

Structure of E^* -Unitary Categorical Inverse Semigroups

CHEN Li-min

(Dept. of Math., Zhanjiang Normal College, Guangdong 524048, China)

(E-mail: chenlmin@zhjnc.edu.cn)

Abstract: By introducing the partial actions of primitive inverse semigroups on a set and their globalizations, a structure theorem for E^* -unitary categorical inverse semigroups is obtained.

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1. Introduction

E^* -unitary inverse semigroups are more complicated than E -unitary inverse semigroups. As yet, there is no structure theorem for general E^* -unitary inverse semigroups. In [1], Bulman-Fleming, Fountain and Gould introduced strongly E^* -unitary inverse semigroups, and gave the structure theorem for strongly E^* -unitary inverse semigroups. But the structure theorem obtained in [1] is unsatisfactory for the reason that corresponding to a strongly E^* -unitary inverse semigroup, there are many McAlister 0-triples. It is difficult to choose an appropriate McAlister 0-triple. In [5], Kellendonk and Lawson proved that every partial action of a group can be obtained by restricting a group action. Furthermore, they proved that every partial group action admits universal globalization and so obtained an interpretation of Munn's proof of the P -theorem for inverse semigroups by applying their globalization theorem. The aim of this paper is to study E^* -unitary categorical inverse semigroups, a class of strongly E^* -unitary inverse semigroups, characterize them by using partial actions of primitive inverse semigroups on sets, and to generalize Kellendonk and Lawson's results to inverse semigroups with zero.

The paper is organized as follows. In Section 1, we define a partial action of a primitive inverse semigroup T on a set Y , a 0-dual-prehomomorphism from T into $I(Y)$, and the symmetric inverse semigroup on Y , and discuss the relationship between two concepts. In Section 2, we consider the globalization of partial actions of primitive inverse semigroups on sets. Then an exact connection between globalization of partial actions of primitive inverse semigroups and the theory of E^* -unitary categorical inverse semigroups is presented in Section 3.

Throughout the paper, we use the usual notation and the basic results of inverse semigroup theory, see [3], [5] or [6]. Especially, for a semigroup S with zero and a subset A of S , let $E(A)$ denote the set of idempotents of A and let $A^* = A \setminus \{0\}$.

Let T be a primitive inverse semigroup, and Y be a set with a distinguished element $\mathbf{0}$. Let $Y^* = Y \setminus \{\mathbf{0}\}$. A partial function from $T \times Y$ to Y is denoted by $(t, x) \mapsto t \cdot x$ and we write $\exists t \cdot x$ to mean that $t \cdot x$ is defined. We say that $(t, x) \mapsto t \cdot x$ defines a partial action of T on Y if the following conditions hold:

- (PA₁) $\forall t \in T, \exists t \cdot \mathbf{0}$ and $t \cdot \mathbf{0} = \mathbf{0}$;
- (PA₂) $\exists t \cdot x$ implies $\exists t^{-1} \cdot (t \cdot x)$ and $t^{-1} \cdot (t \cdot x) = x$;
- (PA₃) $\forall s, t \in T, x \in Y, \exists s \cdot (t \cdot x)$ implies $\exists (st) \cdot x$ and $s \cdot (t \cdot x) = (st) \cdot x$;
- (PA₄) $\exists \mathbf{0} \cdot x$ if and only if $x = \mathbf{0}$;
- (PA₅) $\forall x \in Y^*, \exists t \in T^*$ such that $\exists t \cdot x$.
- (PA₆) $\forall t \in T^*, \exists x \in Y^*$ such that $\exists t \cdot x$.

Corollary 1.1 *Let $(t, x) \mapsto t \cdot x$ be a partial action of T on Y . Then, the following conclusions hold.*

- (1) *For each $x \in Y^*$, there exists uniquely $e \in E(T^*)$ such that $e \cdot x = x$. We denote the element e by e_x .*
- (2) *If $t \cdot x = t \cdot y$, then $x = y$.*

Proof (1) By condition (PA₅), there exists $t \in T^*$ such that $\exists t \cdot x$. By conditions (PA₂) and (PA₃), $(t^{-1}t) \cdot x = x$. Clearly, $e = t^{-1}t \in E(T^*)$ and $e \cdot x = x$. If there is $f \in E(T^*)$ such that $f \cdot x = x$, then by (PA₃), $\exists (ef) \cdot x$. By (PA₄), $ef \neq \mathbf{0}$ and so $e = f$, as required.

- (2) If $t \cdot x = t \cdot y$, then by (PA₂) and (PA₃), $x = (t^{-1}t) \cdot x = (t^{-1}t) \cdot y = y$. □

A function $\theta : T \rightarrow S$ between inverse semigroups with zero is said to be a 0-dual-prehomomorphism if the following axioms hold:

- (DPH₁) $\theta^{-1}(\mathbf{0}) = \mathbf{0}$;
- (DPH₂) $\theta(t^{-1}) = \theta^{-1}(t)$ for all $t \in T$;
- (DHP₃) $\theta(u)\theta(v) \leq \theta(uv)$ for all $u, v \in T$.

Let Y be a set with a distinguish element $\mathbf{0}$. For every $\alpha \in I(Y)$, $\alpha : A \rightarrow B$ is a bijection where A and B are subsets of Y . Given such an α we denote its domain A by $\text{dom } \alpha$ and its image B by $\text{im } \alpha$. $\alpha \in I(Y)$ is said to be 0-restricted if both $\text{dom}(\alpha)$ and $\text{im}(\alpha)$ contain $\mathbf{0}$ and $\alpha(\mathbf{0}) = \mathbf{0}$. It is easy to show that the set $L(Y)$ of all 0-restricted injective partial transformations on Y is an inverse subsemigroup of $I(Y)$ and that the unique map with domain and image $\{\mathbf{0}\}$ is the zero of $L(Y)$.

It is easy to show the following proposition.

Proposition 1.2 *The partial actions of a primitive inverse semigroup T on a set Y with a distinguished element are equivalent to 0-dual-prehomomorphisms θ from T to $L(Y)$ such that $Y = \cup_{t \in T} \text{dom} \theta(t)$.*

2. Globalization

Let T be a primitive inverse semigroup, X be a set with a distinguished element $\mathbf{0}$. Let $X^* = X \setminus \{\mathbf{0}\}$. A partial function $T \times X \rightarrow X$, $(t, x) \mapsto t \cdot x$ is a global action, if the following

conditions hold:

- (GA₁) $\forall t \in T, \exists t \cdot \mathbf{0}$ and $t \cdot \mathbf{0} = \mathbf{0}$;
- (GA₂) $\exists t \cdot x$ implies $\exists t^{-1} \cdot (t \cdot x)$ and $t^{-1} \cdot (t \cdot x) = x$;
- (GA₃) $\forall s, t \in T, x \in X, \exists s \cdot (t \cdot x)$ if and only if $\exists (st) \cdot x$, in which case $s \cdot (t \cdot x) = (st) \cdot x$;
- (GA₄) $\exists \mathbf{0} \cdot x$ if and only if $x = \mathbf{0}$;
- (GA₅) $\forall x \in X, \exists t \in T^*$ such that $\exists t \cdot x$;
- (GA₆) $\forall t \in T^*, \exists x \in X^*$ such that $\exists t \cdot x$.

A function $\theta : T \rightarrow S$ between inverse semigroups with zero is called 0-restricted if $\theta^{-1}(0) = 0$.

It is routine to verify the following proposition:

Proposition 2.1 *The global actions of a primitive inverse semigroup T on a set X with a distinguished element are equivalent to 0-restricted homomorphisms θ from T to $L(X)$ such that $X = \cup_{t \in T} \text{dom}\theta(t)$.*

Let $\theta : T \rightarrow L(Y)$ be a partial action of the primitive inverse semigroup T on the set Y with a distinguished element $\mathbf{0}$. A *globalization* of θ is a pair (ι, φ) consisting of an injection $\iota : Y \rightarrow X$ and a 0-restricted homomorphism $\varphi : T \rightarrow L(X)$ such that $\theta(t) = \iota^{-1}\varphi(t)\iota$ for every $t \in T$. (Note that we regard ι as a partial bijection from Y to X .) In the following, we shall prove that every partial action of primitive inverse semigroups can be globalized in a universal way.

Let $\theta : T \rightarrow L(Y)$ be a partial action of the primitive inverse semigroup T on the set Y with a distinguished element $\mathbf{0}$. If $\theta(t)(x)$ is defined, then we write $\theta(t)(x) = t \cdot x$. Define the relation \sim on the set $T^* \times Y$ by $(t, x) \sim (t', x')$ if and only if either $(t, x) = (t', x')$ or there exists $s \in T$ such that $t = t's$ and $s \cdot x = x'$. It is direct (or see [5]) to show that \sim is an equivalence relation on $T^* \times Y$. Denote the \sim -equivalence class containing the element (t, x) by $[t, x]$. Let

$$Y_T = \{[t, x] : (t, x) \in T^* \times Y\} \cup \{\emptyset\}.$$

Define a partial function δ from $T \times Y_T$ to Y_T by $\delta(s, \emptyset) = \emptyset$ and $\delta(s, [t, x]) = [st, x]$ if $st \neq \mathbf{0}$. It is routine to show that δ is a well-defined global action.

Define $\iota : Y \rightarrow Y_T$ by $\iota(\mathbf{0}) = \emptyset$ and $\iota(x) = [e_x, x]$ if $x \neq \mathbf{0}$.

Lemma 2.2 *The function ι is injective.*

Proof It suffices to show that for all $x, y \in Y^*$, if $\iota(x) = \iota(y)$, then $x = y$. Suppose that $\iota(x) = \iota(y)$. That is $[e_x, x] = [e_y, y]$. By definition, there exists $t \in T^*$ such that $e_x = e_y t$ and $t \cdot x = y$. Hence $t = e_y t = e_x$ and so $y = t \cdot x = e_x \cdot x = x$, by Corollary 1.1. \square

Denote by φ the 0-restricted homomorphism from T to $L(Y_T)$ corresponding to the global action. Then we have

Theorem 2.3 (1) *The global action of T on Y_T is a globalization of the partial action of T on Y .*

(2) Let $X = T \cdot \iota(Y)$. Then δ can be viewed as a partial function from $T \times X$ to X and so we obtain a global action of T on X . Moreover, the global action of T on X is also a globalisation of the partial action of T on Y .

Proof (1) Let $x \in Y^*$, and suppose that $t \cdot x = x'$ is defined. By definition, $e_x = t^{-1}t$ and $t \cdot \iota(x) = t \cdot [e_x, x] = [t, x]$. Since $\exists t^{-1} \cdot x', e_{x'} = tt^{-1}$ and so $\iota(x') = [e_{x'}, x'] = [tt^{-1}, t \cdot x] = [t, x]$. Thus $\theta(t) \subseteq \iota^{-1}\varphi(t)\iota$. Conversely, suppose $t \cdot [e_x, x] \in \iota(Y)$. Then $te_x \neq 0$ and $t \cdot [e_x, x] = [t, x] = [e_y, y]$ for some $y \in Y$. Thus there exists $s \in T$ such that $t = e_y s$ and $s \cdot x = y$. Hence $t = s$ and so $t \cdot x = y$. Thus $\iota^{-1}\varphi(t)\iota \subseteq \theta(t)$. Hence $\iota^{-1}\varphi(t)\iota = \theta(t)$ for every $t \in T$.

(2) Let $t \cdot [e_x, x] \in X$. Then $te_x \neq 0$ and $t \cdot [e_x, x] = [t, x]$. For every $s \in T$, if $\delta(s, [t, x])$ exists, then $st \neq 0$ and so $ste_x \neq 0$. It follows that

$$\delta(s, [t, x]) = [st, x] = st \cdot [e_x, x] \in X.$$

Thus δ is a partial function from $T \times X$ to X . By the same argument as in (1), we can prove that the global action of T on X is also a globalisation of the partial action of T on Y . \square

3. The P -theorem

In this section, we shall give the structure of E^* -unitary categorical inverse semigroups. To do this, we shall need to consider a special class of partial action of primitive inverse semigroups, namely, partial actions of primitive inverse semigroups on semilattices, together with their globalizations. We begin by recalling some definitions and results^[1].

Let \mathcal{X} be a partially ordered set containing a least element 0. A partial (order)-automorphism of \mathcal{X} is an order-isomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{A} and \mathcal{B} are subsets of \mathcal{X} . Given such an α we denote its domain \mathcal{A} by $\text{dom } \alpha$ and its image \mathcal{B} by $\text{im } \alpha$. A partial automorphism α is 0-restricted if $\text{dom } \alpha$ and $\text{im } \alpha$ both contain 0, and $\alpha(0) = 0$. Denote by $\mathcal{L}(\mathcal{X})$ the set of all 0-restricted partial automorphisms of \mathcal{X} whose domain and image are order ideals of \mathcal{X} . Then $\mathcal{L}(\mathcal{X})$ is an inverse subsemigroup of the symmetric inverse semigroup on \mathcal{X} . Note that the unique map with domain and image $\{0\}$ is the zero of $\mathcal{L}(\mathcal{X})$.

We say that a primitive inverse semigroup T globally acts (partially acts) on \mathcal{X} means that a 0-restricted homomorphism (0-dual-prehomomorphism) $\varphi : T \rightarrow \mathcal{L}(\mathcal{X})$. A partial action φ of T on \mathcal{X} is effective, if $\mathcal{X} = \bigcup_{t \in T} \text{dom } \varphi(t)$. When T partially acts on \mathcal{X} we write $t * A$ for $\varphi(t)(A)$.

For any subset \mathcal{A} of \mathcal{X} , we denote the set $\mathcal{A} \setminus \{0\}$ by \mathcal{A}^* . Let $\varphi : T \rightarrow \mathcal{L}(\mathcal{X})$ be an action of T on \mathcal{X} and let \mathcal{Y} be a subset of \mathcal{X} . The triple $(T, \mathcal{X}, \mathcal{Y})$ is a Primitive triple if conditions (P_1) to (P_4) are satisfied:

- (P₁) \mathcal{Y} is a subsemilattice and order ideal of \mathcal{X} ;
- (P₂) For all $e \in E(T^*)$ and all $P, Q \in \text{dom}(\varphi(e)) \cap \mathcal{Y}^*$, we have $P \wedge Q \neq 0$;
- (P₃) $T * \mathcal{Y} = \mathcal{X}$;
- (P₄) $(t * \mathcal{Y}^*) \cap \mathcal{Y}^* \neq \emptyset$ for all $t \in T^*$.

Now let

$$M = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(P, t) \in \mathcal{Y}^* \times T^* : t^{-1} * P \in \mathcal{Y}^*\} \cup \{\bar{0}\},$$

where $(T, \mathcal{X}, \mathcal{Y})$ is a Primitive triple. We define a multiplication on M by the rule that

$$(P, u)(Q, v) = \begin{cases} (P \wedge u * Q, uv), & uv \neq 0 \\ \bar{0} & uv = 0 \end{cases}$$

and

$$(P, u)\bar{0} = \bar{0}(P, u) = \bar{0}\bar{0} = \bar{0}.$$

The main result of this paper is the following theorem.

Theorem 3.1 *Let $M = \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ be defined as above, then M is an E^* -unitary categorical inverse semigroup. Conversely, every E^* -unitary categorical inverse semigroup can be constructed in this way.*

The proof of the direct part of Theorem 3.1 is straightforward. In the following, we shall give the proof of the converse part of Theorem 3.1 by using the theory developed in this paper.

Let (Y, \wedge) be a meet semilattice with least element 0 and let $\mathcal{L}(Y)$ denote the inverse semigroup with zero consisting of all order isomorphisms between order ideals of Y . Note that the zero of $\mathcal{L}(Y)$ is the map whose domain and image are $\{0\}$. Let $\theta : T \rightarrow \mathcal{L}(Y)$ be an effective partial action of T on Y and $\varphi : T \rightarrow \mathcal{L}(Y_T)$ be the globalisation of θ constructed in Section 2. Then, for each $t \in T^*$, $\varphi(t)$ is a partial injection on Y_T defined by $\varphi(t)(\emptyset) = \emptyset$, and $\varphi(t)([s, x]) = [ts, x]$ if $ts \neq 0$. In the following, $\theta(t)(x)$ and $\varphi(t)([s, x])$ will be abbreviated to $t \cdot x$ and $t * [s, x]$, respectively.

It is easy to show the following Lemma with the above notations:

Lemma 3.2 (1) *Define the relation \leq on Y_T by $\emptyset \leq [u, x]$ and $[u, x] \leq [v, y]$ if and only if there exist $(v', y') \in [v, y]$ and $x' \leq y'$ such that $(v', x') \in [u, x]$. Then \leq is a partial order on Y_T .*

(2) *The semilattice Y is order-isomorphic with the partially ordered set $\iota(Y)$ with the order \leq induced from Y_T .*

Corollary 3.3 *If $[u, x] \leq [v, y]$, then for every $(w, c) \in [v, y]$, there exists $d \leq c$ such that $(w, d) \in [u, x]$.*

Lemma 3.4 (1) $\varphi(t)$ is order-preserving.

(2) *Both $\text{dom}\varphi(t)$ and $\text{im}\varphi(t)$ are order ideals of Y_T . Hence φ is a 0-restricted homomorphism from T to $\mathcal{L}(Y_T)$.*

Proof (1) Suppose $[u, a], [v, b] \in \text{dom}\varphi(t)$ and $[u, a] \leq [v, b]$. By Corollary 3.3, there exists $d \leq b$ such that $(v, d) \in [u, a]$. Thus

$$t * ([u, a]) = t * ([v, d]) = [tv, d] \leq [tv, b] = t * ([v, b]),$$

and so $\varphi(t)$ is order-preserving.

(2) Clearly, $\text{dom}\varphi(t) = \{[s, x] \in Y_T : ts \neq 0\} \cup \{\emptyset\}$, $\text{Im}\varphi(t) = \text{dom}\varphi(t^{-1})$. Suppose $[u, y] \leq [s, x] \in \text{dom}\varphi(t)$. Then by Corollary 3.3, there exists $k \in Y$ such that $k \leq x$ and $(s, k) \in [u, y]$. It follows that there exists $v \in T^*$ such that $s = uv$ and $y = v \cdot k$. Since $ts \neq 0$, $tv \neq 0$ and so

$[u, y] \in \text{dom}\varphi(t)$. Hence $\text{dom}\varphi(t)$ is an order ideal of Y_T . Similarly, $\text{dom}\varphi(t^{-1})$ is an order ideal of Y_T . Hence $\text{im}\varphi(t)$ is also an order ideal of Y_T . \square

Let $X = T * \iota(Y)$. Then φ can be viewed as a 0-restricted homomorphism from T to $\mathcal{L}(X)$. Furthermore, we have

Theorem 3.5 *Suppose that for each $e \in T^*$ and $x, y \in \text{dom}\theta(e) \cap Y^*$, we have $x \wedge y \neq 0$. Then $(T, X, \iota(Y))$ is a Primitive triple.*

Proof (1) Clearly, $X = \{[t, x] \in Y_T : x \in Y^*, e_x = t^{-1}t\} \cup \{\emptyset\}$. We claim that $\iota(Y)$ is an order ideal of X . Since $[t, x] \in X$ and $[t, x] \leq [e_y, y] \in \iota(Y)$. Then by Corollary 3.3, there exists $y' \leq y$ such that $(e_y, y') \in [t, x]$. Thus $[t, x] = [e_y, y']$. Since $y \in \text{dom}\theta(e_y)$ and $\text{dom}\theta(e_y)$ is an order ideal of Y , $y' \in \text{dom}\theta(e_y)$ and so $\exists e_{y'} \cdot y'$. Since $x \neq 0$ and $[t, x] = [e_y, y'], y' \neq 0$. Hence $e_{y'}$ exists. Thus

$$e_y \cdot y' = (e_y e_{y'}) \cdot y' = e_{y'}^{-1} \cdot (e_y \cdot y') = y' = e_{y'} \cdot y'.$$

By Corollary 1.1, $e_y = e_{y'}$. Hence $[t, x] = [e_y, y'] = [e_{y'}, y'] \in \iota(Y)$. Thus $\iota(Y)$ is an order ideal of X .

(2) For all $e \in E(T^*)$ and $[s, x], [t, y] \in \text{dom}(\varphi(e)) \cap \iota(Y^*)$, we have $es \neq 0, et \neq 0$ and there exist $[e_a, a], [e_b, b] \in \iota(Y)$ such that $[s, x] = [e_a, a], [t, y] = [e_b, b]$. Thus $e = e_a = e_b$. From $\exists e_a \cdot a$ and $\exists e_b \cdot b$, we have that $a, b \in \text{dom}\theta(e)$. By the assumption, $a \wedge b \in \text{dom}(\theta(e)) \cap Y^*$. Thus $\exists e \cdot (a \wedge b)$ and so $[e, a \wedge b] \in \iota(Y^*)$. Clearly, $[e, a \wedge b] \leq [e_a, a] = [s, x]$ and $[e, a \wedge b] \leq [e_b, b] = [t, y]$. Thus $[s, x] \wedge [t, y] \neq \emptyset$.

(3) $X = T * \iota(Y)$ is clear.

(4) For each $t \in T^*$, by (PA_6) there exists $y \in Y^*$ such that $t \cdot y$ exists and so $\exists (t^{-1}t) \cdot y$. Hence $\iota(y) = [t^{-1}t, y]$. Clearly, $t * (\iota(y)) = t * [t^{-1}t, y] = [t, y] = [tt^{-1}, t \cdot y]$. Since $\exists t^{-1} \cdot (t \cdot y)$ and $\exists t \cdot (t^{-1} \cdot (t \cdot y)), tt^{-1} = e_{t \cdot y}$. Thus $[tt^{-1}, t \cdot y] = \iota(t \cdot y)$. It follows that $t * \iota(Y^*) \cap \iota(Y^*) \neq \emptyset$. \square

The proof of the converse part of Theorem 3.1 Let S be an E^* -unitary categorical inverse semigroup. Put $T = S/\beta$ and $Y = E(S)$, where β is the least 0-restricted primitive congruence on S . Define a function $\theta : T \rightarrow \mathcal{L}(Y)$ as follows: For each $t \in T$, let

$$\text{dom}\theta(t) = \{a^{-1}a : \beta(a) = t\} \cup \{0\}$$

and define $\theta(t)(0) = 0$ and $\theta(t)(a^{-1}a) = aa^{-1}$. Then we can prove that θ is a 0-dual-prehomomorphism and $Y = \cup_{t \in T} \text{dom}\theta(t)$. Thus we obtain an effective partial action of T on Y . For each $y \in Y^*, e_y = \beta(y)$. Clearly, θ satisfies the assumption of Theorem 3.5. By Theorem 3.5, $(T, T * \iota(Y), \iota(Y))$ is a primitive triple.

Define a map δ from S to $M = \mathcal{M}(T, T * \iota(Y), \iota(Y))$ by $\delta(0) = \bar{0}$ and $\delta(s) = ([\beta(ss^{-1}), ss^{-1}], \beta(s))$ if $s \neq 0$. Then δ is well-defined.

First, let us prove that δ is injective. Assume $\delta(a) = \delta(b)$ for some $a, b \in S^*$. Then $\beta(a) = \beta(b)$ and $[\beta(aa^{-1}), aa^{-1}] = [\beta(bb^{-1}), bb^{-1}]$. It follows that $(a, b) \in \beta$ and $aa^{-1} = bb^{-1}$. By the Proposition 14 in page 91 of [5], $a = b$ and so δ is injective.

Next, we shall prove that δ is surjective. Let $([e_y, y], t) \in M$ such that $t^{-1} * [e_y, y] \in \iota(Y^*)$. Then $e_y = tt^{-1}$ and $[t^{-1}, y] = [e_z, z] \in \iota(Y^*)$. It follows that there exists $u \in T$ such that $t^{-1} = e_z u$ and $u \cdot y = z$. By the definition of θ , there is $b \in S^*$ such that $\beta(b) = u = t^{-1}$, $y = b^{-1}b$ and $z = bb^{-1}$. Let $s = b^{-1}$. Then $t = \beta(s)$, $e_y = \beta(ss^{-1})$ and $y = ss^{-1}$. Thus $([e_y, y], t) = \delta(a)$ and so δ is surjective.

Finally, let us show that δ is a homomorphism. It suffices to prove that $\delta(ab) = \delta(a)\delta(b)$ for all $a, b \in S^*$. If $ab = 0$, then $\beta(a)\beta(b) = 0$ and so $\delta(a)\delta(b) = \bar{0} = \delta(0) = \delta(ab)$.

If $ab \neq 0$, then $\beta(a)\beta(b) \neq 0$ and so

$$\delta(a)\delta(b) = ([\beta(aa^{-1}), aa^{-1}] \wedge [\beta(abb^{-1}), bb^{-1}], \beta(ab)).$$

It remains to show that

$$[\beta(aa^{-1}), aa^{-1}] \wedge [\beta(abb^{-1}), bb^{-1}] = [\beta(abb^{-1}a^{-1}), ab(ab)^{-1}].$$

Since $\beta(abb^{-1}a^{-1}) = \beta(aa^{-1})$ and $ab(ab)^{-1} \leq aa^{-1}$, $[\beta(abb^{-1}a^{-1}), ab(ab)^{-1}] \leq [\beta(aa^{-1}), aa^{-1}]$. Let $y = a^{-1}abb^{-1}$. Then $y \leq bb^{-1}$. Since $(abb^{-1})^{-1}abb^{-1} = y$ and $\beta(a) = \beta(abb^{-1})$, $y \in \text{dom}\theta(\beta(a))$ and so $\beta(a) \cdot y = abb^{-1}(abb^{-1})^{-1} = ab(ab)^{-1}$. Thus $[\beta(a), y] = [\beta(abb^{-1}a^{-1}), ab(ab)^{-1}]$. Hence

$$[\beta(abb^{-1}a^{-1}), ab(ab)^{-1}] = [\beta(a), y] = [\beta(abb^{-1}), y] \leq [\beta(abb^{-1}), bb^{-1}].$$

It follows that

$$[\beta(abb^{-1}a^{-1}), ab(ab)^{-1}] \leq [\beta(aa^{-1}), aa^{-1}] \wedge [\beta(abb^{-1}), bb^{-1}].$$

Conversely, suppose that

$$[u, v] \leq [\beta(aa^{-1}), aa^{-1}] \text{ and } [u, v] \leq [\beta(abb^{-1}), bb^{-1}].$$

Then by Corollary 3.3, there exist $x, y \in Y$ such that $x \leq aa^{-1}$, $y \leq bb^{-1}$ and $[\beta(aa^{-1}), x] = [u, v] = [\beta(abb^{-1}), y]$. It follows that there exists $w \in T$ such that $\beta(abb^{-1}) = \beta(aa^{-1})w$ and $x = w \cdot y$. Thus $w = \beta(a)$ and $y \in \text{dom}\theta(\beta(a))$. By the definition of θ , there exists $c \in S$ such that $\beta(c) = \beta(a)$, $y = c^{-1}c$ and $x = cc^{-1}$. Since $x \leq aa^{-1}$ and $y \leq bb^{-1}$, we have $c = cbb^{-1} = aa^{-1}c$. Thus

$$x = cc^{-1} = aa^{-1}c(cbb^{-1})^{-1} = aa^{-1}cbb^{-1}c^{-1}.$$

But $\beta(c) = \beta(a)$, so that $a^{-1}c \in E(S)$. It follows that

$$x = a(a^{-1}c)(bb^{-1})c^{-1} = abb^{-1}a^{-1}cc^{-1} = ab(ab)^{-1}x.$$

Thus $x \leq ab(ab)^{-1}$ and so

$$[u, v] = [\beta(aa^{-1}), x] = [\beta(abb^{-1}a^{-1}), x] \leq [\beta(abb^{-1}a^{-1}), ab(ab)^{-1}].$$

It follows that

$$[\beta(aa^{-1}), aa^{-1}] \wedge [\beta(abb^{-1}), bb^{-1}] = [\beta(abb^{-1}a^{-1}), ab(ab)^{-1}],$$

as required. □

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E^* -酉范畴逆半群的结构

陈历敏

(湛江师范学院数学系, 广东 湛江 524048)

摘要: 本文通过定义本原逆半群在集合上的部分作用及其整体化, 给出了 E^* -酉范畴逆半群的结构.

关键词: 部分作用; 整体化; E^* -酉范畴逆半群.