha)^3\kappa(1)$:

$$\begin{Bmatrix}M_A\\M_B\end{Bmatrix}=\begin{bmatrix}k_{AA}&k_{AB}\\k_{BA}&k_{BB}\end{bmatrix}\begin{Bmatrix}\varphi_A\\\varphi_B\end{Bmatrix} \qquad (6a)$$

$$k_{AA}=\frac{EI_0}{Cl}\left\{2(1+\alpha)^2\log(1+\alpha)-\alpha(2+3\alpha)\right\}$$

$$k_{AB}=k_{BA}=\frac{EI_0}{Cl}\left\{-2(1+\alpha)^2\log(1+\alpha)+\alpha(1+\alpha)(2+\alpha)\right\} \qquad (6b)$$

$$k_{BB}=\frac{EI_0}{Cl}\left\{2(1+\alpha)^2\log(1+\alpha)-\alpha(1+\alpha)^2(2-\alpha)\right\}$$

The above result is achieved because of expansion on the reduced degrees of freedom.

Meanwhile, the axial force-elongation relation is already known, and which is described about our element (e). Submitted to the axial force H with cross-section area (1a), the differential relation of axial displacement $u_G(\xi)$ to H becomes

$$\frac{du_G}{d\xi}=\frac{Hl}{EA_0}\cdot\frac{1}{(1+\alpha\xi)}$$

By the integration with $u_G(0)=0$

$$u_G(\xi)=\frac{Hl}{EA_0}\cdot\frac{\log(1+\alpha\xi)}{\alpha} \qquad (7a)$$

from which, for $w=u_G(1)$

$$H=k_H w,\quad k_H=\frac{EA_0\alpha}{l\log(1+\alpha)} \qquad (7b)$$

Let the above *linear relations* between $\{H, M_A, M_B\}$ and $\{w, \varphi_A, \varphi_B\}$ be expressed in a matrix form

$$\boldsymbol{f}_{(e)}=\left[k_L^D\right]_{(e)}\boldsymbol{\varepsilon}_{(e)}$$

$$\left[k_L^D\right]_{(e)}=\begin{bmatrix}k_H&0&0\\&k_{AA}&k_{AB}\\\text{Sym.}&&k_{BB}\end{bmatrix} \qquad (8)$$

3. NONLINEAR DISCRETIZATION OF SECOND ORDER

The subsequent expansions are in accordance with Ref.5) for a straight beam of uniform cross section. So far as deformed with small strains in the

Bernoulli-Euler kinematic field, the axial and flexural displacements are compared in dimension as

$$\frac{u_G}{l} \approx \left(\frac{v_G}{l}\right)^2 \approx \left(\frac{h}{l}\right)^2 \approx \varepsilon \tag{9}$$

where ε is the magnitude of normal strains on cross section. In the formulation of second order, all the terms higher or equal to ε with respect to unity are neglected: the normal strain becomes to be expressed by

$$e_{xx}(x,y) = \frac{du_G}{dx} + \frac{1}{2}\left(\frac{dv_G}{dx}\right)^2 - y\frac{d^2 v_G}{dx^2} \tag{10}$$

and the axial force N and the bending moment M are related to those terms into

$$N(x) = EA(x)\left(\frac{du_G}{dx} + \frac{1}{2}\left(\frac{dv_G}{dx}\right)^2\right)$$

$$M(x) = -EI(x)\frac{d^2 v_G}{dx^2} \tag{11a,b}$$

On the simple support (**Fig.1**), the displacements $u_G(\xi)$ and $v_G(\xi)$ are interpolated in terms of $\{w, \varphi_A, \varphi_B\}$. The linear solution (4a) is directly valid to the deflection $v_G(\xi)$, but which is relatively large to $u_G(\xi)$. In case our beam (e) is bent by φ_A and φ_B without axial force, the flexural $v_G(\xi)$ is accompanied with $u_G(\xi)$ such that

$$\left(e_G(x) = e_{xx}(x,0) =\right)\frac{du_G}{dx} + \frac{1}{2}\left(\frac{dv_G}{dx}\right)^2 = 0$$

That is, since differential element dx is sloped by angle $\theta(x)=dv_G/dx$ with no elongation, its projective length onto the horizontal axis x is shortened by $(1-\cos\theta)dx \approx 1/2\cdot(dv_G/dx)^2 dx$, as shown in **Fig.2**. Let this secondary axial displacement be denoted by $u^*_G(x)$, which is expressed by the integration

$$u^*_G(x) = -\frac{1}{2}\int_0^x \left(\frac{dv_G}{dx}\right)^2 dx \tag{12}$$

For the stiffness relations, the above shortening needs to be expanded explicitly in the entire span. After substitution of Eqs.(4b) and (5), the integration is carried out as follows:

$$\Delta^D(\varphi_A, \varphi_B)\left(= \frac{l}{2}\int_0^1 \theta(\xi)^2 d\xi = \frac{l}{2}\left\{\varphi_A^2 \cdot \int_0^1 \Gamma_{\theta A}^2 d\xi + 2\varphi_A\varphi_B \cdot \int_0^1 \Gamma_{\theta A}\Gamma_{\theta B} d\xi + \varphi_B^2 \cdot \int_0^1 \Gamma_{\theta B}^2 d\xi\right\}\right)$$

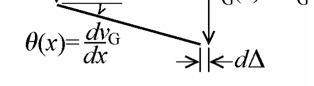

Fig.2 Pure deflection

$$= \frac{1}{2}\langle \varphi_A \ \ \varphi_B \rangle \left[\eta^D\right] \begin{Bmatrix} \varphi_A \\ \varphi_B \end{Bmatrix} \tag{13}$$

in which

$$\left[\eta^D\right] = \begin{bmatrix} \eta_{AA} & \eta_{AB} \\ \eta_{BA} & \eta_{BB} \end{bmatrix} \tag{14a-d}$$

$$\eta_{AA} = \frac{l}{3C^2}\left\{\left(-\frac{\alpha}{(1+\alpha)} + 7\alpha\right) + \left(-5\alpha^2 - 12\alpha\right) \cdot \left(\frac{\log(1+\alpha)}{\alpha}\right) + \left(\alpha^3 + 4\alpha^2 + 6\alpha\right)\left(\frac{\log(1+\alpha)}{\alpha}\right)^2\right\}$$

$$\eta_{AB} = \eta_{BA} = \frac{l(1+\alpha)}{3C^2}\left\{\left(\frac{2\alpha^2}{(1+\alpha)}\right) + \left(\frac{\alpha}{2(1+\alpha)} - \frac{\alpha}{2}(1+\alpha)\right) \cdot \left(\frac{\log(1+\alpha)}{\alpha}\right) - \alpha^2\left(\frac{\log(1+\alpha)}{\alpha}\right)^2\right\}$$

$$\eta_{BB} = \frac{l(1+\alpha)^2}{3C^2}\left\{\left(-\frac{7\alpha}{(1+\alpha)} + \alpha\right) + \left(\frac{5\alpha}{(1+\alpha)} + 7\alpha\right) \cdot \left(\frac{\log(1+\alpha)}{\alpha}\right) + \left(-\frac{\alpha}{(1+\alpha)} - 3\alpha^2 - 5\alpha\right)\left(\frac{\log(1+\alpha)}{\alpha}\right)^2\right\}$$

On the contrary, if the span is fixed at $w=0$ for the lateral $v_G(\xi)$, element (e) is elastically lengthened by Δ^D in the form (7a,b). For the actual $\{w, \varphi_A, \varphi_B\}$, our (e) is elongated by w as well in the same manner. The axial displacement thus becomes to be interpolated by

$$u_G(\xi) = \frac{\log(1+\alpha\xi)}{\log(1+\alpha)} \cdot \left(w + \Delta^D(\varphi_A, \varphi_B)\right) + u^*_G(\varphi_A, \varphi_B; \xi) \tag{15}$$

When this $u_G(\xi)$ is substituted into Eq.(10), the subsidiary $u^*_G(x)$ is trivially canceled, and we have the axial strain expressed by

$$e_G(\xi) = \frac{\alpha}{l\log(1+\alpha)(1+\alpha\xi)} \cdot \left(w + \Delta^D(\varphi_A, \varphi_B)\right) \tag{16}$$

Now the total potential energy of our beam subjected to external forces $\{H, M_A, M_B\}$ is given by

$$W(\boldsymbol{\varepsilon}_{(e)}) = l\int_0^1 \left[\frac{M^2}{2EI} + \frac{N^2}{2EA}\right] d\xi - \boldsymbol{f}_{(e)} \cdot \boldsymbol{\varepsilon}_{(e)} \tag{17}$$

Since $M(\xi)$ and $v_G(\xi)$ are in the same relations to $\{\varphi_A, \varphi_B\}$ as the linear solution, the integration for the bending strain energy becomes to coincide with the result from Eqs. (6) in **Sec.2**:

$$l\int_0^1 \left[\frac{M^2}{2EI}\right] d\xi = \frac{1}{2}\langle \varphi_A \ \ \varphi_B \rangle \begin{bmatrix} k_{AA} & k_{AB} \\ k_{BA} & k_{BB} \end{bmatrix} \begin{Bmatrix} \varphi_A \\ \varphi_B \end{Bmatrix} \tag{18}$$

Substituting the above $e_G(\xi)$ into Eq. (11a), we have the axial force $N(\xi)$ for $\{w, \varphi_A, \varphi_B\}$:

$$N=\frac{EA_0\alpha}{l\log(1+\alpha)}\left(w+\frac{1}{2}\langle\varphi_A\ \ \varphi_B\rangle\left[\eta^D\right]\begin{Bmatrix}\varphi_A\\ \varphi_B\end{Bmatrix}\right) \tag{19}$$

which is certainly constant along the axial line.

After substitution of Eqs. (18) and (19) into function (17), we have the stationary condition of $W(w, \varphi_A, \varphi_B)$ in the matrix form

$$\left(\delta W=\right)\langle\varphi_A\ \ \varphi_B\rangle\begin{bmatrix}k_{AA} & k_{AB}\\ k_{BA} & k_{BB}\end{bmatrix}\begin{Bmatrix}\delta\varphi_A\\ \delta\varphi_B\end{Bmatrix}$$
$$+N\left(\delta w+\langle\varphi_A\ \ \varphi_B\rangle\left[\eta^D\right]\begin{Bmatrix}\delta\varphi_A\\ \delta\varphi_B\end{Bmatrix}\right)-\boldsymbol{f}_{(e)}\cdot\delta\boldsymbol{\varepsilon}_{(e)}=0$$

By the arbitrariness of $\{\delta w, \delta\varphi_A, \delta\varphi_B\}$, the *stiffness relations of second order* are obtained as follows:

$$H(=N)=k_H\left(w+\frac{1}{2}\langle\varphi_A\ \ \varphi_B\rangle\left[\eta^D\right]\begin{Bmatrix}\varphi_A\\ \varphi_B\end{Bmatrix}\right)$$
$$\begin{Bmatrix}M_A\\ M_B\end{Bmatrix}=\left(\begin{bmatrix}k_{AA} & k_{AB}\\ k_{BA} & k_{BB}\end{bmatrix}+N\begin{bmatrix}\eta_{AA} & \eta_{AB}\\ \eta_{BA} & \eta_{BB}\end{bmatrix}\right)\begin{Bmatrix}\varphi_A\\ \varphi_B\end{Bmatrix} \tag{20a,b}$$

Further, by differentiating the above (20a,b), we have the *tangent stiffness matrix* written as

$$\delta\boldsymbol{f}_{(e)}=\left[k_T^D(\boldsymbol{\varepsilon})\right]_{(e)}\delta\boldsymbol{\varepsilon}_{(e)}$$
$$\left[k_T^D(\boldsymbol{\varepsilon})\right]_{(e)}=\begin{bmatrix}k_H & 0 & 0\\ & k_{AA} & k_{AB}\\ \text{Sym.} & & k_{BB}\end{bmatrix}+N\begin{bmatrix}0 & 0 & 0\\ & \eta_{AA} & \eta_{AB}\\ \text{Sym.} & & \eta_{BB}\end{bmatrix}$$
$$+k_H\begin{bmatrix}0 & d_A & d_B\\ & d_A^2 & d_A d_B\\ \text{Sym.} & & d_B^2\end{bmatrix} \tag{21a,b}$$

in which

$$\begin{Bmatrix}d_A\\ d_B\end{Bmatrix}=\left[\eta^D\right]\begin{Bmatrix}\varphi_A\\ \varphi_B\end{Bmatrix} \tag{21c}$$

By the integration of $\delta U_{(e)}=H\delta w+M_A\delta\varphi_A+M_B\delta\varphi_B$, the *strain energy* of element (e) is expressed by

$$U(\boldsymbol{\varepsilon})_{(e)}=\frac{k_H}{2}\left(w+\Delta^D\right)^2$$
$$+\frac{1}{2}\langle\varphi_A\ \ \varphi_B\rangle\begin{bmatrix}k_{AA} & k_{AB}\\ k_{BA} & k_{BB}\end{bmatrix}\begin{Bmatrix}\varphi_A\\ \varphi_B\end{Bmatrix} \tag{22}$$

The above stiffness relations for deformation $\boldsymbol{\varepsilon}_{(e)}$ are sufficient for the geometrically nonlinear analysis of a framed structure: with a segmentation into short elements, any large displacements yielding small strains can be dealt with through the *method of separation into rigid displacement and deformation*.[e.g.6)]

In Ref. 5), meanwhile, a beam of uniform cross section is discretized to the second order directly between the *nodal forces* and *displacements*:

$$\{\boldsymbol{F}\}_{(e)}=\left\{\left(F_x, F_y, M\right)_i\ \left(F_x, F_y, M\right)_j\right\}$$
$$\{\boldsymbol{u}\}_{(e)}=\left\{\left(u, v, \theta\right)_i\ \left(u, v, \theta\right)_j\right\} \tag{23a,b}$$

The corresponding relations can be derived from our results (20) to (22). After the rotation as a rigid body is defined by the *chord angle*

$$\tau=\arctan\left(\frac{v_j-v_i}{l+u_j-u_i}\right)$$
$$\left(=\frac{v_j-v_i}{l}-\frac{1}{l^2}(v_j-v_i)(u_j-u_i)+0\left(\{\boldsymbol{u}\}_{(e)}^3\right)\right),$$

deformations $\{w, \varphi_A, \varphi_B\}$ (see **Fig.3** without P) are geometrically related to $\{\boldsymbol{u}\}_{(e)}$ as

$$\varphi_A=\theta_i-\tau\,,\quad \varphi_B=\theta_j-\tau$$
$$w=\sqrt{(l+u_j-u_i)^2+(v_j-v_i)^2}-l$$
$$\left(=(u_j-u_i)+\frac{1}{2l}\cdot(v_j-v_i)^2+0\left(\{\boldsymbol{u}\}_{(e)}^3\right)\right)$$

Then, neglecting the higher-order terms with regard to dimensional relation (9), we have

$$\tau=\frac{v_j-v_i}{l}$$
$$w=(u_j-u_i)+\frac{1}{2l}\cdot(v_j-v_i)^2 \tag{24a-d}$$
$$\varphi_A=\theta_i-\frac{v_j-v_i}{l}\,,\quad \varphi_B=\theta_j-\frac{v_j-v_i}{l}$$

Through the stiffness equations (20a,b), deformation forces $\{H, M_A, M_B\}$ are estimated for $\{(u, v, \theta)_i, (u, v, \theta)_j\}$.

The nodal forces $\{(F_x, F_y, M)_i, (F_x, F_y, M)_j\}$ are determined by the static equilibrium to $\{H, M_A, M_B\}$: they are kept in the equation of virtual work

$$\boldsymbol{f}_{(e)}^T\delta\boldsymbol{\varepsilon}_{(e)}=\{\boldsymbol{F}\}_{(e)}^T\delta\{\boldsymbol{u}\}_{(e)}$$

where the differentials in Eqs. (24b-d) are written as

$$\delta\boldsymbol{\varepsilon}_{(e)}=\left[Q_X(\boldsymbol{u})\right]_{(e)}\delta\{\boldsymbol{u}\}_{(e)}$$
$$\left[Q_X(\boldsymbol{u})\right]_{(e)}=\begin{bmatrix}-1 & -\tau & 0 & 1 & \tau & 0\\ 0 & \frac{1}{l} & 1 & 0 & -\frac{1}{l} & 0\\ 0 & \frac{1}{l} & 0 & 0 & -\frac{1}{l} & 1\end{bmatrix} \tag{25}$$

By the arbitrariness of $\delta\{\boldsymbol{u}\}_{(e)}$, we have

$$\{\boldsymbol{F}\}_{(e)}=\left[Q_X(\boldsymbol{u})\right]_{(e)}^T\boldsymbol{f}_{(e)} \tag{26}$$

The stiffness relations of second order between $\{\boldsymbol{F}\}_{(e)}$ and $\{\boldsymbol{u}\}_{(e)}$ are obtained by joining the above relations (24) to (26) with the preceding (20) to (22).

4. GEOMETRIC STIFFNESS MATRIX FOR AXIAL FORCE

As well known, the geometric stiffness matrix

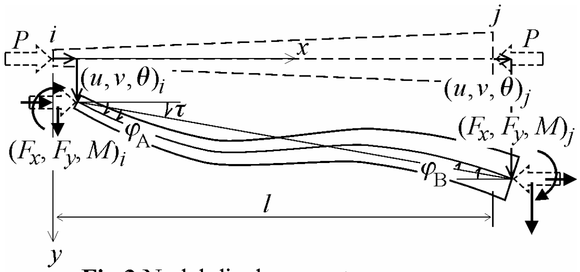

Fig.3 Nodal displacements from a state of axial compression

due to axial force is in a practical usage to deal with the buckling state of framed structures. That linearized effects are here developed for the tapered beam element, directly on the total nodal freedom.

Consider that our tapered element (e) has been in a state of axial compression P, and is then subjected to additional nodal forces $\{\boldsymbol{F}\}_{(e)}$ and displacements $\{\boldsymbol{u}\}_{(e)}$, as shown in **Fig.3**. In so far as $\{\boldsymbol{u}\}_{(e)}$ are small enough, their geometrical and mechanical relations are kept linear to each other. That is, the present $\{\boldsymbol{u}\}_{(e)}$, τ and $\boldsymbol{\varepsilon}_{(e)}$ correspond to $\delta\{\boldsymbol{u}\}_{(e)}$, $\delta\tau$ and $\delta\boldsymbol{\varepsilon}_{(e)}$ upon $\{(v,\theta)_i, (v,\theta)_j\}=\{0,0,0,0\}$ in **Sec. 3**: from differential relations (25)

$$\boldsymbol{\varepsilon}_{(e)} = \left[Q_{0X}\right]_{(e)}\{\boldsymbol{u}\}_{(e)}, \quad \tau = \frac{v_j - v_i}{l}$$

$$\left[Q_{0X}\right]_{(e)} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{l} & 1 & 0 & -\frac{1}{l} & 0 \\ 0 & \frac{1}{l} & 0 & 0 & -\frac{1}{l} & 1 \end{bmatrix} \qquad \text{(27a-c)}$$

Then displacement $u_G(\xi)$ and $v_G(\xi)$ of the axial line are interpolated by superposing the displacements as a rigid body to the linear solutions of deformation in **Sec. 2**:

$$u_G(\xi) = u_i + \frac{\log(1+\alpha\xi)}{\log(1+\alpha)}\cdot w \qquad \text{(28a,b)}$$

$$v_G(\xi) = v_i + \xi\left(v_j - v_i\right) + \Gamma_{vA}(\xi)\varphi_A + \Gamma_{vB}(\xi)\varphi_B$$

$$\left(\ \theta(\xi) = \tau + \Gamma_{\theta A}(\xi)\varphi_A + \Gamma_{\theta B}(\xi)\varphi_B\ \right)$$

From the sate of axial compression P, additional axial force $N(x)$ and bending moment $M(x)$ are linearly related to axial elongation $e_G=1/l\cdot du_G/d\xi$ and curvature $\kappa=1/l\cdot d\theta/d\xi$ by $N=EAe_G$ and $M=-EI\kappa$, respectively. Thus, based on the preceding configuration, the total potential energy is written as

$$W(\{\boldsymbol{u}\}_{(e)}) = \int_0^l \left[\frac{M^2}{2EI} + \frac{(-P+N)^2}{2EA}\right]dx - \left\{(P+F_x)u + F_y v + M\theta\right\}_i - \left\{(-P+F_x)u + F_y v + M\theta\right\}_j \qquad \text{(29)}$$

In the linear relations between $\{\boldsymbol{F}\}_{(e)}$ and $\{\boldsymbol{u}\}_{(e)}$, the energy quantities are to be developed up to the quadratic terms of increments.

In the above (29), let us focus on the integration

$$\int_0^l \frac{(-P+N)^2}{2EA}dx = \int_0^l \frac{N^2}{2EA}dx - \int_0^l \frac{PN}{EA}dx + \int_0^l \frac{P^2}{2EA}dx$$

The first term is sufficed by the linear expression of N. But, while the third term itself is a constant, as for the second term with that P, the increment N has to be expanded up to the quadratic terms. By applying the 2nd-order expression (11a) for N, **Sec.3**, we have

$$-\int_0^l \frac{PN}{EA}dx = -P\int_0^l \left(\frac{du_G}{dx} + \frac{1}{2}\left(\frac{dv_G}{dx}\right)^2\right)dx$$

$$= -P\left(u_G(l) - u_G(0)\right) - \frac{P}{2}\int_0^l \left(\frac{dv_G}{dx}\right)^2 dx$$

It is seen that, in the total potential (29), the above first term is canceled by the external Pu_i and Pu_j, and that the second term indicates an external potential of the preceding P in the span shortening due to deflection $v_G(x)$.

For the interpolation of $v_G(\xi)$ by Eq.(28b), the span shortening Δ is written as

$$\Delta = \frac{l}{2}\int_0^1 \left(\tau + \Gamma_{\theta A}(\xi)\varphi_A + \Gamma_{\theta B}(\xi)\varphi_B\right)^2 d\xi$$

$$= \frac{l}{2}\int_0^1 \tau^2 d\xi + l\int_0^1 (\tau\Gamma_{\theta A}(\xi)\varphi_A + \tau\Gamma_{\theta B}(\xi)\varphi_B)d\xi$$

$$+ \frac{l}{2}\int_0^1 (\Gamma_{\theta A}(\xi)\varphi_A + \Gamma_{\theta B}(\xi)\varphi_B)^2 d\xi$$

In this expression, since $\Gamma_{\theta A}(\xi)$ and $\Gamma_{\theta B}(\xi)$ are the shape functions for end slope φ_A and φ_B on the simple support (see **Fig.1**), their integrations over the entire span result into zero, $l\int_0^1\Gamma_{\theta A}d\xi(=\Gamma_{vA}(1))=0$ and $l\int_0^1\Gamma_{\theta B}d\xi(=\Gamma_{vB}(1))=0$; and the third term is identical to the expression by (13) and (14). It thus appears that rotation τ as a rigid body and deformation by $\{\varphi_A, \varphi_B\}$ are not correlated for Δ, and, with substitution of Eqs.(27a-c), the span shortening by defection $v_G(\xi)$ is developed in terms of $\{\boldsymbol{u}\}_{(e)}$ into

$$\Delta\left(= \frac{l}{2}\tau^2 + \frac{1}{2}\langle\varphi_A\ \ \varphi_B\rangle\left[\eta^D\right]\begin{Bmatrix}\varphi_A \\ \varphi_B\end{Bmatrix}\right)$$

$$= \frac{1}{2}\{\boldsymbol{u}\}_{(e)}^T\left[\eta\right]_{(e)}\{\boldsymbol{u}\}_{(e)} \qquad \text{(30a,b)}$$

$$\left[\eta\right]_{(e)} = \begin{bmatrix} 0 & 0 & 0 \\ & \frac{\eta_{AA}+2\eta_{AB}+\eta_{BB}}{l^2} + \frac{1}{l}, & \frac{\eta_{AA}+\eta_{AB}}{l}, \\ & & \eta_{AA}, \\ & \text{Sym.} & \end{bmatrix}$$

$$
\left.\begin{matrix}
0 & 0 & 0 \\
0 & -\dfrac{\eta_{AA}+2\eta_{AB}+\eta_{BB}}{l^2}-\dfrac{1}{l}, & \dfrac{\eta_{AB}+\eta_{BB}}{l} \\
0 & -\dfrac{\eta_{AA}+\eta_{AB}}{l}, & \eta_{AB} \\
0 & 0 & 0 \\
 & \dfrac{\eta_{AA}+2\eta_{AB}+\eta_{BB}}{l^2}+\dfrac{1}{l}, & -\dfrac{\eta_{AB}+\eta_{BB}}{l} \\
 & & \eta_{BB}
\end{matrix}\right]
$$

As for the square terms of M and N in Eq.(29), since both M and N are enough in the linear relations to $\{w, \varphi_A, \varphi_B\}$, their integrations become to coincide with the result from the linear solution (8): through the geometrical relations (27a,c)

$$
\int_0^l \left[\frac{M^2}{2EI}+\frac{N^2}{2EA}\right]dx \left(=\frac{1}{2}\boldsymbol{\varepsilon}_{(e)}^{\mathrm{T}}\left[k_{\mathrm{L}}^{\mathrm{D}}\right]_{(e)}\boldsymbol{\varepsilon}_{(e)}\right.
$$

$$
\left.=\frac{1}{2}\{\boldsymbol{u}\}_{(e)}^{\mathrm{T}}[Q_{0\mathrm{X}}]_{(e)}^{\mathrm{T}}\left[k_{\mathrm{L}}^{\mathrm{D}}\right]_{(e)}[Q_{0\mathrm{X}}]_{(e)}\{\boldsymbol{u}\}_{(e)}\right)
$$

$$
=\frac{1}{2}\{\boldsymbol{u}\}_{(e)}^{\mathrm{T}}\left[k_{\mathrm{L}}\right]_{(e)}\{\boldsymbol{u}\}_{(e)} \qquad (31\mathrm{a,b})
$$

$$
\left[k_{\mathrm{L}}\right]_{(e)}=\left[\begin{matrix}
k_H, & 0 & 0 \\
 & \dfrac{k_{AA}+2k_{AB}+k_{BB}}{l^2}, & \dfrac{k_{AA}+k_{AB}}{l}, \\
 & & k_{AA}, \\
 & \text{Sym.} &
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-k_H, & 0 & 0 \\
0 & -\dfrac{k_{AA}+2k_{AB}+k_{BB}}{l^2}, & \dfrac{k_{AB}+k_{BB}}{l} \\
0 & -\dfrac{k_{AA}+k_{AB}}{l}, & k_{AB} \\
k_H, & 0 & 0 \\
 & \dfrac{k_{AA}+2k_{AB}+k_{BB}}{l^2}, & -\dfrac{k_{AB}+k_{BB}}{l} \\
 & & k_{BB}
\end{matrix}\right]
$$

Eventually, the total potential is expressed in terms of $\{\boldsymbol{u}\}_{(e)}$ as follows:

$$
W(\{\boldsymbol{u}\}_{(e)})=\frac{1}{2}\{\boldsymbol{u}\}_{(e)}^{\mathrm{T}}\left[k_{\mathrm{L}}\right]_{(e)}\{\boldsymbol{u}\}_{(e)}
-\frac{P}{2}\{\boldsymbol{u}\}_{(e)}^{\mathrm{T}}[\eta]_{(e)}\{\boldsymbol{u}\}_{(e)}-\{F\}_{(e)}^{\mathrm{T}}\{\boldsymbol{u}\}_{(e)}+\text{const.} \qquad (32)
$$

From the stationary conditions of function $W(\{\boldsymbol{u}\}_{(e)})$, the stiffness matrix between $\{\boldsymbol{F}\}_{(e)}$ and $\{\boldsymbol{u}\}_{(e)}$ is obtained in the ordinary form, that is, in the sum of the *linear elastic* $[k_{\mathrm{L}}]_{(e)}$ and the *geometric* $[k_{\mathrm{G}}]_{(e)}$:

$$
\{\boldsymbol{F}\}_{(e)}=\left(\left[k_{\mathrm{L}}\right]_{(e)}+\left[k_{\mathrm{G}}\right]_{(e)}\right)\{\boldsymbol{u}\}_{(e)} \qquad (33)
$$

where $[k_{\mathrm{L}}]_{(e)}$ has been given by Eq.(31b), and the geometric stiffness matrix is associated with $[\eta]_{(e)}$ of Eqs. (30a,b) as

$$
\left[k_{\mathrm{G}}\right]_{(e)}=-P[\eta]_{(e)} \qquad (34)
$$

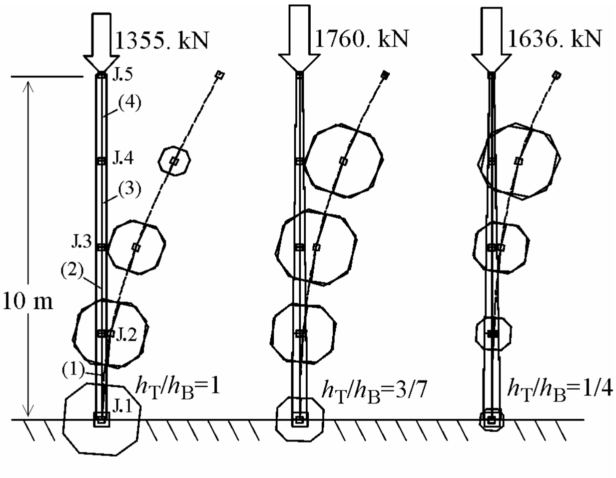

Fig. 4 Columns tapered in cross-section height

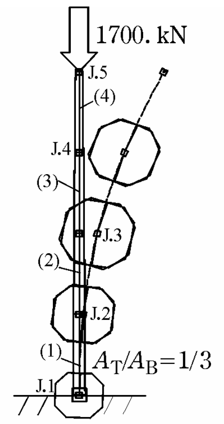

Fig. 5 Both h and b tapered

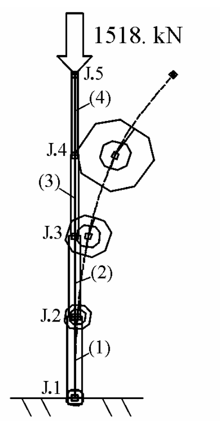

Fig. 6 Stepped assembly

The peculiarities of the tapered beam are represented only by the coefficients $\{k_H, k_{AA}, k_{AB}(=k_{BA}), k_{BB}\}$ and $\{\eta_{AA}, \eta_{AB}(=\eta_{BA}), \eta_{BB}\}$, and from which matrix $[k_{\mathrm{L}}]_{(e)}$ and $[\eta]_{(e)}$ on the total nodal freedom are composed in common with all the straight beams such as a uniform one. It has been our case where the height of cross section is sloped. In **Appendix-A**, the components of $[k_{\mathrm{L}}^{\mathrm{D}}]_{(e)}$ and $[\eta^{\mathrm{D}}]$ are expanded for another typical case where both the height and width are tapered.

5. NUMERICAL EXAMPLE

(1) Tapered columns with one end built-in

For a fundamental example, a uniform column of rectangular cross section is first considered by b_0 =40, h_0=20cm, l=10m and E=20600kN/cm^2, and from which tapered ones are assumed without change of volume and length l. With the bottom end clamped, their buckling loads at the top end are

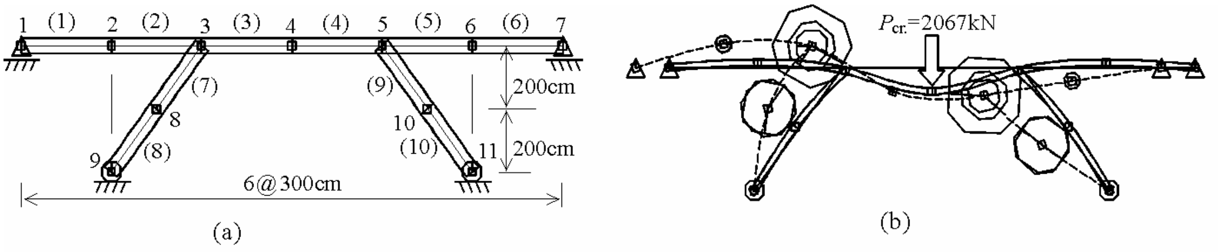

Fig.7 A slant-leg frame of uniform cross-section. (a) Member framing. (b) Buckling mode and preceding deformation

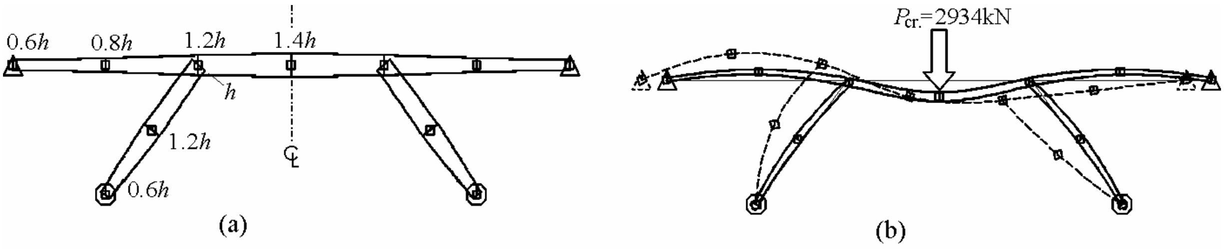

Fig.8 Modified into varying cross-sections. (a) Arrangement of heights. (b) Result of buckling

computed from the eigenvalue problem between the elastic and geometric stiffness matrices, **Sec. 4**.

With four elements of equal length, $P_{cr.}$=1355 kN is obtained for the uniform one. When the height of cross section is straightly sloped as $\{h_T, h_B\}= h_0\{1-\beta, 1+\beta\}$, β=0.1, 0.2,$\cdots$, 0.6, the buckling load is maximized to $P_{cr.}$=1760kN at β=0.4, and is decreased over that ratio. Shown in **Fig.4** are three of them, in which, from curvature κ of the buckling mode, the rate of sectional deformation is estimated by nondimensional $|\kappa|\sqrt{I/A}$, and that relative magnitudes between joints in each column are indicated by circumradius of the octagons: for the uniform cross section, the buckling deformation is concentrated in the lower elements; and, for β=0.6, in the upper elements. In case both *b* and *h* are tapered by $\{A_T, A_B\}=A_0\{1-\gamma, 1+\gamma\}$, γ=0.1, 0,2,$\cdots$, 0.7, the maximum load is found as $P_{cr.}$=1700kN at γ=0.5, with the buckling mode shown in **Fig.5**.

The preceding result for h_T/h_B=1/4 is compared with the conventional treatment into a stepped assembly of uniform elements, to which the heights at mid points of our tapered ones are given: $h_{(1)}$=29, $h_{(2)}$=23, $h_{(3)}$=17 and $h_{(4)}$=11cm. As shown in **Fig.6**, with different deformation rates between the adjoining cross-sections, the critical load is decreased from 1636 to 1518kN.

(2) π-shaped frame

The one-story frame shown in **Fig.7** is in a uniform cross-section: A=37.66, I=4050 cm^4 and E=206 GPa. For vertical loading at joint 4, the frame is well deformed prior to the buckling, and so to which the nonlinear stiffness relations of second order are applied (Eqs. (20) to (26) in **Sec.3**). On the equilibrium path computed in an iteration method,[7)] the sway-mode buckling is approached at $P_{cr.}$= 2067 kN, after certain displacements such as v_4=77.4 cm.

The frame is then considered in varying heights of cross section to improve the load-carrying capacity. With reference to the rates of buckling deformation in **Fig.7(b)**, the heights are arranged as shown in **Fig.8(a)**, without change of the total volume. Through the same nonlinear analysis, it is found that the buckling load is raised up to $P_{cr.}$=2934 kN with v_4=55.3 cm.

Let the above two bucklings in the nonlinear analysis be contrasted with the results from the linear eigenvalue problem: $P'_{cr.}$=2240 (v_4=53.2) for the first one, and $P'_{cr.}$=3122 kN (v_4=40.6 cm) for the tapered one.

6. CONCLUSION

It is proper for the discrete relations of a beam element to be developed from the linear solution of its ordinary differential equations, but, for a varying cross section, the direct expansion on the total nodal freedom is extremely complicated. In this study, it is found that the linear differential equations can be actually solved for the reduced freedom of deformation $\{w, \varphi_A, \varphi_B\}$ on the simple support.

When their geometrical magnitudes and effects are estimated in the respective nonlinear discretizations, the consistent shape functions can be composed from the linear solution and the displacement as a rigid body. Through our actual formulation, the following facts are disclosed: a tapered straight beam is distinguished from others only by the components of $[k_L^D]_{(e)}$ and $[\eta^D]$ for its deformation; and from which the geometric and second-order stiffness relations on the total freedom are constituted in the same form with all the straight beams.

APPENDIX A. TAPERED BEAMS IN BOTH DIMENSIONS OF CROSS-SECTION

Consider a straight beam such that its height and width of cross-section are tapered at an equal rate α: area $A(x)$ and moment of inertia $I(x)$ are written as

$$A(x) = A_0 (1+\alpha \frac{x}{l})^2, \quad I(x) = I_0 (1+\alpha \frac{x}{l})^4 \qquad \text{(A1a,b)}$$

Then, instead of Eq.(2) in **Sec.2**, the differential equation for deflection $v_G(x)$ becomes

$$\frac{d^2 v_G}{d\xi^2} = \frac{A}{(1+\alpha\xi)^4} + \frac{B}{(1+\alpha\xi)^3} \qquad \text{(A2)}$$

After the twice integrations on the simple support (**Fig.1**), constant A and B are related to φ_A and φ_B through compatibility $\theta(0)=\varphi_A$ and $\theta(1)=\varphi_B$: in the same expressions (4a,b), the shape functions are obtained as

$$\Gamma_{\theta A}(\xi)\left(=\frac{1}{l}\cdot\frac{d\Gamma_{vA}}{d\xi}\right) = \frac{1+\alpha}{\alpha^2}\left\{\frac{2(1+\alpha)}{(1+\alpha\xi)^3} - \frac{(3+\alpha)}{(1+\alpha\xi)^2} + \frac{1}{1+\alpha}\right\}$$

$$\Gamma_{\theta B}(\xi)\left(=\frac{1}{l}\cdot\frac{d\Gamma_{vB}}{d\xi}\right) = \frac{(1+\alpha)^2}{\alpha^2}\left\{\frac{2(1+\alpha)}{(1+\alpha\xi)^3} - \frac{(3+2\alpha)}{(1+\alpha\xi)^2} + 1\right\} \qquad \text{(A3a,b)}$$

The end moments M_A and M_B are determined from the curvature $\kappa(\xi)=1/l \cdot d\theta/d\xi$ at $\xi=0$ and $\xi=1$: the components in the bending stiffness matrix (6a) are

$$k_{AA} = \frac{4EI_0}{l}(1+\alpha), \quad k_{AB} = k_{BA} = \frac{2EI_0}{l}(1+\alpha)^2$$

$$k_{BB} = \frac{4EI_0}{l}(1+\alpha)^3 \qquad \text{(A4a-c)}$$

When the beam of cross-section area (A1a) is subjected to the axial force H at the roller end, the displacement $u_G(\xi)$ is in the differential equation

$$\frac{EA_0}{l}(1+\alpha\xi)^2 \frac{du_G}{d\xi} = H \text{ : const.}$$

By the integration with $u_G(0)=0$

$$u_G(\xi) = \frac{Hl}{EA_0} \cdot \frac{\xi}{1+\alpha\xi}$$

from which, for $w=u_G(1)$

$$H = k_H w, \quad k_H = \frac{EA_0}{l}(1+\alpha) \qquad \text{(A5)}$$

The span shortening Δ^D due to deflection $v_G(x)$ is obtained by substitution of the above $\Gamma_{\theta A}(\xi)$ and $\Gamma_{\theta B}(\xi)$ into the integration (13): in the same expression (14)

$$\eta_{AA} = \frac{2l}{15(1+\alpha)}, \quad \eta_{AB} = \eta_{BA} = -\frac{l}{30}, \quad \eta_{BB} = \frac{2l(1+\alpha)}{15} \qquad \text{(A6a-c)}$$

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(Received September 26, 2006)