Stability of Neutral Delay Differential Equations —— Boundary Criteria

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Abstract: We are concerned with the asymptotic stability of the Neutral Delay Differential Equation $x'(t) = Lx(t) + Mx(t-\tau) + Nx'(t-\tau)$, where L, M, and $N \in C^{d\times d}$ are constant complex matrices and $\tau > 0$ stands for a constant delay. We obtain two criteria through the evaluation of a harmonic function on the boundary of a cerain region.

Key words: eigenvalue; matrix norm; spectral radius; boundary criteria; asymptotic stability; harmonic function; logarithmic norm

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1 Introduction

Let W denote a bounded region in the complex plane. The symbols ∂W and $\overline{W}$ represent the boundary and the closure of W, respectively. That is, $\overline{W} = \partial W \cup W$, and

$$f(z) = f(x,y) = u(x,y) + iv(x,y), \tag{1}$$

is an arbitrary analytical function for $z \in \overline{W}$. Here, we adopt the notations $i^2 = -1$, $z = x + iy$, $u(x,y) = \mathrm{Re} f(z)$, and $v(x,y) = \mathrm{Im} f(z)$.

We consider the question of whether there exist zeros of $f(z)$ for $z \in \overline{W}$. The following two theorems give sufficient conditions for the non-existence of zeros of $f(z) \in \overline{W}$.

Theorem 1[1] If for any $(x,y) \in \partial W$, the real part $u(x,y)$ in (1) does not vanish, then $f(x,y) \neq 0$, for any $(x,y) \in W$.

Theorem 2[1] Assume that for any $(x,y) \in \partial W$, there exists a real constant λ satisfying $u(x,y) + \lambda v(x,y) \neq 0$. Then

$$f(z) = u(x,y) + iv(x,y) \neq 0, \text{ for any } (x,y) \in \overline{W}.$$

Theorem 2 is an extension of Theorem 1. These two theorems only require the evaluation on the boundary ∂W of the harmonic functions corresponding to f. Hence they are called boundary criteria.

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2 Delay Independent Stability of NDDEs

Now we deal with the asymptotic stability of NDDEs.

$$x'(t) = Lx(t) + Mx(t-\tau) + Nx'(t-\tau),\ \tau > 0, \tag{2}$$

where L, M, and $N \in C^{d\times d}$ are constant complex matrices and $\tau > 0$ stands for a constant delay.

For the stability of the system (2), we investigate its characteristic equation

$$P(z) = \det[zI - L - Me^{-z\tau} - zNe^{-z\tau}] = 0, \tag{3}$$

where z is a root of the equation.

The above characteristic equation (3) may be written as

$$\det[zI - L - Me^{-z\tau} - zNe^{-z\tau}] = U(x,y) + iV(x,y), \tag{4}$$

where $z = x + iy$.

The following two lemmata are well-known.

Lemma 1[3] If the real parts of all the characteristic roots of (3) are less than zero, then the system (2) is asymtotically stable; that is, every solution $x(t)$ of (2) satisfies $x(t) \to 0$ as $t \to \infty$.

Lemma 2[2] Let $A \in C^{d\times d}$ and $B \in R^{d\times d}$. If the inequality $|A| \leqslant B$ holds, then the inequality $\rho(A) \leqslant \rho(B)$ is valid. Here the order relation of matrices of the same dimensions should be interpreted componentwise. $|A|$ stands for the matrix whose component is replaced by the modulus of the corresponding component of A, and $\rho(A)$ means the spectral radius of A.

For a complex matrix W, let $\mu(W)$ be the logarithmic norm of W.

$$\mu(W) = \lim_{\Delta\to 0^+} \frac{\|I + \Delta W\| - 1}{\Delta}.$$

$\mu(W)$ depends on the chosen matrix norm. Let $\|W\|$ denote the matrix norm of W subordinate to a certain vector norm. In order to specify the norm, the notation $\|\cdot\|_p$ is used. And the notation $\mu_p(\cdot)$ is also adopted to denote the logarithmic norm associated with $\|\cdot\|_p$.

Lemma 3[2] For each eigenvalue of a matrix $W \in C^{d\times d}$, the inequality

$$-\mu_p(-W) \leqslant \mathrm{Re}\lambda_i(W) \leqslant \mu_p(W)$$

holds.

We have the following results[2]

$$\mu_1(W) = \max_k\Big[\mathrm{Re}(\omega_{kk}) + \sum_{i,i\neq k} |\omega_{ik}|\Big],$$

$$\mu_2(W) = \frac{1}{2}\max_i[\lambda_i(W + W^*)],$$

$$\mu_\infty(W) = \max_i\Big[\mathrm{Re}(\omega_{ii}) + \sum_{k,k\neq i} |\omega_{ik}|\Big].$$

Here, " $*$ " denotes the conjugation symbol.

The following lemma states a sufficient condition for the stability of (2).

Lemma 4 Let $\|N\| < 1$. If the condition

$$\mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|} < 0 \tag{5}$$

holds, the system (2) is asymptotically stable.

Proof Assume that the condition of the lemma is satisfied and that the system (2) is unstable. There is a root z of $P(z)$ satisfying $\mathrm{Re}(z) \geqslant 0$.

Note that z is also an eigenvalue of the matrix $L + Me^{-z\tau} + zNe^{-z\tau}$. The inner product

$$z = \langle Lx, x\rangle + \langle Mx, x\rangle e^{-z\tau} + z\langle Nx, x\rangle e^{-z\tau}, \text{ where } x \in C^d,\ \|x\| = 1,$$

implies

$$|z| \leqslant \|L\| + \|M\| + |z|\cdot\|N\|,$$

or

$$|z| \leqslant \frac{\|L\| + \|M\|}{1 - \|N\|}. \tag{6}$$

Applying the properties of the logarithmic norm and lemma 3, we have the following inequalities:

$$0 \leqslant \mathrm{Re}(z) \leqslant \mu(L + Me^{-z\tau} + zNe^{-z\tau}) = \lim_{\Delta\to 0^+}\frac{\|I + \Delta(L + Me^{-z\tau} + zNe^{-z\tau})\| - 1}{\Delta} \leqslant$$

$$\mu(L) + \|M\| + |z|\cdot\|N\| \leqslant \mu(L) + \|M\| + \|N\|\cdot\left(\frac{\|L\| + \|M\|}{1 - \|N\|}\right) =$$

$$\mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}.$$

This, however, contradicts the condition (5). Hence the proof is completed.

The following Theorem 3 gives a region including all the roots of (3) with nonnegative real parts when the condition of Lemma 4 fails.

Theorem 3 Let $\|N\| < 1$. Suppose that there exists a root of (3) whose real part is nonnegative.

(i) If we have the estimation

$$\mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|} > 0,$$

the inequalities

$$0 \leqslant \mathrm{Re}(z) \leqslant \mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}$$

and

$$\mu(iL) - \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|} \leqslant \mathrm{Im}(z) \leqslant \mu(-iL) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}$$

hold.

(ii) If we have the estimation

$$-\mu(-L) - \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|} > 0,$$

define a positive number β satisfying

$$-\mu(-L) - \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}e^{-\beta\tau} = \beta.$$

Then the inequalities

$$\beta \leqslant \mathrm{Re}(z) \leqslant \mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}e^{-\beta\tau}$$

and

$$-\mu(iL) - \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}e^{-\beta\tau} \leqslant \mathrm{Im}(z) \leqslant \mu(-iL) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}e^{-\beta\tau}$$

are valid.

Proof (i) A proof similar to that of Lemma 4 yields

$$0 \leqslant \mathrm{Re}(z) \leqslant \mu(L) + \frac{\|M\| + \|L\|\cdot\|N\|}{1 - \|N\|}.$$

Next, the imaginary part of an eigenvalue of a matrix L is equal to the real part of the eigenvalue of $-iL$. The second inequality holds.

(ii) By Lemma 3

$$-\mu(-L-Me^{-z\tau}-zNe^{-z\tau})\leqslant \mathrm{Re}(z)\leqslant \mu(L+Me^{-z\tau}+zNe^{-z\tau}). \tag{7}$$

A derivation similar to that in (i) leads to

$$-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\leqslant \mathrm{Re}(z)\leqslant \mu(L)+\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}. \tag{8}$$

Set
$$\beta_0=-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}.$$

By (7),

$$\mathrm{Re}(z)\geqslant-\mu(-L-Me^{-z\tau}-zNe^{-z\tau})=-\lim_{\Delta\to0^+}\frac{\|I+\Delta(-L-Me^{-z\tau}-zNe^{-z\tau}\|-1}{\Delta}\geqslant$$

$$-\lim_{\Delta\to0^+}\frac{\|I+\Delta(-L)\|+\|\Delta(-Me^{-z\tau}-zNe^{-z\tau}\|-1}{\Delta}=-\mu(-L)-(\|M\|+|z|\cdot\|N\|)|e^{-z\tau}|\geqslant$$

$$-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot|e^{-z\tau}|=-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{\mathrm{Re}(z)\tau}.$$

Hence, in virtue of (8),

$$\mathrm{Re}(z)\geqslant-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{-\beta_0\tau}.$$

Let $\beta_1=-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{-\beta_1\tau}$. Then we have $\mathrm{Re}(z)\geqslant\beta_1\geqslant\beta_0$. Let $\beta_1=-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{-\beta_2\tau}$. Again we have $\mathrm{Re}(z)\geqslant\beta_2\geqslant\beta_1\geqslant\beta_0$.

The iteration

$$-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{-\beta_j\tau}=\beta_{j+1}\geqslant\beta_j,\ (j=0,1,\cdots)$$

and the monotonicity

$$\beta_0\leqslant\beta_1\leqslant\cdots\leqslant\beta_j\leqslant\beta_{j+1}\leqslant\cdots\leqslant\mathrm{Re}(z)\leqslant\mu(L)+\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}$$

assure that the limit of the series $\{\beta_j\}$ exists and is equal to β, where β is a positive number satisfying

$$-\mu(-L)-\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot e^{-\beta_j\tau}=\beta.$$

Therefore, the first inequality holds. In a similar manner, we can get the second inequality.

Theorem 4 Let $\|N\|\leqslant1$. If z is a characteristic root of (3) with nonnegative real part, the inequality

$$|z|\leqslant\rho(|L|+|M|+\frac{\|M\|+\|L\|\cdot\|N\|}{1-\|N\|}\cdot|N|)$$

holds.

Proof By the assumption above, there exists an integer j $(1\leqslant j\leqslant d)$ such that

$$z=\lambda_j(L+Me^{-z\tau}+zNe^{-z\tau}).$$

This implies the inequality

$$|z|\leqslant\rho(L+Me^{-z\tau}+zNe^{-z\tau}).$$

It is obvious that

$$|L+Me^{-z\tau}+zNe^{-z\tau}|\leqslant|L|+|M|\cdot|e^{-z\tau}|+|z|\cdot|N|\cdot|e^{-z\tau}|\leqslant|L|+|M|+\frac{\|L\|+\|M\|}{1-\|N\|}\cdot|N|.$$

Therefore, due to Lemma 2, we have the conclusion.

3 Boundary Criteria for NDDEs

Let

$$\gamma = \mu(L) + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|}.$$

By virtue of Lemma 4, if $\gamma < 0$, the system(2) is delay-independent asymptotically stable. If $\gamma \geqslant 0$, the system(2) may be stable or unstable. We consider the stability of (2) when $\gamma \geqslant 0$.

Let $\beta_0 = -\mu(-L) - \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|}$ and $\gamma \geqslant 0$. We define the following quantities according to the sign of β. (See Theorem 3)

(i) If $\beta_0 \leqslant 0$, we put

$$E_0 = 0,\ F_0 = -\mu(iL) - \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|},$$

$$E = \mu(L) + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|},\ F = \mu(-iL) + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|}.$$

(ii) If $\beta_0 > 0$, we put $E_0 = \beta$, $E = \mu(L) + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|} \cdot e^{\beta\tau}$, $F_0 = -\mu(iL) - \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|} \cdot e^{\beta\tau}$ and $F = \mu(-iL) + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|} \cdot e^{\beta\tau}$, where β is a root of the equation

$$-\mu(iL) - \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|} \cdot e^{\beta\tau} = \beta.$$

Under the above notations we turn our attention to the following three kinds of bounded regions in the z-plane.

Definition 1 Let l_1, l_2, l_3 and l_4 denote the segments $\{(E_0, y): F_0 < y < F\}$, $\{(x, F): E_0 \leqslant x \leqslant E\}$, $\{(E, y): F_0 \leqslant y \leqslant F\}$ and $\{(x, F_0): E_0 \leqslant x \leqslant E\}$, respectively. Furthermore, $l = l_1 \cup l_2 \cup l_3 \cup l_4$ and let D be the rectangular region surrounded by l.

Definition 2 Let $R = \rho(|L| + |M| + \frac{\|M\| + \|L\| \cdot \|M\|}{1 - \|N\|} \cdot |N|.)$ Let K denote the circular region with radius R centered at the origin of the plane of C.

$$K = \{(r, \theta):\ r \leqslant R,\ 0 \leqslant \theta \leqslant 2\pi\}.$$

Definition 3 Let T represent the intersection $D \cap K$. The boundary of T is denoted by ∂T and $\overline{T} = T \cup \partial T$.

The following two theorems give criteria for the delay-dependent stability of system(3).

Theorem 5 If for any $(x, y) \in \partial T$, the real part $U(x, y)$ in (4) does not vanish, then the system (2) is asymptotically stable.

Proof Assume that the condition is satisfied and that the system(2) is unstable. This means the existence of a characteristic root z of (3) with nonnegative real part. According to Lemma 1, it suffices to prove $P(z) \neq 0$ for $\mathrm{Re}(z) \geqslant 0$. Applying Theorem 3 and Theorem 4 and Definition 3, it is sufficient to consider $z \in \overline{T}$.

From the assumption of this theorem and the statement of Theorem 1, this contradicts with $P(z) = 0$ for $z \in \overline{T}$. Hence $P(z) \neq 0$ for $\mathrm{Re}(z) \geqslant 0$ and the proof is completed.

Due to Theorem 2, we can further extend the above result as follows.

Theorem 6 Assume that for any $(x,y) \in \partial T$, there exists a real constant λ satisfying

$$U(x,y) + \lambda V(x,y) \neq 0,$$

Then the system(2) is asymptotically stable.

The proof is analogous to Theorem 5.

We gave two criteria for the delay-dependent stability of the linear delay system(2). Theorem 3 and Theorem 4 show that the unstable characteristic roots of the system(2) are located in some specified bounded region in the complex plane, while Theorem 5 and Theorem 6 show that it is sufficient to check certain conditions on its boundary to exclude the possibility of such roots from the region. Theorem 1 and Theorem 2 provide general and simple criteria for nonexistence of zeros of an analytic function in any bounded region.

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中立型方程的稳定性：边界准则

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摘要：研究了中立型方程 $x'(t) = Lx(t) + Mx(t-\tau) + Nx'(t-\tau)$ 的渐近稳定性，其中 $L,M,N \in C^{d \times d}$ 是常数复阵，τ 为常数延时量. 利用在一个区域边界上对一种相应的调和函数的估计，得到了判别其稳定性的两种稳定性准则.

关键词：特征值；矩阵范数；谱半径；边界准则；渐近稳定性；调和函数；对数范数