Propagation of Velocity Discontinuities on Potential Vorticity Fronts*

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ABSTRACT

The propagation of finite-length velocity discontinuities along a potential vorticity front is studied. The front is oriented in a fundamentally north–south direction on the β-plane. The potential vorticity on either side of the front is constant.

According to a semigeostrophic theory, the shear disturbances propagate with a speed which depends on the *difference* of the deformation radius across the front and on the local depth, h_B, of the moving fluid layer at the front. The disturbance steepens as it propagates. Breaking is predicted by the semigeostrophic theory at a time which depends inversely on the propagation speed and the initial steepness of the perturbation to the layer depth. Small velocity discontinuities will grow as they propagate.

1. Introduction

The baroclinic theory of the formation region of western boundary currents, such as the Gulf Stream, identifies the point of separation of the current as the position along the coast where isopycnal surfaces at the base of the current rise to the surface, e.g., Charney (1955). The simplest such models describe a western boundary current in which the potential vorticity is uniform in a single, uppermost, moving layer (the "1½ layer" model). In such a case considerations only of mass conservation and the assumption of geostrophic balance for the downstream velocity imply that the stream will separate a latitude where the Coriolis parameter, f, is exactly twice the value of the Coriolis parameter, f_0, at the starting latitude of the flow. This point is also coincident with the point where the southward propagation of a Kelvin wave is just arrested by the northward velocity of the current. A similar but increasingly severe limitation of the solution when more than one layer is considered was found by Blandford (1965) and, again, the critical points correspond to the point of arrest of the slowest Kelvin wave.

On the other hand, a Gulf Stream model with nonuniform potential vorticity will have a different point of separation. Indeed for the 1½ layer model referred to above, the same simple mass considerations imply that the latitude where the interface between the swiftly flowing current and the resting deep layer will surface along the coast is given by the condition

$$\int_1^{f/f_0} \frac{q_\infty^2(f_0)}{q_\infty^2(f')}\, d\left(\frac{f'}{f_0}\right) = 1, \qquad (1.1)$$

where f_0 is the Coriolis parameter at the latitude where the stream initiates and q_∞ is the potential vorticity, f/h, of the oncoming, westward flow of the upper layer outside the boundary layer. If q_∞ is independent of latitude, the condition (1.1) implies that $f/f_0 = 2$ as mentioned above. However, if q_∞ increases (say) northward, the point of separation of the stream will occur farther north.

It is natural to wonder whether this retardation is in any way related to the presence of a new class of waves in the current associated with the presence of a potential vorticity gradient.

Perhaps the simplest model in which to examine such time-dependent dynamics is one in which the potential vorticity variation is confined to a single front separating two regions each with uniform potential vorticity. Even this problem is rather complex, and the present paper reports on some interesting dynamical features of the frontal problem in the absence of the coastal wall and its associated Kelvin wave. Thus, although the separation produced the motivation for the current study, the results of this paper unfortunately are not directly applicable to that problem.

Nevertheless, the frontal dynamical problem presents sufficient features of interest of its own. The model to be described represents the motion of fluid of density ρ over a deep resting layer of density $\rho + \Delta\rho$. Two regions, each of uniform potential vorticity, q_n ($n = 1, 2$), are separated by the line $x = x_B(y, t)$ where x is the eastward coordinate, y the northward coordinate,

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and t is time. The front separating the two regions is oriented north–south on the β-plane, i.e., the earth's sphericity is modeled by allowing the northward increase of the Coriolis parameter, i.e., $f = f(y)$. A semigeostrophic model is used in which the downstream, but not the weaker cross-stream velocity, is in geostrophic balance. Such an approximation is valid whenever the downstream scale is large compared to the cross-stream scale.

The theory developed below describes the nonlinear propagation of shear zones, i.e., discontinuities in along-front velocity, along the frontal boundary. In particular, the propagation speed of such shear zones is shown to depend on the difference in the potential vorticity across the front. Such disturbances are shown to steepen while propagating downstream forming "shocks" at a point determined by the steepness of the initial disturbance, the cross-front potential vorticity jump, and the β-effect. The effect is entirely absent in quasi-geostrophic models in which the deformation radius is necessarily independent of horizontal position (e.g., Pratt and Stern 1986). The steepening will proceed to the point where the downstream scale contracts to the order of the deformation radius forming a folded zone of strong shear.

Section 2 describes the basic model and the appropriate mathematical formulation of the problem. Section 3 describes the solution. Conclusions and further discussion are presented in section 4.

2. The model

Consider the situation depicted in Fig. 1. A layer of fluid of density ρ lies above a deep, resting fluid of density $\rho + \Delta\rho$. The line $x = x_B(y, t)$ separates two zones of different but uniform potential vorticity. The layer's velocity components are (u_n, v_n) to the east and north, respectively, while subscripts n refer to the western ($n = 1$) and eastern ($n = 2$) regions of constant potential vorticity. The frontal line $x_B(y, t)$ is oriented essentially north–south, but its position is a function of both latitude and time. The thickness of the upper layer is h_n, $n = 1, 2$, and it clearly must be continuous at $x = x_B$ to ensure continuity of pressure, determined by the hydrostatic approximation across the front at x_B, i.e.,

$$h_1 = h_2 \quad \text{at} \quad x = x_B. \tag{2.1}$$

The downstream velocity, v_n, is in geostrophic balance with the cross-stream pressure gradient. Assuming the hydrostatic approximation and the condition that the lower layer be at rest, this implies that

$$fv_n = \gamma \frac{\partial h_n}{\partial x}, \quad n = 1, 2, \tag{2.2}$$

where f is the Coriolis parameter and $\gamma = g(\Delta\rho/\rho)$ is the "reduced" gravity.

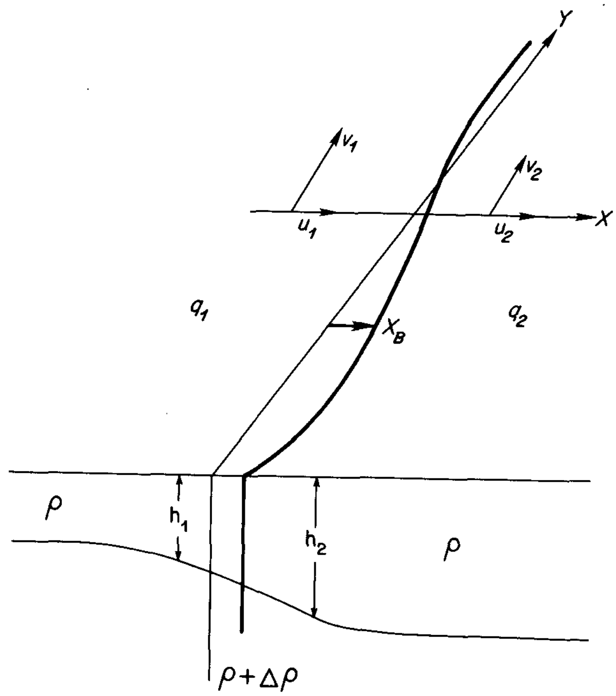

FIG. 1. A schematic showing the front at position $x = x_B(y, t)$. The front separates two zones of constant potential, q_1 and q_2. The fluid has density ρ and lies over a resting layer of density $\rho + \Delta\rho$.

The equation of motion in the downstream direction is

$$\frac{\partial v_n}{\partial t} + u_n \frac{\partial v_n}{\partial x} + v_n \frac{\partial v_n}{\partial y} + fu_n = -\gamma \frac{\partial h_n}{\partial y}, \quad n = 1, 2, \tag{2.3}$$

while mass conservation implies that

$$\frac{\partial h_n}{\partial t} + \frac{\partial}{\partial x}(u_n h_n) + \frac{\partial}{\partial y}(v_n h_n) = 0, \quad n = 1, 2. \tag{2.4}$$

These equations imply that the potential vorticity, q_n, will be conserved where in the present context

$$q_n = \frac{[f + (\partial v_n/\partial x)]}{h_n}, \quad n = 1, 2. \tag{2.5}$$

The potential vorticity in each zone is initially constant but, in general, $q_1 \neq q_2$. Since q_n is conserved within each region, q_n will remain constant for fluid on either side of the boundary line $x_B(y, t)$. On $x = x_B$, h_n is continuous as manifested by (2.1). The second required boundary condition is the kinematic condition. Since $x_B(y, t)$ is the boundary between fluid of differing potential vorticities and since q_n is conserved, the boundary must move with the fluid. Thus,

$$u_n = \frac{\partial x_B}{\partial t} + v_n \frac{\partial x_B}{\partial y}, \quad \text{on} \quad x = x_B(y, t). \tag{2.6}$$

This equation can be rewritten in a form that is more suitable in the present problem. Assuming that q_n is not zero, i.e., that $f + (\partial v_n/\partial x) \neq 0$, multiplication of (2.6) by the absolute vorticity $f + (\partial v_n/\partial x)$ yields with the aid of (2.3)

$$u_n\left(f + \frac{\partial v_n}{\partial x}\right) = \left(f + \frac{\partial v_n}{\partial x}\right)\left(\frac{\partial x_B}{\partial t} + v_n \frac{\partial x_B}{\partial y}\right)$$
$$= -\left(\frac{\partial v_n}{\partial t} + v_n \frac{\partial v_n}{\partial y} + \gamma \frac{\partial h_n}{\partial y}\right), \quad (2.7)$$

or

$$f \frac{\partial x_B}{\partial t} + \left(\frac{\partial v_n}{\partial t} + \frac{\partial v_n}{\partial x}\frac{\partial x_B}{\partial t}\right) + v_n\left(\frac{\partial v_n}{\partial y} + \frac{\partial v_n}{\partial x}\frac{\partial x_B}{\partial y}\right)$$
$$+ f v_n \frac{\partial x_B}{\partial y} + \gamma \frac{\partial h_n}{\partial y} = 0, \quad \text{on} \quad x = x_B. \quad (2.8)$$

Since

$$\frac{\partial}{\partial t} v_n(x_B, y, t) = \frac{\partial v_n}{\partial x}\frac{\partial x_B}{\partial t} + \frac{\partial v_n}{\partial t}$$
$$\frac{\partial v_n(x_B, y, t)}{\partial y} = \frac{\partial v_n}{\partial x}\frac{\partial x_B}{\partial y} + \frac{\partial v_n}{\partial y},$$

the boundary condition (2.6) becomes

$$f \frac{\partial x_B}{\partial t} + \frac{\partial v_n(B)}{\partial t} + v_n(B)\frac{\partial v_n(B)}{\partial y}$$
$$+ \gamma \frac{\partial h_n(B)}{\partial y} = 0, \quad n = 1, 2, \quad (2.9)$$

where

$$v_n(B) \equiv v_n(x_B(y, t), y, t)$$
$$h_n(B) \equiv h_n(x_B(y, t), y, t).$$

In deriving (2.9), the geostrophic balance for v_n, (2.2), has been used to combine the last two terms in (2.8). In (2.9), the t and y derivatives are understood to be taken *after* the functions v_n and h_n are evaluated on $x = x_B$. Thus the y derivatives in (2.9) are not at constant x but follow the local position of the front.

Since x_B is independent of n (2.9) implies that

$$\frac{\partial}{\partial t}[v_1(B) - v_2(B)]$$
$$+ \frac{\partial}{\partial y}\left[\frac{v_1^2(B) - v_2^2(B)}{2}\right] = 0 \quad (2.10)$$

where the continuity of h_n at $x = x_B$ has been used.

It follows from (2.10) that if v_n is continuous all along the front at any time t, then it must remain continuous. However, in this paper, situations will be considered for which $v_1(B) \neq v_2(B)$ at $t = 0$ for some interval in y, and it is the propagation of that shear zone along the front that we wish to describe.

Within each zone the potential vorticity is a constant. Thus with (2.2) this yields as the equation for h_n

$$\frac{\partial^2}{\partial x^2} h_n - \frac{f q_n}{\gamma} h_n = -\frac{f^2}{\gamma} \quad (2.11)$$

where q_n is constant. Further, $q_n > 0$.

The solutions which remain finite as $|x - x_B| \to \infty$ are

$$h_1 = \frac{f}{q_1} + A_1 e^{\lambda_1(x - x_B)} \quad (2.12a)$$

$$h_2 = \frac{f}{q_2} + A_2 e^{-\gamma_2(x - x_B)} \quad (2.12b)$$

where

$$\lambda_n = \left(\frac{f q_n}{\gamma}\right)^{1/2}$$

is the inverse of the deformation radius in each zone. Note that in quasi-geostrophic theory this factor must be independent of horizontal position, but it is not required within the present dynamical context. For large $|x - x_B|$

$$h_n \to \frac{f}{q_n} \equiv h_{n\infty},$$

i.e., the base of the layer asymptotes to a constant in the x-direction which differs from one side of the front to the other. Of course, these asymptotic values for h increase northward as f increases.

Continuity of h_n (pressure) at $x = x_B$ yields

$$A_1 + \frac{f}{q_1} = A_2 + \frac{f}{q_2} \equiv h_B(y, t), \quad (2.13)$$

while using (2.12a,b) in (2.9) yields

$$f \frac{\partial x_B}{\partial t} + \frac{\gamma \lambda_1}{f}\frac{\partial A_1}{\partial t}$$
$$+ \frac{\partial}{\partial y}\left[\frac{\gamma^2 \lambda_1^2}{2f^2} A_1^2 + \gamma\left(A_1 + \frac{f}{q_1}\right)\right] = 0, \quad (2.14a)$$

$$f \frac{\partial x_B}{\partial t} - \frac{\gamma \lambda_2}{f}\frac{\partial A_2}{\partial t}$$
$$+ \frac{\partial}{\partial y}\left[\frac{\gamma^2 \lambda_2^2}{2f^2} A_2^2 + \gamma\left(A_2 + \frac{f}{q_2}\right)\right] = 0, \quad (2.14b)$$

whose difference, with the aid of (2.13) yields

$$\frac{\partial}{\partial t}\left(\frac{\lambda_1 + \lambda_2}{f} A_1\right) + \frac{\partial}{\partial y}\left[\frac{\gamma}{2f^2}(\lambda_1^2 - \lambda_2^2) A_1\right.$$
$$\left. \times \left(A_1 + 2\frac{f}{q_1}\right) - \frac{f q_2 (q_2 - q_1)^2}{2 q_1^2 q_2^2}\right] = 0. \quad (2.15)$$

Noting that λ_n and f are independent of t, (2.15) becomes

$$\frac{\lambda_1+\lambda_2}{f}\frac{\partial h_B}{\partial t}+\frac{\partial}{\partial y}\left[\frac{q_1-q_2}{2f}h_B^2-\frac{f}{2}\frac{(q_1-q_2)}{q_1q_2}\right]=0, \tag{2.16}$$

while (2.14a) may be similarly rewritten as

$$f\frac{\partial x_B}{\partial t}+\frac{\gamma\lambda_1}{f}\frac{\partial h_B}{\partial t}+\gamma\frac{\partial}{\partial y}\left[\frac{q_1h_B^2}{2f}+\frac{f}{2q_1}\right]=0. \tag{2.17}$$

The term $\frac{\partial}{\partial y}\frac{h_B^2}{2f}$ can be eliminated between (2.16) and (2.17) to obtain

$$f\frac{\partial x_B}{\partial t}-\frac{\gamma}{f}\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\frac{\partial h_B}{\partial t}+\frac{\gamma\beta}{2}\left\{\frac{1}{q_1}+\frac{1}{q_2}\right\}=0,$$

where $\beta\equiv\partial f/\partial y$. Thus

$$x_B=\frac{\gamma}{f^2}\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}h_B-\frac{\gamma\beta t}{2f}\left\{\frac{1}{q_1}+\frac{1}{q_2}\right\}+\text{const}. \tag{2.18}$$

A simple linear relation between h_B and x_B therefore obtains modified by a westward shift of the frontal zone. This shift occurs even if h_B itself is independent of time. Thus, in a qualitative way, (2.16) may also be thought of as an equation for the frontal line itself aside from the β-shift described above. A steady solution of (2.16) exits for which

$$v_1(x_B)=\frac{\lambda_1\gamma}{f}A_1=v_2(x_B)=-\lambda_2\frac{\gamma}{f}A_2,$$

which with (2.13) implies that

$$h_B=\frac{f}{(q_1q_2)^{1/2}}, \tag{2.19a}$$

$$v_B=f(\lambda_2^{-1}-\lambda_1^{-1}). \tag{2.19b}$$

This solution consists of a jet flowing along the front in the direction such that an observer looking downstream sees higher potential vorticity on the left. The jet is *steady relative to the front* while the whole system drifts westward at the rate

$$c_\beta=-\frac{\gamma\beta}{2f}\left\{\frac{1}{q_1}+\frac{1}{q_2}\right\}. \tag{2.20}$$

3. Propagation of velocity discontinuities

In this section we will consider the richer solutions of the propagation equation, which occur when the interface x_B is deformed from a straight line. From the discussion of the previous section it is clear that this implies that v_n is discontinuous at x_B whenever $f/h_B \neq (q_1q_2)^{1/2}$. The situation is as depicted in Fig. 2. A disturbance is made to the front between the two regions over some finite region in y. If the frontal position increases, h_B increases by (2.18) and since

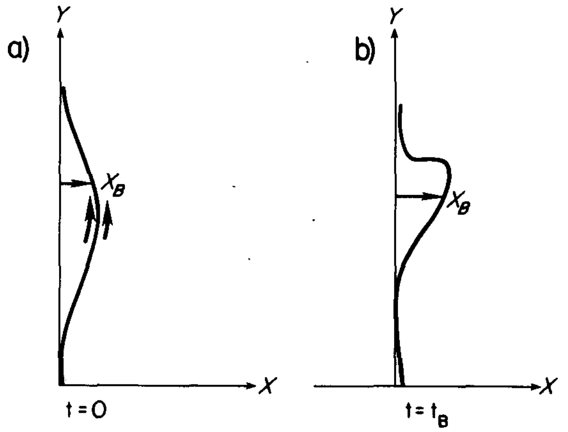

FIG. 2. A schematic of the frontal disturbance showing (a) the initial shear at time $t = 0$ and (b), the shape of the front at $t = t_B$.

$$\frac{\partial}{\partial t}(v_1(x_B)-v_2(x_B))=(\lambda_1+\lambda_2)\frac{\gamma}{f}\frac{\partial h_B}{\partial t}, \tag{3.1}$$

it yields an increasingly anticyclonic shear zone at the front. Such regions of sharp shear can be considered as local peaks in vorticity, akin to point vortices. We can imagine them as anomalous peaks in potential vorticity introduced at the front, and the problem to be considered is how they propagate and how $x_B(y, t)$ and $h_B(y, t)$ evolve as the shear zone propagates.

Rewriting (2.16) as

$$\frac{\lambda_1+\lambda_2}{f}\frac{\partial h_B}{\partial t}+(q_1-q_2)\frac{\partial}{\partial y}\left[\frac{1}{2f}\left(h_B^2-\frac{f^2}{q_1q_2}\right)\right]=0, \tag{3.2}$$

it is clearly useful to define the new dependent variable

$$p=\left(h_B^2-\frac{f^2}{q_1q_2}\right)\Big/2f \tag{3.3}$$

which when inserted in (3.2) yields, after recalling that $\lambda_n^2=fq_n/\gamma$,

$$\frac{\partial p}{\partial t}+\frac{\gamma h_B(\lambda_1-\lambda_2)}{f}\frac{\partial p}{\partial y}=0. \tag{3.4}$$

The propagation speed for the disturbance may be written in a number of ways:

$$c=\frac{\gamma h_B}{f}(\lambda_1-\lambda_2), \tag{3.5a}$$

$$=h_B\left(\frac{\gamma}{f}\right)^{1/2}\{q_1^{1/2}-q_2^{1/2}\}, \tag{3.5b}$$

$$= (\gamma h_B)^{1/2}\left[\left(\frac{h_B}{h_{1\infty}}\right)^{1/2} - \left(\frac{h_B}{h_{2\infty}}\right)^{1/2}\right], \quad (3.5c)$$

$$= \gamma^{1/2}(q_1^{1/2} - q_2^{1/2})\left[2p + \frac{f}{q_1q_2}\right]^{1/2}. \quad (3.5d)$$

Although the last of the forms is the most useful for calculation, (3.5a) reveals that the propagation speed, which is a function of the disturbance amplitude, h_B, depends on the difference of the deformation radii on either side of the front. Consequently, as direct analysis shows, a quasi-geostrophic theory in which the deformation radius must be independent of horizontal position would give rise to *zero* propagation speed.

The system (3.3), (3.5d) can be simply solved by the method of characteristics. Introducing the characteristic variable, s, such that

$$\frac{dt}{ds} = 1 \quad (3.6a)$$

$$\frac{dy}{ds} = \gamma^{1/2}(q_1^{1/2} - q_2^{1/2})\left\{2p + \frac{f}{q_1q_2}\right\}^{1/2}, \quad (3.6b)$$

it follows that

$$\frac{dp}{ds} = 0. \quad (3.6c)$$

Thus in integrating (3.6b), p is held constant so that the characteristic equations become, using $f = f_0 + \beta y$

$$t = s$$

and

$$y = y_0 + \left[\left(\frac{\gamma}{q_2}\right)^{1/2} - \left(\frac{\gamma}{q_1}\right)^{1/2}\right] s\{2pq_1q_2 + f(y_0)\}^{1/2} + \frac{\beta s^2}{4}\gamma\{q_1^{-1/2} - q_2^{-1/2}\}^2, \quad (3.7)$$

where y_0 is the position at $t = 0$ of the particular characteristic under consideration. On the other hand, (3.6c) implies that

$$p = p(y_0). \quad (3.8)$$

The evolution of the disturbance is governed in the classical way (see, for example, Whitham, 1974, Chapter 2) by the crowding together of characteristics in the y–t plane. At any time t the slope of p is given by

$$\frac{\partial p}{\partial y} = \frac{\dfrac{\partial p}{\partial y_0}(y_0)}{\left(\dfrac{\partial y}{\partial y_0}\right)}. \quad (3.9)$$

If we note that

$$2q_1q_2p(y_0) + f(y_0) = \frac{h_B^2(y_0)}{f(y_0)}q_1q_2,$$

then

$$\frac{\partial y}{\partial y_0} = 1 + t[(\gamma q_1)^{1/2} - (\gamma q_2)^{1/2}]\frac{\partial}{\partial y_0}\left\{\frac{h_B(y_0)}{f^{1/2}(y_0)}\right\}$$

$$= 1 + t[\lambda_1(y_0) - \lambda_2(y_0)]\frac{\gamma h_B}{f(y_0)}\left[\frac{1}{h_B}\frac{\partial h_B}{\partial y_0} - \frac{\beta}{2f}\right]$$

$$= 1 + tc(y_0)\left[\frac{1}{h_B}\frac{\partial h_B}{\partial y_0} - \frac{\beta}{2f(y_0)}\right], \quad (3.10)$$

where $h_B(y_0)$ is the shape of the disturbance (the depth of the moving layer at the front) at the initial instant, $t = 0$. According to the long-wave theory developed here, the disturbance will break whenever

$$c(y_0)\left[\frac{1}{h_B}\frac{\partial h_B}{\partial y_0} - \frac{\beta}{2f(y_0)}\right] < 0. \quad (3.11)$$

Thus if, for example, $c(y_0) > 0$ ($q_1 > q_2$), then long-wave breaking, which represents a folding of the shear zone, requires only that

$$\frac{1}{h_B}\frac{\partial h_B(y_0)}{\partial y_0} < \frac{\beta}{2f(y_0)}. \quad (3.12)$$

On an f-plane, this condition is equivalent to the classical situation in which the wave steepens on the forward face of the wave ($\partial h_B/\partial y < 0$). The time at which breaking will occur is

$$t_B = -\left\{c(y_0)\left[\frac{1}{h_B}\frac{\partial h_B}{\partial y_0} - \frac{\beta}{2f(y_0)}\right]\right\}_{\max}^{-1}, \quad (3.13)$$

where the subscript, max, refers to the maximum value of the quantity in the enclosed bracket as a function of y_0. Equivalently,

$$t_B^{-1} = \left[(\gamma q_1)^{1/2} - (\gamma q_2)^{1/2}\right]\left[\frac{\partial}{\partial y_0}\frac{h_B(y_0)}{f^{1/2}(y_0)}\right]_{\max}, \quad (3.14)$$

so that aside from the effect of the β-effect, it is the steepness of the slope in the y direction of the base of the layer in the initial data that determines the time to breaking of the front. The time, t_B, is also inversely proportional to $q_1^{1/2} - q_2^{1/2}$, i.e., to the strength of the potential vorticity jump across the front.

If, for example, we choose $h_{1\infty}$ and $h_{2\infty}$ to be 400 and 800 meters, respectively, and h_B to be initially 600 meters with a characteristic scale of downstream variation of the order of 200 km, then c is of order 90 cm/sec, and the time to breaking, ignoring the negligible effect of β on these scales, would be of the order of two days, i.e., quite fast. The time t_B will simply linearly increase with the scale length

$$\left(\frac{1}{h_B}\frac{dh_B}{dy_0}\right)^{-1}.$$

Of course, the theoretical prediction of wave breaking at $t = t_B$ is based entirely on the semi-geostrophic dynamics embodied in (2.2). If the motion initially satisfies the requirements of semigeostrophy, i.e., that $\partial/\partial y \ll \partial/\partial x$, it will increasingly less do so with time as the steepening implicit in (3.6) occurs. Motion on smaller y scales will be dispersive. All that can be said with confidence is that disturbances which are originally long compared with a deformation radius will fairly swiftly steepen to scales that are short even in the along-front direction.

Consider now the situation in which initially the front is stationary (aside from the weak β drift) and the velocity is continuous at the front. Then from (2.19a,b) and (3.5a), it follows that

$$v_B = f\left\{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right\} = c; \qquad (3.15)$$

so that the velocity of the fluid in this special case is coincident with the propagation speed. Naturally, there is no disturbance at the end point of this limit. Suppose we consider a small perturbation to this system, i.e., let

$$v_1(x_B) = V + v_1'$$
$$v_2(x_B) = V + v_2' \qquad (3.16)$$

where $V = f(\lambda_2^{-1} - \lambda_1^{-1})$ and where $|v_n'| \ll V$ but $v_2' \neq v_1'$. That is, we impose a small shear discontinuity on the front. For now, let us ignore the y variation of f so that V is independent of y. Then, for as long as the disturbance remains small, $\delta v \equiv v_1 - v_2$ will satisfy

$$\frac{\partial}{\partial t}\delta v + V\frac{\partial}{\partial y}\delta v = 0. \qquad (3.17)$$

Thus small shear perturbations will propagate with the mean speed, V, which, when it is continuous across the front, is identical to c. However, the solution to the full nonlinear equation will, by (3.9) and (3.10), lead inexorably to an increase in both $\partial h_B/\partial y$ and $\partial h_B/\partial t$. As a consequence, from (3.1), it follows that the shear discontinuity across the front will increase with time as the wave steepens. In this sense, flows along the potential vorticity front are unstable with respect to small "tears" in the along-front velocity. These tears will increase as the frontal perturbation propagates and sharpens.

4. Discussion

A very simple, semigeostrophic model of perturbations to a potential vorticity front oriented north–south on the β-plane allows a straightforward analytical description of the propagation, steepening, and growth of strong shear zones. These shear zones are modeled as velocity discontinuities, finite in length, which propagate along the front at a rate which depends on the difference in the value of the potential vorticity across the front. The propagation is an effect absent in a quasi-geostrophic long-wave theory, depending, as it does, on the horizontal difference of the Rossby deformation radius which enters the dispersion relation for disturbances that propagate along the front. As in most nonlinear hyperbolic systems, the perturbation in both fluid depth and frontal excursion steepen with time. The long-wave theory, while predicting breaking, loses its validity when the long-front scale shrinks to the cross-front scale (deformation radius). Hence, the time for breaking is most conservatively to be considered as the time for the development of small-scale along-front structures produced rather rapidly from initially long perturbations. Whether this process has anything to do with observed small-scale structures in coastal regions is unclear at this stage. Direct shear instabilities on small long-front scales could also, a priori, provide a further mechanism for the production of such structures. Also, very large initial meander amplitudes can lead to front roll-up (breaking) even in quasi-geostrophic frontal models in which the cross-front velocity remains continuous (Pratt 1988). This mechanism will vanish if, as in the case studied here, the initial slope of the front in the x, y plane is small.

It would be of interest, in view of the original motivation of this work described in the Introduction, to include the presence of a wall at some finite distance from the front. Then the Kelvin wave natural to the wall region and the vorticity wave on the potential vorticity front would interact. The consequences of that interaction on the propagation and steepening of both waves would be interesting, especially as the direction of the Kelvin wave is fixed but that of the vorticity wave depends on the ratio q_2/q_1.

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