Long-Wave Generation Due to the Refraction of Short-Wave Groups over a Shear Current

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ABSTRACT

The generation of long waves by short-wave groups propagating over a shear current is studied. The incident wave groups consist of two co-linear short waves with slightly different frequencies. These short waves are refracted by the shear currents. For certain angles of incidence caustics may exist and the short waves are reflected back by the shear current. Similarly, caustics could also appear in the wave envelope, which propagates in a different direction from that of the short waves. In the present paper, the treatment of caustics is not considered. In the region where the shear current vanishes, the short-wave groups are accompanied by locked long waves that propagate with the wave envelope of short waves at their group velocity. In the shear current region, the refraction effects not only separate the propagation directions of the short waves and the wave envelope but also generate free long waves, which propagate at the speed of $\sqrt{gh}$. The free long waves could also be trapped over the shear current region.

1. Introduction

The dynamical effects of a current on surface waves are of interest in the refraction of swell by oceanic currents such as the Gulf Stream (Kenyon 1971), the warm-core rings (Mapp et al. 1985), and internal-wave-induced surface currents (Thompson, Gotwols and Sterner 1988). Comprehensive reviews of refraction theories of short waves over currents and their applications have been made by Peregrine (1976) and more recently by Jonsson (1989). However, the incident sea is seldom uniform and the amplitudes of short waves modulate as wave groups. The refraction of wave groups by a current and the generation of long waves have not been carefully examined.

Using the concept of radiation stresses, Longuet-Higgins and Stewart (1962) demonstrated that when a train of periodical water wave groups propagates over a constant depth, a second-order locked long wave exits and propagates with the group velocity of carrier waves. The propagation direction of the locked long waves is the same as that of the wave envelope of the wave groups. If the wave groups are diffracted by a vertical structure or are refracted over a slowly varying depth, a different kind of second-order long waves could be generated (Mei and Benmoussa 1984; Zhou and Liu 1987). These long waves propagate with a speed of $\sqrt{gh}$, where h is the local depth, in a direction unrelated to the direction of propagation of the wave groups and are, therefore, called free long waves. These long waves, although second-order in magnitude, could play a significant role in coastal oceanography problems, such as sediment transport and bar generation, if they are trapped and resonated in the nearshore area (Bowers 1977; Symonds and Bowen 1984; Roelvink and Stive 1989).

In this paper we focus on the generation and propagation of second-order long waves due to the interaction of modulated waves and a shear current. The current velocity field is assumed to be unidirectional and varies slowly within a characteristic wave length. The obliquely incident wave groups propagate with the currents. The reflection of the wave groups by the shear currents is assumed to be negligible. It is shown herein that caustics could exist for both carrier waves and the wave envelope. However, our present analysis for the long waves excludes the caustics.

In the following section we summarize the governing equations for the wave envelope of the wave groups and the associated long waves. The assumptions and approximations used in the paper are elaborated in this section. In section 3 the results for the carrier

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(short) waves over a shear current region are presented. The conditions for the existence of caustics in both carrier waves and wave envelope are discussed. The solutions for long waves are investigated in sections 4 and 5. The free long waves could either radiate away from or be trapped within the shear current region, depending on the angle of wave incidence and the current velocity. Some numerical results are given in section 6.

2. Governing equations

Consider a train of modulated sinusoidal water waves propagating over a slowly varying shear current. The length- and time-scales of the wave groups are much longer than those of the carrier waves. Denoting k_0 and ω_0 as the carrier wavenumber and frequency, respectively, we assume that the wavenumber and frequency of the wave envelope are ϵk_0 and $\epsilon\omega_0 C_{g0}/C_0$, where $\epsilon \ll 1$ and C_0 and C_{g0} are the phase and group velocity of the short waves propagating over a constant depth h. The small parameter ϵ characterizes the wave slope of the carrier waves. The current velocity is assumed to be $O(\mu^{-1})$ times the orbital velocity of the carrier waves and in the same order of magnitude as the phase velocity C_0. We further assume that the current velocity changes slowly within a typical wavelength; i.e.,

$$O\left(\frac{1}{k_0|\mathbf{U}|}|\nabla\mathbf{U}|\right) = O(\delta), \qquad (2.1)$$

where $\mathbf{U}$ denotes the current velocity vector. We consider the case where all small parameters are of the same order of magnitude, i.e., $O(\mu) = O(\delta) = O(\epsilon)$. Typical numerical values for different physical variables involved are: carrier wave period = O(10 s), group and long-wave period = O(100 s), depth = O(20 m), the length scale for the current = O(200 m), carrier wave amplitude = O(2 m) and the maximum current velocity = O(10 m s^{-1}).

Introducing the slow variables

$$\mathbf{X} = \epsilon\mathbf{x}, \quad \text{with} \quad \mathbf{x} = (x, y); \quad T = \epsilon t, \qquad (2.2)$$

we can express the first-order free-surface displacement and velocity potential in the following form:

$$\zeta = \frac{1}{2}A(\mathbf{X}, T)\exp\left[i\left(\int^{\mathbf{x}}\mathbf{k}(\mathbf{x})\cdot d\mathbf{x} - \omega_0 t\right)\right] + *, \qquad (2.3)$$

$$\phi = -\frac{igA}{2\sigma}\frac{\cosh k(z+h)}{\cosh kh} \times \exp\left[i\left(\int^{\mathbf{x}}\mathbf{k}(\mathbf{x})\cdot d\mathbf{x} - \omega_0 t\right)\right] + *, \qquad (2.4)$$

where * denotes the complex conjugate of the preceeding term, $A(\mathbf{X}, t)$ is the modulated wave envelope, and σ the intrinsic wave frequency

$$\sigma = \omega_0 - \mathbf{k}\cdot\mathbf{U}. \qquad (2.5)$$

The wavenumber $\mathbf{k}(\mathbf{X})$ is governed by the dispersion relation

$$\sigma^2 = gk\tanh kh, \quad \text{with} \quad k = |\mathbf{k}|, \qquad (2.6)$$

and the irrotationality of the wavenumber vector,

$$\nabla\times\mathbf{k} = 0. \qquad (2.7)$$

The complex wave amplitude A varies slowly in both space and time. For the case without currents Liu and Dingemans [1989, Eq. (4.7)] derived an equation for A. In a similar way it can be shown that A now satisfies the following complex equation:

$$\frac{\partial}{\partial T}\left(\frac{A^2}{\sigma}\right) + \nabla\cdot\left[\left(C_g\frac{\mathbf{k}}{k} + \mathbf{U}\right)\frac{A^2}{\sigma}\right] = 0, \qquad (2.8)$$

where $\nabla = (\partial/\partial X, \partial/\partial Y)$ and $C_g = d\sigma/dk$ is the local group velocity. The real part of (2.8) leads to the well-known wave action equation. At second order, the mean water level $\xi(\mathbf{X}, T)$ and the wavegroup induced long-wave velocity potential Φ are governed by the following equations (Kirby 1983):

$$\xi = -\frac{1}{g}\frac{D\Phi}{DT} - \frac{\sigma^2|A|^2}{4g\sinh^2 kh}, \qquad (2.9)$$

$$\frac{\partial\xi}{\partial T} + \nabla\cdot\left[\xi\mathbf{U} + h\nabla\Phi + \frac{\mathbf{k}g|A|^2}{2\sigma}\right] = 0, \qquad (2.10)$$

where

$$\frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U}\cdot\nabla. \qquad (2.11)$$

Taking the horizontal gradient of (2.9) yields

$$\frac{D}{DT}\nabla\Phi + \nabla\left[g\xi + \frac{\sigma^2|A|^2}{4\sinh^2 kh}\right] = 0, \qquad (2.12)$$

which is the depth-averaged horizontal momentum equation, while (2.10) is the depth-averaged continuity equation. Eliminating ξ from (2.10) and (2.12), we obtain

$$\frac{D^2\Phi}{DT^2} + (\nabla\cdot\mathbf{U})\frac{D\Phi}{DT} - g\nabla\cdot(h\nabla\Phi) = \frac{g^2}{2}\nabla\cdot\left(\mathbf{k}\frac{|A|^2}{\sigma}\right) - \frac{gk}{2\sinh 2kh}\frac{\partial|A|^2}{\partial T} - \nabla\cdot\left[\frac{gk|A|^2}{2\sinh 2kh}\mathbf{U}\right]. \qquad (2.13)$$

The long-wave equation (2.13) has been derived by Kirby (1983) using a mean Lagrangian method. If the current velocity vanishes, (2.13) reduces to

$$\frac{\partial^2\Phi}{\partial T^2} - g\nabla\cdot(h\nabla\Phi) = \frac{g^2}{2}\nabla\cdot\left(\mathbf{k}\frac{|A|^2}{\omega}\right) - \frac{gk}{2\sinh 2kh}\frac{\partial|A|^2}{\partial T}, \quad (2.14)$$

which has been derived by Chu and Mei (1970) using the multiple-scales perturbation method. We stress that governing equations (2.8) and (2.13) are valid only in the range $O(\mathbf{X}, T) = O(1)$; for larger time and spatial domains, higher-order effects must be considered.

In this paper the long waves generated by the refraction of wave groups over a one-dimensional shear current are investigated. The current velocity is specified as

$$\mathbf{U} = (0, V(X)). \quad (2.15)$$

The long-wave equation, (2.13), can be simplified to be

$$\frac{\partial^2\Phi}{\partial T^2} + 2V\frac{\partial^2\Phi}{\partial T\partial Y} + (V^2 - gh)\frac{\partial^2\Phi}{\partial Y^2} - gh\frac{\partial^2\Phi}{\partial X^2} = \frac{g^2}{2}\nabla\cdot\left(\mathbf{k}\frac{|A|^2}{\sigma}\right) - \frac{gk}{2\sinh 2kh}\left(\frac{\partial|A|^2}{\partial T} + V\frac{\partial|A|^2}{\partial Y}\right), \quad (2.16)$$

in which the local short wavenumber $\mathbf{k}$ can be determined from the refraction

$$\mathbf{k}(X) = (k_x, k_y), \quad k_x = k\cos\alpha, \quad k_y = k\sin\alpha = k_0\sin\alpha_0, \quad (2.17)$$

where $\alpha(X)$ is the local angle of the carrier wave propagation and the subscript zero denotes quantities associated with the incident waves.

The following normalized variables will be used in the rest of this paper:

$$\mathbf{X} \to \frac{\mathbf{X}}{k_\infty}, \quad T \to \frac{T}{\omega_0}, \quad \sigma \to \omega_0\sigma, \quad h \to \frac{h}{k_\infty},$$

$$k \to k_\infty k, \; A \to aA, \; C_g \to \frac{\omega_0}{k_\infty}C_g, \; \xi \to k_\infty(2a)^2\xi,$$

$$\Phi \to (2a)^2\omega_0\Phi, \quad \mathbf{U} \to \frac{\omega_0}{k_\infty}\mathbf{U}, \quad (2.18)$$

where a is the incident carrier wave amplitude and $k_\infty = \omega_0^2/g$.

3. Short waves over a one-dimensional shear current

In this paper, we focus our attention on one class of shear currents where the velocity increases from zero at $X = X_0$ to a maximum value, and then decreases to zero again at $X = X_1$. We consider only incident waves propagating with the currents. Therefore, k_y is always a positive constant. It is well known from the geometric optic theory that the wavenumber of the short waves, k, over the shear current is always smaller than the incident wavenumber, k_0, and has a minimum value at the location of maximum current speed. If the minimum wavenumber is greater than k_y, wave rays of the short waves pass through the shear current; the direction of the wave propagation in the region $X > X_1$ is the same as that of the incident waves. On the other hand, if the minimum wavenumber is less than k_y, k_x becomes zero at a certain location within the shear current region and the rays are bent backwards. The effects of reflection become important. In this paper we will not study the cases involving caustics.

The incident wave groups considered here are the superposition of two co-linear periodical wave trains of slightly different frequencies, $1 \pm \Omega_0$, (the corresponding wavenumbers are $\mathbf{k}_0 \pm \mathbf{K}_0$) with normalized amplitudes 1 and b, respectively. Within the spatial and time domain, $O(\mathbf{X}, T) = O(1)$, the incident wave envelope can be expressed as (see Liu et al. 1989):

$$A^I(X, Y, T) = \exp(i\chi_0) + b\exp(-i\chi_0), \quad X < X_0$$

with

$$\chi_0(X, Y, T) = K_{x0}(X - X_0) + K_{y0}Y - \Omega_0 T \quad (3.1)$$

where

$$K_{x0} = K_0\cos\alpha_0, \quad K_{y0} = K_0\sin\alpha_0,$$

$$\Omega_0 = K_0 C_{g0} = K_0\frac{d\omega}{dk_0}, \quad (3.2)$$

in which α_0 is the angle of incidence of the carrier waves. We may choose either Ω_0 or K_0 so that (3.1) is completely determined. For simplicity, we shall use $K_0 = k_0$ in the following analysis.

In the shear current region, $X_0 \leq X \leq X_1$, the wave envelope can be written as

$$A(X, Y, T) = \tilde{A}(X)[\exp(i\chi) + b\exp(-i\chi)], \quad X_0 < X < X_1$$

with

$$\chi = \int_{X_0}^{X} K_x dX + K_{y0}Y - \Omega_0 T \quad (3.3)$$

where the amplitude function $\tilde{A}(X)$ and the wavenumber component $K_x(X)$ are real functions of X. Substituting (3.3) into equation (2.8), we obtain

$$\tilde{A}(X) = \left[\frac{C_{gx0}}{C_{gx}}\right]^{1/2}\sigma^{1/2}, \quad (3.4)$$

$$\Omega_0 = (V + C_{g_y})K_{y0} + K_x C_{g_x}, \quad (3.5)$$

where

$$\sigma = 1 - k_y V, \quad (3.6)$$

$$C_{g_x} = C_g \cos\alpha, \quad C_{g_y} = C_g \sin\alpha. \quad (3.7)$$

Using (3.2) and the definition for α, (3.7), in (3.5) yields

$$K_x(X) = k_x + \frac{k^2}{k_x}\left[\frac{\Sigma}{kC_g} - 1\right], \quad (3.8)$$

where

$$\Sigma = \Omega_0 - K_{y0}V \quad (3.9)$$

is the intrinsic frequency of the wave envelope. The effect of the Doppler shift on the envelope propagation is included in (3.8). Because k_x and K_x are different, the direction of propagation of the wave envelope, defined as

$$\theta = \arctan\left(\frac{K_{y0}}{K_x}\right),$$

is different from that of the carrier waves.

As pointed out before, caustics could exist in the carrier wave field and k_x becomes zero along the caustics. From the constancy of the y-component of the wavenumber vector it follows that the waves are trapped when $(k_0/k)\sin\alpha_0 \geq 1$, so that $\sin\alpha = (k_0/k)\sin\alpha_0$ has no real solution for α. The corresponding critical angle of incidence is

$$\alpha_{cr} = \arcsin\left(\frac{k}{k_0}\right). \quad (3.10)$$

From (3.8) it can be shown that K_x could also become zero, if

$$\sin^2\alpha = \frac{\Sigma}{kCg} \quad (3.11)$$

is satisfied. Hence, the wave envelope could also experience caustics for certain combinations of angle of wave incidence and current speed. Equating (3.2) and (3.5) and using $K_x = 0$ and (2.17), the result can be written as a quadratic equation for $\sin\alpha_0$, and we find the critical angle of incidence for the appearance of caustics in wave envelope to be

$$\bar{\alpha}_{cr} = \arcsin\left\{\frac{1}{2}\frac{kV}{k_0C_g}\left[1 + \left(1 + \frac{4k_0C_{g0}C_g}{kV^2}\right)^{1/2}\right]\right\}. \quad (3.12)$$

K_x and k_x are computed for dimensionless depths [see (2.18)] $h = 0.5$ and 1.0 with different current speed V. As shown in Fig. 1, the caustics of the wave envelope occur at a smaller current speed than that of the carrier waves. Of course, the present theory becomes invalid when any caustics appear.

4. Long waves in the regions of zero current

In this section the second-order long waves are obtained for the regions where current vanishes. Substitution of (3.3) into the nondimensional form of (2.16) yields

$$\left(\frac{\partial}{\partial T} + V\frac{\partial}{\partial Y}\right)^2\Phi - h\left(\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2}\right) = \frac{(1+b^2)}{8}\frac{d}{dX}\left[\frac{k_x}{\sigma}|\tilde{A}|^2\right] + \frac{b}{4}\left\{\frac{1}{2}\frac{d}{dX}\left[\frac{k_x}{\sigma}|\tilde{A}|^2\right] + ik|\tilde{A}|^2\Sigma\left(\frac{1}{\sinh 2kh} + \frac{1}{\sigma C_g}\right)\right\}\cdot\exp\left[2i\left(\int_{X_0}^{X} K_x dX + K_{y0}Y - \Omega_0 T\right)\right] + *, \quad (4.1)$$

which suggests that the long-wave potential Φ should be decomposed into the steady-state and the dynamic components:

$$\Phi = \Phi_0(X) + \frac{1}{2}\phi(X)\exp[2i(K_{y0}Y - \Omega_0 T)] + *. \quad (4.2)$$

For the steady-state component, (4.1) may be integrated once to obtain

$$\frac{d\Phi_0}{dX} = -\frac{(1+b^2)}{8h}\frac{k_x}{\sigma}|\tilde{A}|^2 + c, \quad (4.3)$$

where c is an integration constant. Equation (4.3) represents the wave-induced mass transport velocity component in the x-direction. The steady-state velocity potential does not affect the mean free-surface displacement ξ in (2.9).

The amplitude function of the unsteady long-wave potential, ϕ, satisfies

$$\frac{d^2\phi}{dX^2} + \frac{4}{h}(\Sigma^2 - K_{y0}^2 h)\phi = -\frac{b}{2h}\left\{\frac{1}{2}\frac{d}{dX}\left[\frac{k_x}{\sigma}|\tilde{A}|^2\right] + ik|\tilde{A}|^2\Sigma\left(\frac{1}{\sinh 2kh} + \frac{1}{\sigma C_g}\right)\right\}\exp\left[2i\int_{X_0}^{X} K_x dX\right]. \quad (4.4)$$

In the region of zero current velocity, $X > X_0$ and $X < X_1$, (4.4) becomes

$$\frac{d^2\phi}{dX^2} + \frac{4}{h}(\Omega_0^2 - K_{y0}^2 h)\phi = -\frac{ib}{2h}k_0\Omega_0\left(\frac{1}{\sinh 2k_0h} + \frac{1}{C_g}\right)\Bigg\}\exp\left[2i\int_{X_0}^{X} K_x dX\right], \quad (4.5)$$

where $K_x = K_{x0}$ when $X > X_1$ and $X < X_0$. The long-wave potential can be further split into two parts: the locked long waves and free long waves. The locked long wave is the particular solution of (4.5), while the free long wave is the homogeneous solution of (4.5). Thus, the solution for (4.5) can be expressed as

$$\phi = \phi_L \exp\left[2i \int_{X_0}^{X} K_x dX\right] + \phi_F(X), \qquad (4.6)$$

with

$$\phi_L = -\frac{ib}{8(\Omega_0^2 - K_0^2 h)} k_0 \Omega_0 \left(\frac{1}{\sinh 2k_0 h} + \frac{1}{C_{g0}}\right), \qquad (4.7)$$

where $K_0^2 = K_{y0}^2 + K_{x0}^2$. The corresponding mean free-surface level can be calculated from (2.9)

$$\xi_L = -\frac{(1+b^2)}{16 \sinh^2 k_0 h} + \frac{b}{4(\Omega_0^2 - K_0^2 h)}\left(\frac{k_0 \Omega_0^2}{C_g} + \frac{K_0^2 h}{2 \sinh^2 k_0 h}\right) \exp\left[2i\left(\int_{X_0}^{X} K_x dX + K_{y0} Y - \Omega_0 T\right)\right] + *. \qquad (4.8)$$

Here the first term on the right-hand side of the above equation represents the steady mean free-surface set-down without wave groups. Mei and Benmoussa (1984) also derived an equation equivalent to (4.8). The locked long waves propagate in the same direction and with same speed as those of the wave envelope.

The free long-wave potential ϕ_F in the region of zero current velocity satisfies the following equation:

$$\frac{d^2\phi_F}{dX^2} + 4k_0^2\left(\frac{C_{g0}^2}{h} - \sin^2\alpha_0\right)\phi_F = 0, \qquad (4.9)$$

for $X > X_1$ and $X < X_0$. If the coefficient in front of ϕ_F is positive, the free long waves must be outgoing waves propagating away from the current region.

$$(\phi_F)_j = B_j \exp[2i|\lambda| \cdot |X - X_j|], \qquad (4.10)$$

where

$$\lambda^2 = k_0^2\left(\frac{C_{g0}^2}{h} - \sin^2\alpha_0\right), \qquad (4.11)$$

and $j = 0$ and 1, for $X < X_0$ and $X > X_1$, respectively. Since $C_{g0}^2/h < 1$, there always exist an incident angle α_0 such that λ^2 becomes negative. In those cases $(\phi_F)_j$ attenuates exponentially

$$(\phi_F)_j = B_j \exp[-2|\lambda| \cdot |X - X_j|], \quad j = 0, 1. \qquad (4.12)$$

The long waves are trapped over the shear current region.

5. Long waves over the shear current region

Over the shear current region, the long-wave potential can not always be obtained analytically. Numerical solutions are usually required. Substituting (3.4) into (4.4), we obtain

$$\frac{d^2\phi}{dX^2} + \frac{4}{h}(\Sigma^2 - K_{y0}^2 h)\phi = -\frac{b}{2h}\left\{\frac{C_{gx0}}{2}\frac{d}{dX}\left[\frac{k}{C_g}\right] + ik\Sigma\frac{C_{gx0}}{C_{gx}}\left(\frac{\sigma}{\sinh 2kh} + \frac{1}{C_g}\right)\right\}\exp\left[2i\int_{X_0}^{X} K_x dX\right], \qquad (5.1)$$

where

$$\frac{d}{dX}\left(\frac{k}{C_g}\right) = \frac{1}{C_g}\frac{dk}{dX} - \frac{k}{C_g^2}\frac{dC_g}{dX}, \quad \frac{dk}{dX} = -\frac{k_y}{C_g}\frac{dV}{dX},$$

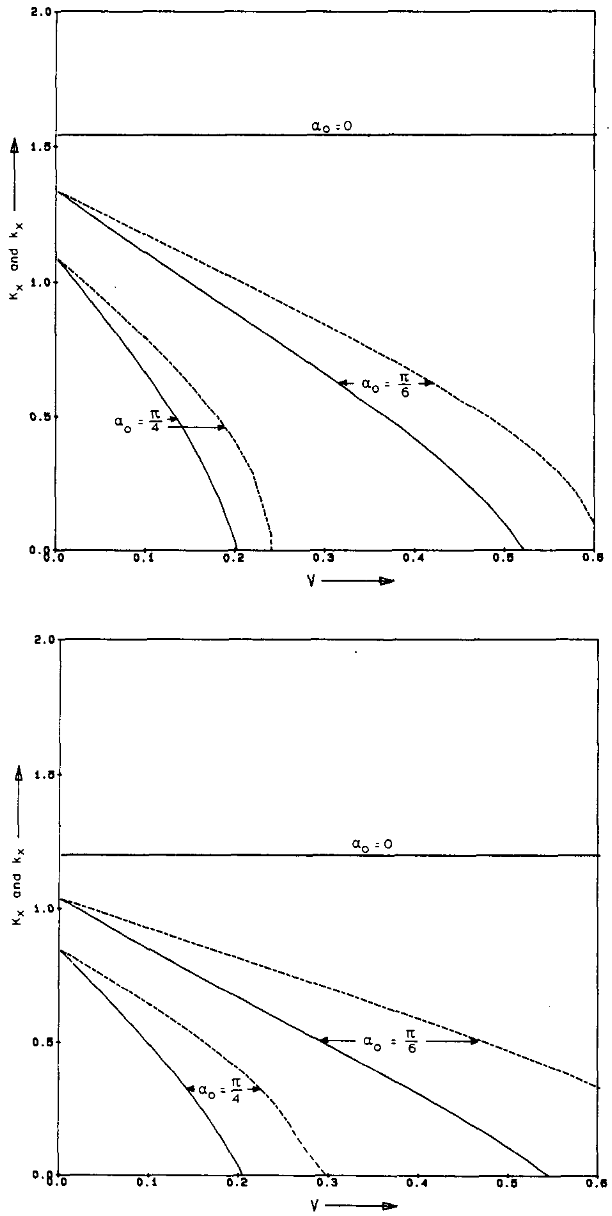

FIG. 1. Wavenumber components for carrier waves, k_x (dashed) and wave envelope, K_x (solid) for different angle of incidence, α_0; (a) $h = 0.5$ and (b) $h = 1.0$.

and

$$\frac{dC_g}{dX} = -\frac{k_y}{\sigma} C_g \frac{dV}{dX} \times \left[\frac{C_g}{k} + \frac{\sigma h}{k \sinh 2kh}\left(\frac{2kh}{\tanh 2kh} - 1\right)\right]\frac{dk}{dX}. \quad (5.2)$$

The boundary conditions require the continuity of ϕ and $d\phi/dX$ along $X = X_0$ and X_1. Thus

$$\phi_L + B_0 = \phi(X_0) \quad (5.3)$$

$$2iK_{x0}\phi_L + 2B_0\binom{-i}{1}|\lambda| = \frac{d\phi(X_0)}{dX}, \quad \text{for} \quad \lambda^2 \gtrless 0, \quad (5.4)$$

$$\phi_L \exp\left(2i\int_{X_0}^{X_1} K_x dX\right) + B_1 = \phi(X_1), \quad (5.5)$$

$$2iK_{x0}\phi_L \exp\left(2i\int_{X_0}^{X_1} K_x dX\right) + 2B_1\binom{i}{-1}|\lambda| = \frac{d\phi(X_1)}{dX}, \quad \text{for} \quad \lambda^2 \gtrless 0, \quad (5.6)$$

where ϕ_L is given in (4.7) and K_x in (3.8). Eliminating B_0 from (5.3) and (5.4), and B_1 from (5.5) and (5.6), we obtain

$$\frac{d\phi(X_0)}{dX} - 2\binom{-i}{1}|\lambda|\phi(X_0) = 2\left[iK_{x0} - \binom{-i}{1}|\lambda|\right]\phi_L, \quad \text{for} \quad \lambda^2 \gtrless 0, \quad (5.7)$$

$$\frac{d\phi(X_1)}{dX} - 2\binom{i}{-1}|\lambda|\phi(X_1) = 2\left[iK_{x0} - \binom{i}{-1}|\lambda|\right]\phi_L \exp\left(2i\int_{X_0}^{X_1} K_x dX\right),$$
$$\text{for} \quad \lambda^2 \gtrless 0. \quad (5.8)$$

Equations (5.1), (5.7), and (5.8) can be expressed as simultaneous three-point difference equations with a tridiagonal coefficient matrix. Once solutions for ϕ are obtained, B_0 and B_1 can be calculated from (5.3) and (5.5), respectively. Numerical examples are given in the following section.

6. Numerical examples

Numerical computations were carried out for the following current velocity profile:

$$V(X) = \frac{V_m}{2}\left\{1 + \cos\left[\pi\left(\frac{2(X - X_0)}{X_1 - X_0} - 1\right)\right]\right\}, \quad (6.1)$$

where V_m represents the maximum current velocity at $X = (X_1 - X_0)/2$. For $0 < \alpha_0 < \pi/4$, V_m was chosen as 0.2 for $h = 0.5$ and 1.0 such that caustics did not appear in both the carrier wave field and the wave envelope. In Fig. 2, the long-wave potentials over the shear current region are plotted for $(X_1 - X_0) = 1.0$. For normal incidence, the wave groups are not affected by the shear current. The total long-wave potential for $\alpha_0 = 0$ shown in Fig. 2 is the locked long-wave potential associated with the incident wave groups. The corresponding amplitudes for the free long waves propagating away from the current region, $|B_0|$ and $|B_1|$, are given in Table 1. The amplitudes of the free long wave are zero for the normal incident wave. The long

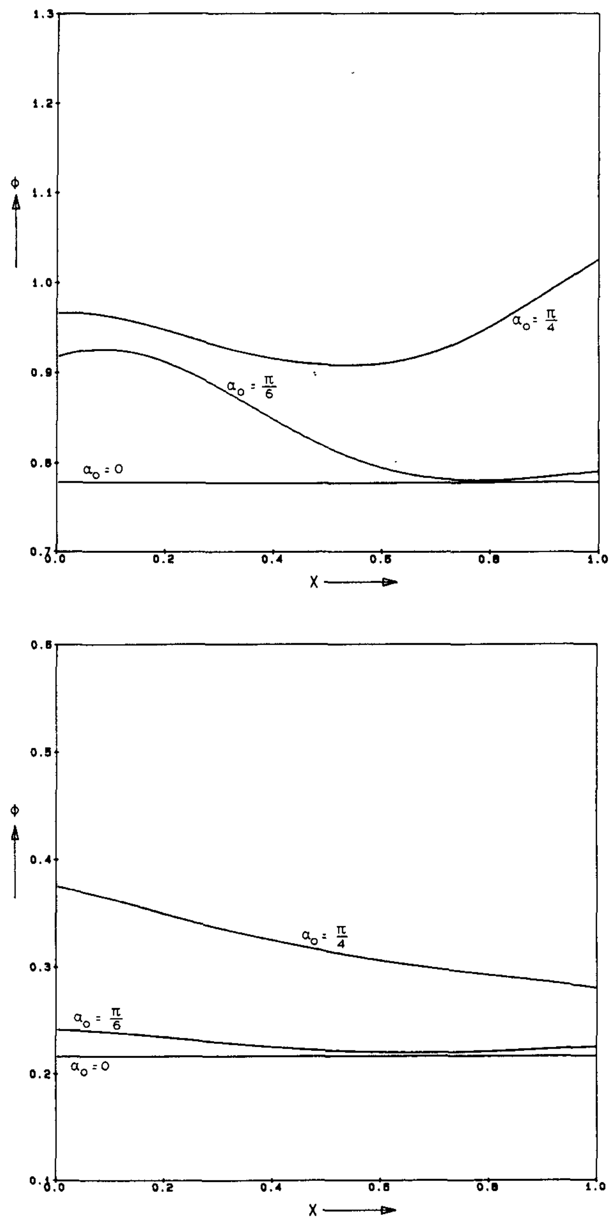

FIG. 2. Total long-wave potential over a shear current region for different angle of incidence, α_0. The normalized maximum velocity is 0.2; (a) $h = 0.5$ and (b) $h = 1.0$.

TABLE 1. Wave amplitudes of free long waves at X_0 and X_1, $|B_0|$ and $|B_1|$, respectively, for $Vm = 0.2$, and $(X_1 - X_0) = 1.0$.

	h			
	0.5		1.0	
α_0	$\pi/6$	$\pi/4$	$\pi/6$	$\pi/4$
$\|B_0\|$	0.146904	0.187476	0.026048	0.160288
$\|B_1\|$	0.075690	0.438783	0.018943	0.063238

TABLE 3. Wave amplitudes of free long waves at X_0 and X_1, $|B_0|$ and $|B_1|$, respectively, for $V_m = 0.2$, and $h = 1.0$.

	$(X_1 - X_0)$			
	0.5		2.0	
α_0	$\pi/6$	$\pi 4$	$\pi/6$	$\pi/4$
$\|B_0\|$	0.013903	0.144985	0.042669	0.176895
$\|B_1\|$	0.009837	0.088388	0.031651	0.049856

wave amplitudes increase as the angles of incidence increase. The wave amplitudes also increase as the depth becomes shallower. For the case where $h = 1$ and $\alpha_0 = \pi/4$, the free long waves are trapped over the current region.

In Table 2, the amplitudes of free long waves for different maximum current velocity are shown. For a fixed width of the shear current region, the amplitudes increase as the current velocity increases. On the other hand, when the maximum velocity, V_m, is fixed, the free long-wave amplitudes decrease as the width of the current region decreases for $\alpha_0 = \pi/6$ and $\pi/4$ with the exception that for $\alpha_0 = \pi/4$ $|B_1|$ increases slightly when the width decreases (Table 3).

7. Concluding remarks

The long waves generated by a train of modulated periodical wave groups propagating over a uni-directional current field are investigated. It is shown that the carrier (short) waves, the wave envelope, and the free long waves propagate in different directions with different speeds over the shear current region. The locked long waves, however, always propagate with the wave envelope. The free long waves could radiate away from the current region or be trapped within the region, depending on the angle of incidence and the current velocity.

In the present paper, the wave reflection has been ignored. In the cases where the current velocity has a discontinuity, such as a top-hat profile, the reflection become important (e.g., Mei and Lo 1984; Kirby 1986). Similar to the approach used by Liu, Kostense, and Dingemans (1989) for studying the long wave generation by wave groups over a step, the present analysis may be extended to include the reflection caused by the currents.

TABLE 2. Wave amplitudes of free long waves at X_0 and X_1, $|B_0|$ and $|B_1|$, respectively, for $h = 1.0$, $\alpha_0 = \pi/6$, and $(X_1 - X_0) = 1.0$.

	Vm			
	0.2	0.3	0.4	0.5
$\|B_0\|$	0.026048	0.034895	0.048923	0.057287
$\|B_1\|$	0.018943	0.027441	0.044201	0.063170

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