A classification of points on the Sierpinski gasket

Ali Deniz, Mustafa Saltan and Bunyamin Demir

Abstract. In this article we classified the points of the well-known fractal set Sierpinski Gasket (SG) according to their addresses. We also gave a characterization of points of SG that describes the relation between their addresses and their components in $\mathbb{R}^2$.

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1 Introduction

The Sierpinski Gasket (also known as Sierpinski Triangle) was described by Waclaw Sierpinski in 1915, and it became an important sample of fractal set. There are many works about Sierpinski Gasket [1],[3],[4]. But, to our knowledge, no classification of points was made according to their addresses. In order to achieve such classifications, we will present a method to construct it.

Let P_1, P_2, P_3 be three points in the plane which are not collinear. Let $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $f_i(x) = \frac{1}{2}(x + P_i)$, for $i = 1, 2, 3$. Obviously, these functions are contractions and P_i is the only fixed point of f_i ($i = 1, 2, 3$).

Let $\mathcal{H}(\mathbb{R}^2) = \Big\{A \mid A \neq \varnothing \text{ , and } A \subset \mathbb{R}^2 \text{ compact subset}\Big\}$. It is well known that $\mathcal{H}(\mathbb{R}^2)$ is a complete metric space with Hausdorff metric (see [1]).

Define $F : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2)$, $F(A) = f_1(A) \cup f_2(A) \cup f_3(A)$. It is well known that F is a contraction function on $\mathcal{H}(\mathbb{R}^2)$ and by the Banach fixed point theorem, F has a unique fixed point (as a set) in $\mathcal{H}(\mathbb{R}^2)$ and the sequence

$$B, F(B), F(F(B)) = F^2(B), \dots, F^n(B), \dots$$

converges to the fixed point for any $B \in \mathcal{H}(\mathbb{R}^2)$. This fixed point, say K, is called the Sierpinski Gasket (SG), (see [1]).

Without loss of generality we consider the points P_1, P_2, and P_3 as $(0, 1), (0, 0), (1, 0)$ respectively. Figure 1 shows the fixed point of F for these points.

In order to describe the points on SG in a particular way, there is a so-called code space representation of SG. Consider the set

$$\Omega = \{S = \sigma_1\sigma_2 \cdots \sigma_n \cdots \mid \sigma_i \in \{1, 2, 3\} \text{ for } i = 1, 2, 3, \dots\}.$$

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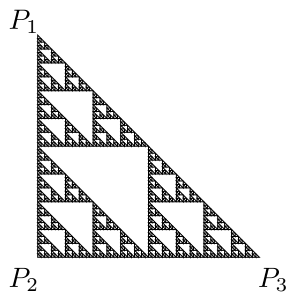


Figure 1: The fixed point of F

For an element $S \in \Omega$, the set $\bigcap_{n=1}^{\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(K)$ is a singleton (i.e. set of a single point). If we denote this singleton by $\{x\} \subset K$, then the function $\Phi : \Omega \longrightarrow K$,

$$\Phi(\sigma_1 \sigma_2 \cdots \sigma_n \cdots) = x$$

is well-defined. $S = \sigma_1 \sigma_2 \cdots \sigma_n \cdots$ is said to be the address of x. The function Φ is onto but not one to one, for example the addresses $1222 \cdots$ and $2111 \cdots$ represent the same point.

Now, let $S = \sigma_1 \sigma_2 \cdots \sigma_n \cdots$ be the address of a point $q \in K$. In order to find q in K, we iterate K step by step, by taking the compositions of f_i's ($i = 1, 2, 3$) according to the order of S. At the first step we get $f_{\sigma_1}(K)$. Since f_i's are contraction functions with contraction constant $\frac{1}{2}$, K shrinks by f_{σ_1} towards the point P_{σ_1} in ratio of $\frac{1}{2}$. So, $f_{\sigma_1}(K)$ is one of the three smaller copies of K. More precisely, $f_1(K)$ is the upper part of K, $f_2(K)$ is the left-bottom part of K and $f_3(K)$ is the right-bottom part of K.

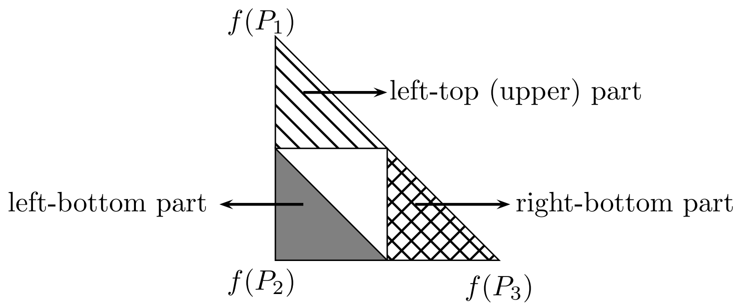


Figure 2: Components of $f(K)$.

If we continue with the iteration, at the n-th step, we obtain $f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(K)$. For simplicity, say $f = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_{n-1}}$. Thus, if $\sigma_n = 1$ then $f \circ f_{\sigma_n}(K)$ is the upper part of $f(K)$, if $\sigma_n = 2$ then $f \circ f_{\sigma_n}(K)$ is the left-bottom part of $f(K)$, and if

$\sigma_n = 3$ then $f \circ f_{\sigma_n}(K)$ is the right-bottom part of $f(K)$(see Fig. 2). Table 1 shows which part is chosen at the n-th step.

σ_i	Parts	
1	left	top
2	left	bottom
3	right	bottom

Table 1: The image of f_{σ_i} depending on σ_i.

Example 1: $S = 121212\cdots$ represents the point $(0, \frac{2}{3})$ on K. (see Fig. 3.)

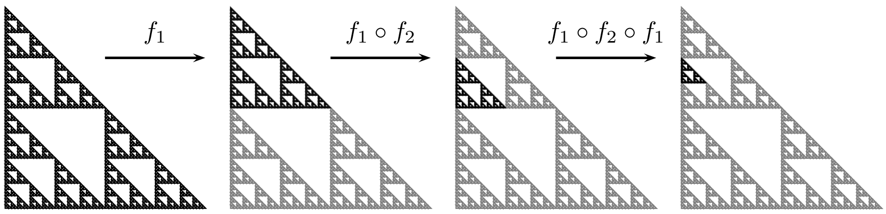


Figure 3: Iteration of K with respect to address $121212\cdots$

The components of the point $q = \{x, y\}$ determined by its address is given by the infinite sums:

$$x = \sum_{i=1}^{\infty} x_i \frac{1}{2^i}, \quad y = \sum_{i=1}^{\infty} y_i \frac{1}{2^i}$$

where

$$x_i = \begin{cases} 0 & , \quad \sigma_i = 1 \text{ or } 2 \\ 1 & , \quad \sigma_i = 3 \end{cases} , \quad y_i = \begin{cases} 1 & , \quad \sigma_i = 1 \\ 0 & , \quad \sigma_i = 2 \text{ or } 3. \end{cases}$$

Thus, diadic expansions of x and y are written as follows:

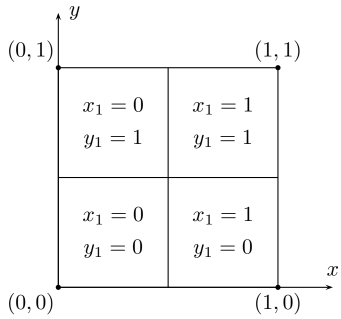
$$x = 0, x_1 x_2 x_3 \cdots x_n \cdots \text{ and } y = 0, y_1 y_2 y_3 \cdots y_n \cdots$$

Conversely, one can obtain the address of a given point $q \in K$ from the binary expansions of its components. Let x and y be the binary expansion of components of q , say

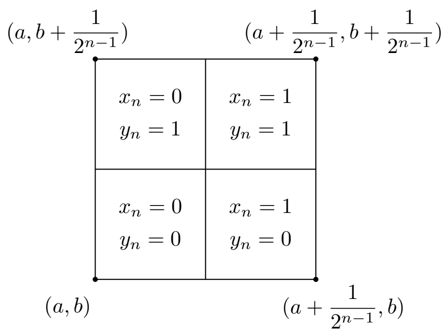
$$x = 0, x_1 x_2 x_3 \cdots x_n \cdots \text{ and } y = 0, y_1 y_2 y_3 \cdots y_n \cdots$$

where $x_i, y_i \in \{0, 1\}$, and $(i = 1, 2, 3, \dots)$.

Depending on the first digits, namely x_1 and y_1, q is in at least one the following small squares.



Thus we can get the first term of the address of q . If $x_1 = 0, y_1 = 1$, then q must be in upper part of K, so $\sigma_1 = 1$, if $x_1 = 0, y_1 = 0$, then $\sigma_1 = 2$, and if $x_1 = 1, y_1 = 0$, then $\sigma_1 = 3$. In the case of $x_1 = y_1 = 1$, q must be $(\frac{1}{2}, \frac{1}{2})$ and one can choose a convenient representation of q such that $x_1 \neq 1$ or $y_1 \neq 1$. Repeating this process, at the n-th step, depending on x_n and y_n, q is in at least one of the following small squares:



where $a = x_1\frac{1}{2} + x_2\frac{1}{2^2} + \cdots + x_{n-1}\frac{1}{2^{n-1}}$ and $b = y_1\frac{1}{2} + y_2\frac{1}{2^2} + \cdots + y_{n-1}\frac{1}{2^{n-1}}$. Again, in the case of $x_n = 1$ and $y_n = 1$ a convenient representation of q can be chosen, then we have

$$\sigma_n = \begin{cases} 1 \quad , \quad (x_n, y_n) = (0, 1) \\ 2 \quad , \quad (x_n, y_n) = (0, 0) \\ 3 \quad , \quad (x_n, y_n) = (1, 0). \end{cases}$$

2 Classification of points on the Sierpinski Gasket

We will classify the points of SG with respect to their addresses. First we need to give following definition.

Definition 1 (Junction Point). Let P_1, P_2, P_3 be the vertices of Sierpinski Gasket and $q \in K$. If there exist finitely many numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that

$$f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(P_i) = q$$

where $i = 1, 2$, or 3 and $\sigma_j \in \{1, 2, 3\}$ for $j = 1, 2, \ldots n$, then q is called a junction point on SG.

Our classification consists of three types of points: the junction points, the points on a segment of SG, the points which are not on a segment of SG.

Theorem 1. *Let S be the address of $q \in K$. Then,*

i) If S is eventually fixed then q is a junction point.

ii) If an element of the set $\{1, 2, 3\}$ appears only finitely many times in S, then q is on a segment of SG.

iii) If each of the elements of $\{1, 2, 3\}$ appears infinitely many times in S then q is not on a segment of SG.

Proof. *i)* Let $S = \sigma_1\sigma_2\cdots\sigma_n\sigma\sigma\sigma\cdots$ be the address of $q \in K$. Consider first n terms of S which is not fixed and say $f = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}$.

Applying f to SG, we get the part of SG whose vertices are $f(P_1), f(P_2)$ and $f(P_3)$. In the next step of the iteration, we obtain $f \circ f_\sigma(K)$ by applying f_σ to $f(K)$. By definition, depending on the value of σ, $f \circ f_\sigma(K)$ corresponds to a smaller part of the triangle $f(K)$ divided according to the rule given above. This has been illustrated as the shaded triangle in Figure 4, when $\sigma = 1$.

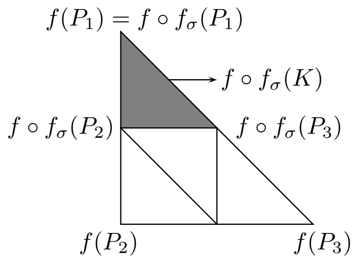


Figure 4: $(n + 1)$-th step of the iteration when $\sigma = 1$.

Continuing with the iteration, in the $(n + k)$-th step, we obtain $f \circ f_\sigma^k(K)$ for $(k = 1, 2, \dots)$. (Here f_σ^k denotes the composition $f_\sigma \circ f_\sigma \circ \cdots f_\sigma$, k-times). Each time we apply f_σ to $f \circ f_\sigma^k(K)$, we get a small one third part of $f \circ f_\sigma^k(K)$ getting closer to the vertex $f(P_\sigma)$.

Thus, choosing k large enough, we can make $f \circ f_\sigma^k(K)$ close to $f(P_\sigma)$ as close as we want. This means if k goes to infinity, $f \circ f_\sigma^k(K)$ converges to the vertex $f(P_\sigma)$, so q is a junction point.

ii) Let $S = \sigma_1\sigma_2\cdots\sigma_n\cdots$ be the address of $q \in K$ and one of the elements of $\{1, 2, 3\}$ appears only finitely many times in S. Let us suppose that the finitely used element is 3, the other cases are similar.

Let σ_n be the last digit where 3 occurs. Thus we can say if $k > n$ then $\sigma_k \in \{1, 2\}$. This means that after the n-th step of the iteration, only f_1 and f_2 are used to determine the location of q.

Let $f = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}$. Then $f(K)$ is the part of SG whose vertices are $f(P_1), f(P_2), f(P_3)$. In $(n+1)$-th step, if $\sigma_{n+1} = 1$ then $f \circ f_{\sigma_{n+1}}(K)$ is the upper part of $f(K)$ and if $\sigma_{n+1} = 2$ then $f \circ f_{\sigma_{n+1}}(K)$ is left-bottom part of $f(K)$. In both cases $f \circ f_{\sigma_{n+1}}(K)$ contains a subsegment of the segment $\mid f(P_1)f(P_2) \mid$. The furthest point of $f \circ f_{\sigma_{n+1}}(K)$ to $\mid f(P_1)f(P_2) \mid$ is $f \circ f_{\sigma_{n+1}}(P_3)$ and since we are in the $(n+1)$-th step, its distance is $\frac{1}{2^{n+1}}$. This situation continues in the subsequent steps. That is, for $k \geq n$, $f \circ f_{\sigma_{n+1}} \circ \cdots \circ f_{\sigma_k}(K)$ contains a subsegment of $\mid f(P_1)f(P_2) \mid$. Here $f \circ f_{\sigma_{n+1}} \circ \cdots \circ f_{\sigma_k}(P_3)$ is the furthest point to $\mid f(P_1)f(P_2) \mid$, and its distance is $\frac{1}{2^k}$. As k goes to infinity this distance tends to zero which means q is on $\mid f(P_1)f(P_2) \mid$.

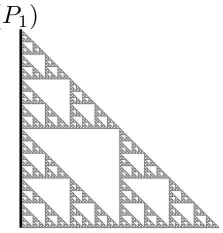


Figure 5: q is on the segment $\mid f(P_1)f(P_2) \mid$, if the number 3 appears only finitely many times in S.

iii) Let $S = \sigma_1\sigma_2 \cdots \sigma_n \cdots$ be the address of $q \in K$ and each of the elements of $\{1, 2, 3\}$ appears infinitely many times in S.

Thus for each $m \in \mathbb{N}$, there exist numbers $N_1, N_2, N_3 \in \mathbb{N}$ such that $N_i > m$ $(i = 1, 2, 3)$ and $\sigma_{N_1} = 1, \sigma_{N_2} = 2$ and $\sigma_{N_3} = 3$. Now, fix $m \in \mathbb{N}$ and let $f = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_m}$. We claim that q is not on the segment $\mid f(P_1)f(P_2) \mid$. To prove this, choose $N_3 > m$ such that $\sigma_{N_3} = 3$. Since $f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_{(N_3-1)}}(K)$ shrinks by $f_{\sigma_{N_3}}$ towards the point $f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_{(N_3-1)}}(P_3)$ in ratio of $\frac{1}{2}$, then

$$d(q, \mid f(P_1)f(P_2) \mid) \geq \frac{1}{2^{N_3}}$$

where $d(q, \mid f(P_1)f(P_2) \mid)$ is the distance between q and the segment $\mid f(P_1)f(P_2) \mid$. Hence, q can not be on the segment $\mid f(P_1)f(P_2) \mid$.

Similarly q can not be on the segments $\mid f(P_1)f(P_3) \mid$ and $\mid f(P_2)f(P_3) \mid$. Since $m \in \mathbb{N}$ is arbitrary, q is not on a segment of SG.

□

Now, we will give another characterization of points of SG that describes the relation between their addresses and their components.

Theorem 2. *Let q be a point on SG. q has rational coordinates in the plane if and only if its address is eventually periodic.*

Proof. Let us prove, first, that a point which has eventually periodic address has rational coordinates. Let

$$S = \sigma_1\sigma_2\cdots\sigma_n\sigma_{n+1}\cdots\sigma_{n+p}\sigma_{n+1}\cdots\sigma_{n+p}\cdots$$

be the address of q with period p, where $\sigma_i \in \{1,2,3\}$. Consider the diadic expansions of the components of $q = (x, y)$:

$$\begin{aligned} x &= 0, x_1x_2x_3\cdots x_n\cdots \\ y &= 0, y_1y_2y_3\cdots y_n\cdots \end{aligned}$$

where $x_i, y_i \in \{0,1\}$ for $i \in \mathbb{N}$. Periodicity of S implies that above expansions are also periodic. Moreover, their periods are the same as the address of q, namely p.

We can write the coordinates of q in the plane as

$$x = \sum_{i=1}^{\infty} x_i \frac{1}{2^i}, \quad y = \sum_{i=1}^{\infty} y_i \frac{1}{2^i}.$$

The sum of the first n terms (aperiodic parts) of these sums are rational, thus we can consider the following remainders:

$$x = \sum_{i=n+1}^{\infty} x_i \frac{1}{2^i}, \quad y = \sum_{i=n+1}^{\infty} y_i \frac{1}{2^i}.$$

For the first component x, this remainder can be written as follows

$$\begin{aligned} \sum_{i=n+1}^{\infty} x_i \frac{1}{2^i} &= x_{n+1}\left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+p+1}} + \cdots\right) + \cdots + x_{n+p}\left(\frac{1}{2^{n+p}} + \frac{1}{2^{n+2p}} + \cdots\right) \\ &= \left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \cdots\right)\left(\frac{x_{n+1}}{2^{n+1}} + \frac{x_{n+2}}{2^{n+2}} + \cdots + \frac{x_{n+p}}{2^{n+p}}\right) \\ &= \frac{2^p}{2^p - 1}\left(\frac{x_{n+1}}{2^{n+1}} + \frac{x_{n+2}}{2^{n+2}} + \cdots + \frac{x_{n+p}}{2^{n+p}}\right) \end{aligned}$$

which is rational. Similarly y is also rational.

Let us now prove the sufficiency: if the components of q are rational then the address of q is eventually periodic. Write the binary expansions of x and y,

$$\begin{aligned} x &= 0, x_1x_2\cdots x_nx_{n+1}\cdots x_{n+p}x_{n+1}\cdots x_{n+p}\cdots \\ y &= 0, y_1y_2\cdots y_ny_{n+1}\cdots y_{n+p}y_{n+1}\cdots y_{n+p}\cdots \end{aligned}$$

where $x_i, y_i \in \{0,1\}$. Forming the pairs (x_n, y_n), we get a new sequence which is also eventually periodic. Using the method in the introduction, we have the address $S = \sigma_1\sigma_2\cdots\sigma_n\cdots$ of q. Periodicity of (x_n, y_n) provides that S is periodic. □

Theorem 2 is stated for the points $P_1 = (0,1), P_2 = (0,0), P_3 = (1,0)$. As a result, we can give the following corollary for three points in the plane which are not collinear.

Corollary 1. Let P_1, P_2, P_3 be three points in the plane which are not collinear. A point q on SG (whose vertices are P_1, P_2, P_3) has eventually periodic address if and only if q is a linear combination of P_1, P_2, P_3 with rational coefficients.

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Authors' address:

Ali Deniz, Mustafa Saltan and Bunyamin Demir
Anadolu University, Department of Mathematics,
26470 Eskisehir, Turkey.
E-mail: adeniz@anadolu.edu.tr, msaltan@anadolu.edu.tr, bdemir@anadolu.edu.tr