Fluctuating Flow through Straits of Variable Depth

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ABSTRACT

A model is presented for the fluctuating flow through a strait of nonuniform depth connecting two semi-infinite oceans.

An analytical solution is found. The solution is applied to several depth profiles to study the effect of the topography on the volume flux through the strait. A nondimensional number $\sigma = (\partial h/\partial x)fW/2\omega h$ is found to determine the importance of the topography of the strait, where f, ω, W, L and h are the Coriolis parameter, fluctuating frequency, and the width, length and depth of the strait, respectively. If $\sigma < 0.6$, the effect of the variation of strait depth is negligible; if σ increases, the effect of the depth variation is to shorten the length of the strait, thus allowing more flux through the strait; at the value of about $\sigma = \pi/2$, the strait is almost invisible to the open oceans as far as the flux is concerned.

The mechanism of the geostrophic control of the flux through the strait is studied. A model of energy balance clearly shows that the flux is limited by the amount of the energy which the two outgoing Kelvin waves can carry: the flux through the strait can not be greater than the flux at the geostrophic-control limit, otherwise it will generate in the open oceans such big Kelvin waves that they would carry away more energy than the strait system can receive from the incoming Kelvin waves.

1. Introduction

The fluctuating flow through straits significantly affects the properties of the oceans connected by the straits, as well as the oceanographic conditions in the straits themselves. The main driving force for these fluctuations is the sea-level difference between the opposite ends of the strait, which may be largely associated with meteorological forcing or tidal motion in the two oceans separated by the strait.

Toulany and Garrett (1984, henceforth referred to as TG; also see Garrett and Toulany 1982) considered a simple algebraic model and offered a formula for the volume flux through a strait, which reveals some important properties of the problem. According to this formula, the flux through a strait can be much less than that in a nonrotating system, and at the low-frequency limit, the flux is inversely proportional to the Coriolis parameter; the term "geostrophic control" of the flux was thus coined. Toulany and Garrett suggested that this control is due to the physical effect that the sea level difference across the strait due to geostrophic setup cannot be greater than the sea level difference between the two connected oceans.

Very recently, Rocha and Clarke (1987, henceforth referred to as RC) gave a more complete solution of the problem. Their solution reveals that the formula for the flux at the geostrophic-control limit $\omega \rightarrow 0$ depend not on the depth of the strait, but only on the depth of the two oceans. Although they did not analyze this property further, their solution suggests that the flux at the limit $\omega \rightarrow 0$ is controlled by the processes in the oceans, rather than the processes in the strait as TG argued. What in the open oceans cause the geostrophic control remains to be found.

Rocha and Clarke's model applies only to straits of uniform depth. A steplike topography appears at both ends of the strait if the uniform depth of the strait is different from those of the oceans. Question arises as to how the flux through the strait depends on depth if a depth profile is continuous between the strait and oceans.

In addition, as the depths of straits in real oceans are rarely uniform, the following questions are often asked: when can the variation of the strait depth be ignored? what are the determining parameters? how does the flux through the strait change if the depth variation in the strait has to be taken into account?

The present study attempts to answer the questions raised above. The paper is organized as follows. Section 2 presents a model and its solution. Section 3 analyzes the solution for several depth profiles and discusses the effect of the variation of the strait depth on the flux. In section 4, the solution is discussed and compared with that of TG and RC. In section 5 a model of energy balance is constructed to analyze the mechanism of

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the geostrophic control of the volume flux. In section 6 the energy transport in the strait system is calculated to further understand the strait system. Section 7 is a summary.

2. Model and solution

The configuration of the model is shown in Fig. 1. The coordinates x and y are directed along and across the strait, respectively, and the origin of the coordinates is at the center of the strait. In the following, the variables in the left ocean are denoted by a subscript l, those in the right ocean by a subscript r, and those in the strait by a subscript s or without subscript. The depth in the strait, $h(x)$, is independent of y; the depths at the two ends of the strait, $h_l = h(-L/2)$ and $h_r = h(L/2)$, are the uniform depths of the left and right oceans, respectively, so that the depth is continuous between the strait and the oceans. As TG and RC have justified that the nonlinear effects are negligible, small-amplitude flows are assumed so that the equations are linearized, and the velocity components and sea-level fluctuation are taken to be proportional to $\exp(i\omega t)$ everywhere. The fluctuating frequency ω can be that of a tide, in which case ω is on the order of f for straits away from the equatorial region, or that of low-frequency meteorologically forced waves, in which case $\omega \ll f$.

a. The solution in the strait

As in TG and RC, a narrow strait is considered, in the sense that

$$W/L_R \ll 1, \tag{2.1}$$

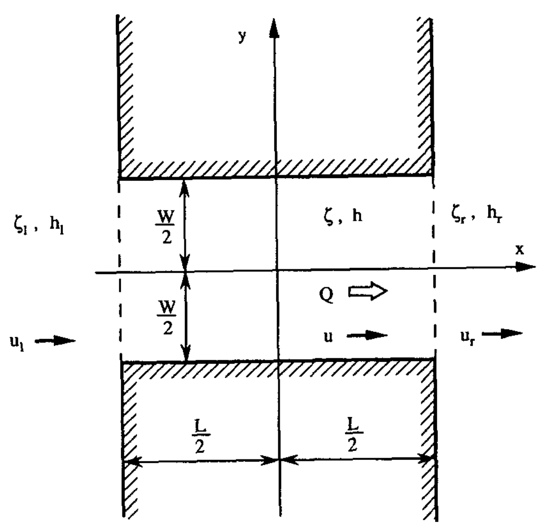

FIG. 1. Schematic diagram of the model. The sea levels ζ, currents u, and the mass flux through the strait Q are periodic with frequency ω.

where W is the strait width and $L_R = (gh)^{1/2}/f$ is the Rossby radius of deformation based on strait depth. The length of the strait is also assumed to be at most on the order of the wavelength of the Kelvin waves in the strait, i.e.,

$$L \leqslant (gh)^{1/2}/\omega. \tag{2.2}$$

These assumptions are realistic for most straits. For instance, for a strait of depth 250 m and a tidal frequency $\omega \approx f = 10^{-4}\ \mathrm{s}^{-1}$, the scale of the Kelvin waves in the strait is $(gh)^{1/2}/\omega \approx (gh)^{1/2}/f \approx 500$ km, which is much greater than the scales of a typical strait.

The linearized governing equations are

$$i\omega v + fu = -g\frac{\partial \zeta}{\partial y} \tag{2.3}$$

$$i\omega u - fv = -g\frac{\partial \zeta}{\partial x} \tag{2.4}$$

$$i\omega\zeta + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0. \tag{2.5}$$

A key assumption to be made here is the cross-strait geostrophic balance; Eq. (2.3) is to be replaced by the following equation,

$$fu = -g\frac{\partial \zeta}{\partial y}, \tag{2.6}$$

which is valid if $\omega v/fu \ll 1$. Two situations will lead to this cross-strait geostrophic balance: a long strait, where the strait length and the length scale of the depth variation are much greater than the width of the strait (thus $u \gg v$, by assuming that v/u is on the order of the aspect ratio of the strait); a low-frequency, meteorologically forced flow, where $\omega \ll f$.

By studying the current meter and hydrographic data, Garrett and Petrie (1981) concluded that the cross-strait balance is geostrophic in both tidal motion and low-frequency flows in the Strait of Belle Isle, where the ratio of the width and the length of the strait is about 10 km/100 km = 0.1 and $\omega/f = 1.8 \times 10^{-5}/1.14 \times 10^{-4} \approx 0.16$ for the typical period of four days of low-frequency flows. In addition, their paper referred to several studies in which the cross-strait geostrophic balance is the case.

The ratio of the first term to the second term of the continuity equation (2.5) is

$$\frac{i\omega\zeta}{\partial(hu)/\partial x} \approx \frac{\omega}{(gh/fWL)\max(1, |Lh_x/h|)} \leqslant \frac{fW}{(gh)^{1/2}}\frac{\omega L}{(gh)^{1/2}}. \tag{2.7}$$

Therefore, for a narrow strait where (2.1) and (2.2) apply, the temporal change in the continuity equation can be ignored, and (2.5) becomes

$$\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}=0. \quad (2.8)$$

A streamfunction ψ is introduced such that

$$\frac{\partial\psi}{\partial y}=-hu,\quad \frac{\partial\psi}{\partial x}=hv,\quad \psi(x, W/2)=0,$$

$$\text{and}\quad \psi(x,-W/2)=Q=\int_{-W/2}^{W/2} hudy. \quad (2.9)$$

Then, ψ and ζ are solved from (2.3) and (2.4) with the boundary condition (2.9):

$$\psi(x,y)=i\frac{Q}{2}\frac{\exp(i\sigma 2y/W)-\exp(i\sigma)}{\sin(\sigma)} \quad (2.10)$$

$$\zeta(x,y)=\frac{Q}{2}\left[i\frac{f}{gh}\frac{\exp(i\sigma 2y/W)-\exp(i\sigma)}{\sin(\sigma)}+\frac{f}{g}\left(\frac{1}{h_l}-\frac{1}{h}\right)-i\frac{2\omega}{Wg}F(x)\right]-\frac{C}{g}, \quad (2.11)$$

where

$$\sigma=fWh_x/2\omega h,$$

$$F(x)=\int_{-L/2}^{x}\frac{\sigma}{h}\operatorname{ctg}(\sigma)dx', \quad (2.12)$$

and the flux Q through the strait and another constant C are to be determined by matching (2.11) at the two ends of the strait with the solutions of the two oceans.

The along-strait velocity is

$$u(x,y)=-\frac{1}{h}\psi_y=\sigma\frac{Q}{hW}\frac{\exp(i\sigma 2y/W)}{\sin(\sigma)}. \quad (2.13)$$

If $\sigma=\pi$, then the flux $Q=0$, in order for the solution to be finite. The reason for this apparently strong constraint on the flux through the strait is that u changes sign across the strait, and the net flux is zero. Evaluation of the ratio $\omega v/fu$ at $y=0$, the center line of the strait, gives

$$\frac{\omega v}{fu}=\frac{i\omega\sigma_x W}{2f\sigma\sin(\sigma)}[1-\cos(\sigma)]. \quad (2.14)$$

If σ is far from π and $\omega\sigma_x W/f\sigma \ll 1$, the cross-strait balance is valid. (Note that σ/σ_x can be considered to be the length scale of the depth variation, since σ varies in x only due to the depth variation.) As σ approaches π, however, the cross-strait geostrophic balance breaks down, as shown in (2.14). Therefore, an assumption that σ is less than and distant from π, to say, $\sigma \leqslant \pi/2$, is added to the model to assure the cross-strait geostrophic balance.

b. The Green-function solution in the two oceans

The sea level elevation for the left semi-infinite ocean, in the absence of the strait, is that of a Kelvin wave, i.e.,

$$\zeta_l=A_l\exp\{[i\omega-f(x-L/2)]/(gh_l)^{1/2}\},\quad x\leqslant -L/2.$$

In the vicinity of the strait mouth, namely $x\approx -L/2$ and $y\approx W$, the following approximation is used

$$\zeta_l=A_l,\quad x\leqslant -L/2.$$

Similarly,

$$\zeta_r=A_r,\quad x\geqslant L/2.$$

In the presence of the strait, the sea-level elevations for the two oceans are

$$\zeta_l(x,y)=-\int_{-W/2}^{W/2} G_l\left(-x-\frac{L}{2},\eta-y\right)\times u_l\left(-\frac{L}{2},\eta\right)d\eta+A_l,\ x\leqslant -L/2, \quad (2.15)$$

$$\zeta_r(x,y)=\int_{-W/2}^{W/2} G_r\left(x-\frac{L}{2},y-\eta\right)u_r\left(\frac{L}{2},\eta\right)d\eta+A_r,\ x\geqslant L/2. \quad (2.16)$$

The Green's functions G_l and G_r are the near-field approximation, which is good near the strait mouth (Buchwald 1971):

$$G_l(0,y)\approx\frac{\omega}{2g}\left[M_l-\frac{f}{\omega}\operatorname{sgn}(y)-\frac{2i}{\pi}\ln\left(\frac{4|y|}{W}\right)\right] \quad (2.17)$$

$$G_r(0,y)\approx\frac{\omega}{2g}\left[M_r-\frac{f}{\omega}\operatorname{sgn}(y)-\frac{2i}{\pi}\ln\left(\frac{4|y|}{W}\right)\right], \quad (2.18)$$

where

$$M_l=\frac{1}{\omega}\max(f,\omega)+i\frac{\omega L_{el}}{W}$$

$$M_r=\frac{1}{\omega}\max(f,\omega)+i\frac{\omega L_{er}}{W},$$

and

$$L_{el}=-\frac{2}{\pi}W\left[\ln(0.60\delta_l)-\sum_{n=0}^{n=\infty}\left(\frac{\omega}{f}\right)^{2n}(2n)^{-1}(2n+1)^{-1}\right]$$

$$L_{er}=-\frac{2}{\pi}W\left[\ln(0.60\delta_r)-\sum_{n=0}^{n=\infty}\left(\frac{\omega}{f}\right)^{2n}(2n)^{-1}(2n+1)^{-1}\right].$$

Toulany and Garrett termed L_{el} and L_{er} as effective lengths, which were supposed to be added to the actual strait length in their formula of the flux through the strait, due to the wave diffraction at both ends of the strait.

c. Solving the matching equations

Matching ζ at the two ends of the strait then gives

$$\zeta\left(-\frac{L}{2}, y\right) = \frac{Q}{2}\frac{if}{gh_l}\frac{\exp(i\sigma_l 2y/W) - \exp(i\sigma_l)}{\sin(\sigma_l)} - \frac{C}{g} = \zeta_l\left(-\frac{L}{2}, y\right) = -\sigma_l \frac{Q}{h_l W}\frac{1}{\sin(\sigma_l)}\int_{-W/2}^{W/2} G_l(0, \eta - y)e^{i\sigma_l 2\eta/W}d\eta + A_l, \quad -W/2 \leq y \leq W/2, \quad (2.19)$$

$$\zeta\left(\frac{L}{2}, y\right) = \frac{Q}{2}\frac{if}{gh_r}\frac{\exp(i\sigma_r 2y/W) - \exp(i\sigma_r)}{\sin(\sigma_r)} - \frac{C}{g} + \frac{Q}{2}\frac{f}{g}\left(\frac{1}{h_l} - \frac{1}{h_r}\right) - i\frac{Q\omega}{Wg}F, = \zeta_r\left(\frac{L}{2}, y\right) = \sigma_r \frac{Q}{h_r W}\frac{1}{\sin(\sigma_r)}\int_{-W/2}^{W/2} G_r(0, y - \eta)e^{i\sigma_r 2\eta/W}d\eta + A_r, \quad -W/2 \leq y \leq W/2, \quad (2.20)$$

where $F' = F(L/2)$ defined by (2.12).

The above two equations are integrated in the y direction from $-W/2$ to $W/2$, yielding the following equations for Q and C

$$\left(\frac{f}{2h_l} + Z_l\right)Q = C + gA_l \quad (2.21)$$

$$\left(\frac{f}{2h_l} - Z_r - Z_s\right)Q = C + gA_r, \quad (2.22)$$

where

$$Z_l = \frac{1}{2h_l}\left\{\max(f, \omega) + i\omega\left[\frac{L_{el}}{W} + I(\sigma_l)\right]\right\} \quad (2.23)$$

$$Z_r = \frac{1}{2h_r}\left\{\max(f, \omega) + i\omega\left[\frac{L_{er}}{W} + I(\sigma_r)\right]\right\} \quad (2.24)$$

$$Z_s = i\omega F'/W, \quad (2.25)$$

and

$$I(\sigma) = -\frac{1}{\pi}\left\{2(\ln 4 - 1) - \frac{1}{\sigma} \times \left[\mathrm{ctg}(\sigma)\int_0^{|2\sigma|}\frac{\cos t - 1}{t}dt + \int_0^{|2\sigma|}\frac{\sin t}{t}dt\right]\right\}. \quad (2.26)$$

The plot of $I(\sigma)$ (Fig. 2) shows that $I(\sigma) < 0.13$ for $\sigma < 1.5$, while TG calculated L_e/W and found that it is between 1 and 3.5. Therefore, $I(\sigma_l)$ and $I(\sigma_r)$ are negligible in the expressions of Z_l and Z_r.

Solving (2.21) and (2.22) gives the solution of C and Q

$$C = \frac{g}{Z_l + Z_r + Z_s}\left[A_l\left(\frac{f}{2h_l} - Z_r - Z_s\right) + A_r\left(\frac{f}{2h_l} + R_l\right)\right] \quad (2.27)$$

$$Q = \frac{g(A_l - A_r)}{Z_l + Z_r + Z_s} = g(A_l - A_r)\Big/\left[\frac{1}{2}\max(f, \omega) \times \left(\frac{1}{h_l} + \frac{1}{h_r}\right) + \frac{i\omega}{W}\left(F' + \frac{L_{el}}{2h_l} + \frac{L_{er}}{2h_r}\right)\right], \quad (2.28)$$

The solution for sea-level elevation and along-strait velocity in the strait are then:

$$\zeta(x, y) = \frac{1}{Z_l + Z_r + Z_s} \times \left\{\left[\frac{if}{2h}\frac{\exp(i\sigma 2y/W) - \exp(i\sigma)}{\sin(\sigma)} - i\frac{\omega}{W}F(x) - \frac{f}{2h}\right] \times (A_l - A_r) + A_l(Z_s + Z_r) + A_r Z_l\right\} \quad (2.29)$$

$$u(x, y) = \frac{g(A_l - A_r)}{hW(Z_l + Z_r + Z_s)}\frac{\sigma\exp(i\sigma 2y/W)}{\sin(\sigma)}. \quad (2.30)$$

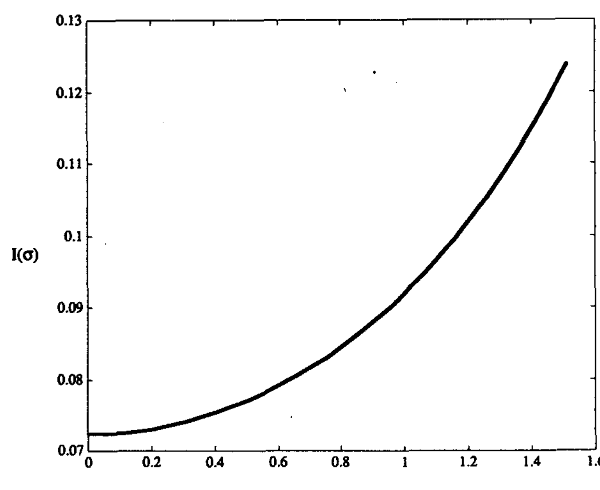

FIG. 2. Function $I(\sigma)$, as defined by (2.26).

3. Effect of variation of strait depth and solutions for several depth profiles

The variation of the strait depth affects the formula of flux Q, (2.28), through the term Z_s of the denominator. The term Z_s (or F', which is proportional to Z_s) depends on the strait depth as an integral of a function of h and σ:

$$Z_s = i\frac{\omega}{W}F' = i\frac{\omega}{W}\int_{-L/2}^{L/2}\frac{\sigma}{h}\operatorname{ctg}(\sigma)dx'. \quad (3.1)$$

For a strait of uniform depth H, $F' = L/H$. Let H denote the average depth of the strait, then the ratio $F'/(L/H)$ indicates how much depth variation affects the volume flux. If $F'/(L/H) > 1$, the strait is more restrictive to the water exchange between the two open oceans than a strait of uniform depth H and length L.

In the following three examples, the depth profiles are expressed so that H is the average depth and Δ_c is the relative depth variation $Lh_x/2h$ at the center of the strait $x = 0$. The value of σ at $x = 0$ is denoted as σ_c.

a. *Uniform slope*

The depth profile of a uniformly-sloped strait is expressed as

$$h = H[1 + \Delta_c x/(L/2)]. \quad (3.2)$$

Substitution of the above $h(x)$ into (2.12) yields

$$F'\frac{H}{L} = \frac{1}{2\Delta_c}\ln\left|\frac{\sin(\sigma_c/(1-\Delta_c))}{\sin(\sigma_c/(1+\Delta_c))}\right|.$$

Figure 3 shows $F'H/L$ vs. Δ_c for different values of σ_c.

It is noticed that $F'(\Delta_c, \sigma_c) = F'(-\Delta_c, -\sigma_c)$, i.e., the flux Q does not depend on the direction of the slope. This is expected from linear theory, for the flux changes direction with time sinusoidally, and the average flux in one period is zero. If the flow is strong so that the nonlinear effect has to be considered, the conservation of potential vorticity favors the flow from the deeper ocean to the shallower ocean, and the symmetry of the flow in two directions is expected to be broken.

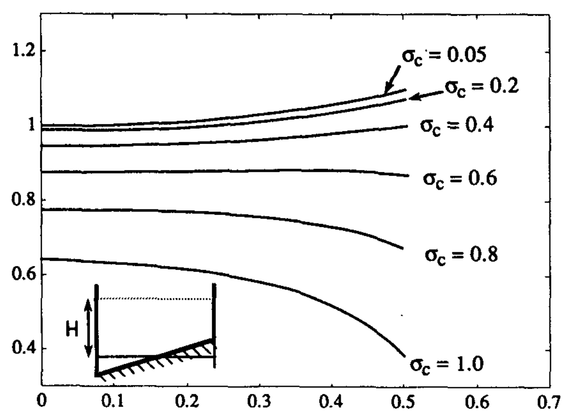

FIG. 3. $F'(H/L)$ for a strait of uniformly sloped depth profile (3.2).

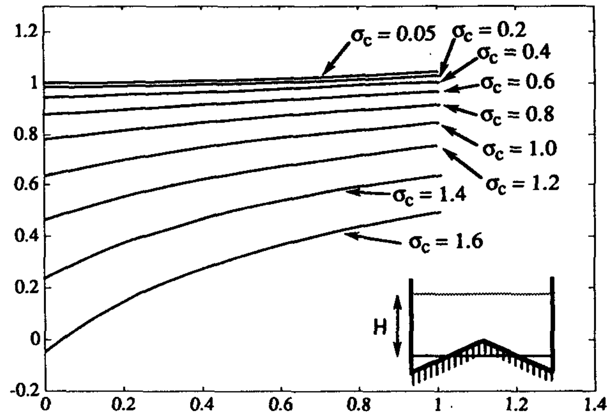

FIG. 4. $F'(H/L)$ for a strait shoaling or deepening in the middle. The depth profile is piecewise linear as expressed in (3.3).

b. *Piecewise linear profiles*

Shoaling or deepening in the middle of the strait can be modeled with

$$h = H\left[1 + \frac{\Delta_c}{1+\frac{1}{2}\Delta_c}\frac{|x| - L/4}{L/2}\right], \quad (3.3)$$

where $\Delta_c > 0$ is for shallowing, and $\Delta_c < 0$ for deepening. Then,

$$F'\frac{H}{L} = \frac{1+\frac{1}{2}\Delta_c}{\Delta_c}\ln\left|\frac{\sin(\sigma_c)}{\sin(\sigma_c/(1+\Delta_c))}\right|.$$

Figure 4 shows $F'H/L$ vs. Δ_c for different values of σ_c.

c. *Piecewise, symmetric exponential profile*

Shallowing or deepening in the middle of the strait can also be modeled with the exponential function

$$h = H\frac{\Delta_c}{\exp(\Delta_c) - 1}\exp\left[\Delta_x\frac{|x|}{L/2}\right]$$

$$(\Delta_c > 0 \text{ or } \Delta_c < 0). \quad (3.4)$$

The solution is

$$F'\frac{H}{L} = \frac{2[ch(\Delta_c) - 1]}{\Delta_c^2}\sigma_c\operatorname{ctg}(\sigma_c).$$

Figure 5 shows $F'H/L$ vs. Δ_c for different values of σ_c.

Figures 3, 4 and 5 show that σ_c is a key parameter to determine the importance of the variation of the strait depth. The physical meaning of σ is examined here. A nondimensional width of the strait, W', is in-

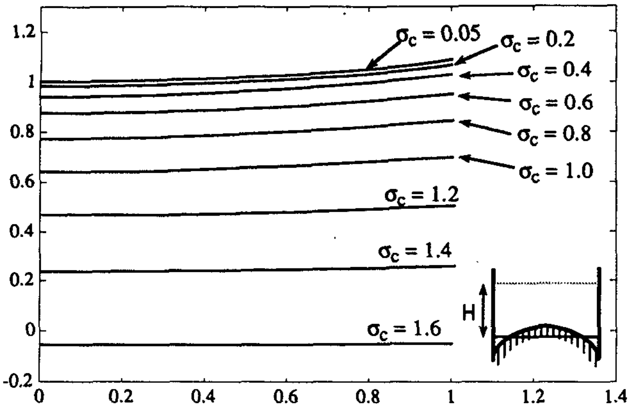

FIG. 5. $F'(H/L)$ for a strait shoaling or deepening in the middle. The depth profile is piecewise exponential as expressed in (3.4).

troduced as the ratio of the width W to the width of Kelvin waves (the Rossby radius of deformation)

$$W' = Wf/(gh)^{1/2}. \tag{3.5}$$

The variable $L_h = 2h/h_x$ is the length scale for the depth variation; its nondimensional form, L'_h, is the ratio of L_h to the wavelength of Kelvin waves, i.e.,

$$L'_h = L_h\omega/(gh)^{1/2}. \tag{3.6}$$

Then the parameter

$$\sigma = W'/L'_h \tag{3.7}$$

is the aspect ratio of the strait based on the nondimensional width and length scale.

From the three examples of this section, we conclude that for a narrow strait in the sense of $\sigma_c < 0.6$, the effect of the depth variation of the strait is negligible $[F'/(H/L) \sim 1]$; for $\sigma_c > 0.6$, the effect of the depth variation is to shorten the length of the strait $[F'/(H/L) < 1]$, thus to increase the volume transport through the strait; for the value of σ_c around 1, the oceans only feel a strait with 65% of its actual length $[F'/(H/L) \approx 0.65]$. The value of $F'/(L/H)$ becomes zero if σ_c increases to $\sigma_c = (1 - \Delta_c^2)\pi/2$ for uniform slopes, $\sigma_c = (1 + \Delta_c)\pi/(2 + \Delta_c)$ for piecewise linear profiles, and $\sigma_c = \pi/2$ for piecewise exponential profiles. Thus, the strait becomes almost invisible to the system, as far as the flux is concerned, and the flux is nearly the same as when the two oceans are separated by a wall and connected by a gap of the wall, such as the case studied in Buchwald and Miles (1974). The physics of this situation needs further study and will not be treated in the present paper.

4. Comparison with previous works and discussions of the solution

a. A limiting case and comparison with RC's model

If the strait is narrow in the sense that $\sigma = W'/L'_h \ll 1$, then the solution becomes

$$Q = g(A_l - A_r)\bigg/\left[\frac{1}{2}\max(f, \omega)\left(\frac{1}{h_l} + \frac{1}{h_r}\right) + \frac{i\omega}{W}\left(\int_{-L/2}^{L/2}\frac{dx}{h} + \frac{L_{el}}{2h_l} + \frac{L_{er}}{2h_r}\right)\right] + O(\sigma^2) \tag{4.1}$$

$$\begin{aligned}\zeta(x, y) = {} & \left\{\frac{1}{2}\max(f, \omega)\left(\frac{1}{h_l} + \frac{1}{h_r}\right) + \frac{i\omega}{W}\left(\int_{-L/2}^{L/2}\frac{dx}{h} + \frac{L_{el}}{2h_l} + \frac{L_{er}}{2h_r}\right)\right\}^{-1} \\ & \times\left\{\left[-\frac{f}{2h}\frac{y}{W/2} - i\frac{\omega}{W}\int_{-L/2}^{x}\frac{dx'}{h}\right](A_l - A_r) + \frac{A_l}{2h_r}\left[\max(f, \omega) + i\omega\frac{L_{er}}{W}\right] \right. \\ & \left. + \frac{A_r}{2h_l}\left[\max(f, \omega) + i\omega\frac{L_{el}}{W}\right] + A_l\frac{i\omega}{W}\int_{-L/2}^{L/2}\frac{dx}{h}\right\} + O(\sigma). \end{aligned} \tag{4.2}$$

This limiting solution is identical to that of the limiting case for a narrow strait in RC, except that, in the present solution, the variables

$$\int_{-L/2}^{x}\frac{dx'}{h} \quad\text{and}\quad \int_{-L/2}^{L/2}\frac{dx}{h}$$

appear as the natural extensions of $(x + L/2)/h$ and L/h, respectively, due to the variation of the strait depth.

If σ is not small, then L/h in RC's solution of Q will be replaced by

$$F' = \int_{-L/2}^{L/2}\frac{\sigma}{h}\,\mathrm{ctg}(\sigma)dx'$$

in the present solution. As discussed in section 3, the value of F' can be close to L/h, or less than L/h, or zero, depending on the key parameter σ. By evaluating F', one can calculate by how much the flux through the strait changes due to the depth variation in the strait. While RC's model is for a strait of uniform depth, and thus this effect can not be considered.

In addition, compared with RC's solution, which is in the form of a series expansion, the present solution, (2.28) and (2.29), or (4.1) and (4.2) for the limiting case $\sigma \ll 1$, is easier to understand and apply.

b. Comparison with TG's algebraic model

Toulany and Garrett presented a simple algebraic model for a strait system, where the depth h is constant over the strait and the two open oceans (Fig. 6). They denoted ζ_3, ζ_4, ζ_5, and ζ_6 as the elevations at the four corners of the strait.

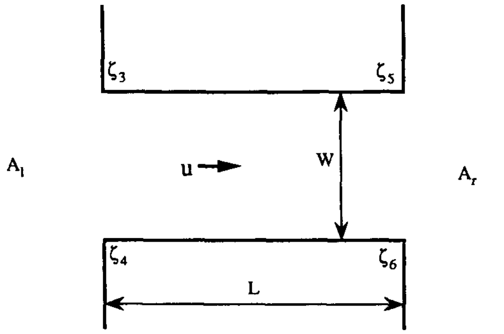

FIG. 6. Schematic diagram of TG's simple algebraic model of a strait of length L, width W connecting two open oceans. The depth h is constant over the strait and open oceans. A_l and A_r are the far-field elevations of the two oceans; ζ_3, ζ_4, ζ_5, and ζ_6 are the elevations on the opposite sides of each end of the strait.

The assumptions of cross-strait geostrophic balance and the along-strait balance between acceleration and the sea surface slope lead to

$$\zeta_4 - \zeta_3 = \zeta_6 - \zeta_5 = (f/g)Wu \qquad (4.3)$$

$$\zeta_5 - \zeta_3 = \zeta_6 - \zeta_4 = -i(\omega/g)Lu. \qquad (4.4)$$

(For simplicity, friction is omitted here, whereas it was retained in TG.) Noting that the Kelvin waves propagating in the two basins impose upstream values on ζ_4 and ζ_5, they assumed that $\zeta_4 = A_l$ and $\zeta_5 = A_r$, where A_l and A_r are the elevations at the two open oceans when the strait is absent. Then the flux through the strait can be solved as

$$Q = Whu = \frac{gh(A_l - A_r)}{(f + i\omega L/W)^{-1}}. \qquad (4.5)$$

Toulany and Garrett speculated that some "end correction" should be added to the actual length L of the strait in (4.5), even though their model is too crude to resolve this.

The present model refines TG's model by solving the differential equations instead of the algebraic equations and by allowing depth change along the strait. If the depth h is constant over the strait and the two open oceans, and $\omega < f$, then the solution for the flux, (2.28), reduces to TG's solution (4.5), except that the strait length L is replaced by

$$L + \frac{1}{2}(L_{el} + L_{er})$$

in the present solution. This verifies TG's conjecture of the end correction. This end correction is about 1 to 3.5 times the width of the strait, depending on the values of ω/f and $Wf/(gh)^{1/2}$.

Examination of solution (2.29) at two points near the corners of the strait reveals that

$$\zeta_s\left(-\frac{L}{2} - \frac{L_{el}}{2}, -\frac{W}{2}\right) = A_l \qquad (4.6)$$

$$\zeta_s\left(\frac{L}{2} + \frac{L_{er}}{2}, \frac{W}{2}\right) = A_r. \qquad (4.7)$$

Although ζ_s is not valid outside the strait, a little risk is taken to extend the solution beyond the strait by $L_e/2$, which is about 0.5 to 1.7 times the width of the strait. The sea levels at the two corners, (4.6) and (4.7), confirm the assumption of TG that ζ_4 and ζ_5 are equal to the far-field elevations A_l and A_r respectively. However, the points 4 and 5 should not be exactly at the corners of the strait because of the existence of the end correction. They are away from the mouths of the strait and toward the open oceans by the amount of $L_e/2$ and thus are the corners of the virtual strait with the length $L + \frac{1}{2}(L_{el} + L_{er})$. This property also suggests locations to sample the ocean waves that are free of the influence of the strait.

If $h_l = h_r$, another property of the solution (2.29) is that

$$\zeta\left(-\frac{L}{2} - \frac{L_{el}}{2}, \frac{W}{2}\right) + \zeta\left(\frac{L}{2} + \frac{L_{er}}{2}, -\frac{W}{2}\right) = A_r + A_l,$$

which appears to say that $\zeta_3 + \zeta_5 = A_r + A_l$, as one can get readily from TG's equation (4.3) and (4.4).

5. Energy balance and the mechanism of geostrophic control

For the low-frequency limit (the geostrophic-control limit), $\omega \to 0$, TG's formula (4.5) reduces to

$$Q = gh(A_l - A_r)/f. \qquad (5.1)$$

They termed the flux at this limit as the flux of geostrophic control, because the flux is inversely proportional to the Coriolis parameter f. They argued that the geostrophic control arises from a simple physics that the sea level difference across the strait can not be greater than the sea level difference between the two open oceans.

For the same low-frequency limit, however, the present solution (2.28) (as well as RC's solution) reduces to

$$Q = g(A_l - A_r)\Big/\left[\frac{f}{2}\left(\frac{1}{h_l} + \frac{1}{h_r}\right)\right]. \qquad (5.2)$$

This solution depends on the depths of the two open oceans instead of on the depth of the strait. This suggests that geostrophic control be due to the process in the open oceans rather than the process in the strait as TG argued. Then what in the open ocean causes this geostrophic control? The present section addresses this question.

The energy balance of the strait system at the geostrophic-control limit $\omega \to 0$ is considered here. For simplicity, $A_r = 0$ is assumed; then the only energy source for the strait system is the energy of the incoming Kelvin wave in the left ocean. The energy fluxes of the system are shown in Fig. 7, where E_i is the rate of energy input from the incoming Kelvin wave, E_{lK} and E_{rK} are the rates of energy carried away by the Kelvin waves in the left and right oceans. The amplitudes of the two outgoing Kelvin waves are denoted by A_{lK} and A_{rK}, and the amplitudes of the incoming Kelvin wave in the left ocean by A_l. Since $\omega \to 0$, Poincare waves are not propagating waves and do not carry energy away from the system (Buchwald 1971). Thus, the energy balance requires

$$E_i = E_{lK} + E_{rK}. \tag{5.3}$$

Now, suppose the flux Q is not known; it will be determined only by the energy balance (5.3).

The far-field solution of the Green's function (Buchwald, 1971) indicates that a unit outflow from a point source generates a Kelvin wave of magnitude f/g. Therefore, the outflow Q in the right ocean generates a Kelvin wave of magnitude

$$A_{rK} = \frac{f}{g}\frac{Q}{h_r}. \tag{5.4}$$

According to the formula of Gill (1982, p. 380), the energy carried by this radiated Kelvin wave is

$$E_{rK} = \frac{\rho g^2 h_r |A_{rK}|^2}{4f} = \frac{\rho f}{4h_r}|Q|^2. \tag{5.5}$$

The outgoing Kelvin wave in the left ocean is the sum of the incoming Kelvin wave passing over the left mouth of the strait and the wave generated by the inflow Q, i.e.,

$$A_{lK} = A_l + \frac{f}{g}\left(-\frac{Q}{h_l}\right). \tag{5.6}$$

The energy carried by this Kelvin wave is then

$$E_{lK} = \frac{\rho g^2 h_l |A_{lK}|^2}{4f} = \frac{\rho f}{4h_l}\left|\frac{gh_l}{f}A_l - Q\right|^2. \tag{5.7}$$

When $\omega \to 0$, the wavelength becomes very long, and the phase difference of different variables of the strait system vanishes, as one can see that the coefficients in (2.3)–(2.5) become real numbers. Thus,

$$Q = \frac{|Q|}{|A_l|}A_l,$$

and

$$E_{lK} = \frac{\rho f}{4h_l}\left(\frac{gh_l}{f}|A_l| - |Q|\right)^2.$$

The only energy source from the incoming Kelvin wave of amplitude A_l is

$$E_i = \frac{\rho g^2 h_l |A_l|^2}{4f}. \tag{5.8}$$

The total energy loss of the system can then be written as

$$E_{lK} + E_{rK} = E_i + \frac{\rho f}{4}\left(\frac{1}{h_l} + \frac{1}{h_r}\right) \times |Q|\left\{|Q| - \left[g|A_l| \Big/ \frac{f}{2}\left(\frac{1}{h_l} + \frac{1}{h_r}\right)\right]\right\}.$$

Applying the energy balance requirement (5.3), one recover the solution at the geostrophic-control limit, (5.2) (for the case of $A_r = 0$),

$$Q = gA_l \Big/ \left[\left(\frac{1}{h_l} + \frac{1}{h_r}\right)\frac{f}{2}\right]. \tag{5.9}$$

Generally, with propagating Poincare waves and friction, the energy requirement is

$$E_i \geqslant E_{lK} + E_{rK},$$

and the "equal sign" in (5.9) becomes "less than or equal to". Thus, the flux at the geostrophic-control limit, (5.9), is the maximum flux that the energy balance allows.

Therefore, the geostrophic control appears to be the result of the energy drain of the two Kelvin waves. In other words, the flux through the strait can not be very

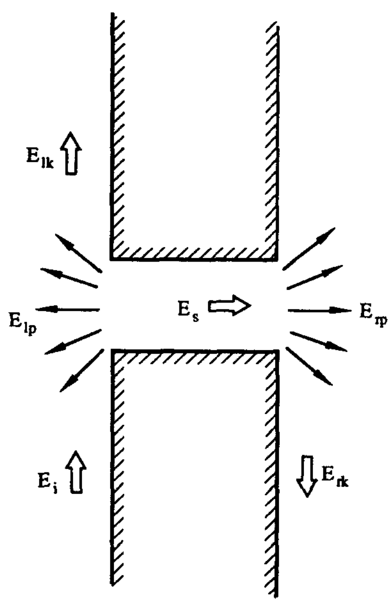

FIG. 7. Diagram of energy flux in a strait system.

big, otherwise it will geostrophically set up such big alternating cross-strait slopes beyond the ends of the strait, which in turn will generate in the open oceans such large Kelvin waves, that they would carry away more energy than the system can receive from the incoming Kelvin waves.

Integration of the cross-strait geostrophic equation yields another expression for Q,

$$Q = -\frac{gh}{f}\int_{-L/2}^{L/2}\frac{\partial\zeta}{\partial y}dy = -\frac{gh(\delta\zeta)}{f}, \quad (5.10)$$

where $\delta\zeta$ is the cross-strait difference of sea level. Eliminating Q from (5.2) and (5.10) gives

$$\frac{\delta\zeta}{A_l - A_r} = \frac{2h_l h_r}{h(h_l + h_r)}.$$

If at a certain section of the strait, $h < h_l$ and $h < h_r$, then at that section the cross-strait sea-level difference will be greater than the sea-level difference between the two oceans. Thus, TG's explanation for the geostrophic control does not stand. For stronger flows where nonlinearity cannot be neglected, however, vortex stretching or compression may not allow the cross-strait sea-level difference to be greater than the sea-level difference between the two oceans, as suggested by C. Garrett (personal communication).

6. Energy balance when $\omega \neq 0$

In this section, the energy analysis of section 5 is extended to the case of finite ω, in order to understand the strait system better and to verify the solution. As in section 5, $A_r = 0$ is assumed. The energy fluxes of the system are shown in Fig. 7, where E_i, E_{lK} and E_{rK} are the same as previously explained, while E_{lP} and E_{rP} are the rates of energy loss through Poincare waves, and E_s is the energy flux through the strait.

By using the solution for ζ and u in the strait, (2.29) and (2.30), the energy E_s passing through the strait is calculated:

$$E_s = \int_{-h_1}^{0} dz \int_{-W/2}^{W/2} dy \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\rho g \times \mathrm{Re}\left[u\left(-\frac{L}{2}, y\right)e^{i\omega t}\right]\mathrm{Re}\left[\zeta\left(-\frac{L}{2}, y\right)e^{i\omega t}\right] = \frac{\rho g^2|A_l|^2}{4|Z|^2}\frac{\max(f, \omega)}{h_r},$$

This E_s is the only energy source for the right-hand open ocean, as $A_r = 0$ is assumed. Thus, in the right-hand ocean, the difference between the energy coming from the strait, E_s, and the energy loss due to the outgoing Kelvin wave, E_{rK}, is the amount of energy radiated away by the Poincare wave, i.e.,

$$E_{rP} = E_s - E_{rK} = \frac{\rho g^2|A_l|^2}{4|Z|^2}\frac{1}{h_r}[\max(f, \omega) - f]. \quad (6.1)$$

If $f < \omega$, then $E_{rP} = 0$; this is consistent with the fact that if $f < \omega$, Poincare waves are not propagating waves and thus can not carry energy away, as noted by Buchwald (1971).

The formula for the energy of the Poincare wave in the left ocean is similar to the formula for the right ocean,

$$E_{lP} = \frac{\rho g^2|A_l|^2}{4|Z|^2}\frac{1}{h_l}[\max(f, \omega) - f].$$

The sum of all the energy carried away from the strait system by the radiated Kelvin waves and Poincare waves is then calculated:

$$E_{lK} + E_{rK} + E_{lP} + E_{rP} = \frac{\rho g^2|A_l|^2 h_l}{4f}$$

where E_{lK} and E_{rK} are expressed by (5.5) and (5.7). The right-hand side of the above expression turns out to be the energy input E_i of the incoming Kelvin wave; the solution in section 2 produces consistent energy transport.

The phases of A_{lK} and A_{rK}, as expressed by (5.4) and (5.6), are the ones at the mouths of the strait. Knowing the Kelvin-wave speed, one can calculate the phases of the outgoing Kelvin waves at any distance from the mouth. This opens the way for possible applications to multistrait systems. If there is another strait located next to the first strait, then the magnitude and phase of the outgoing Kelvin wave from the first strait toward the second one can be used to calculate the magnitude and phase of an incoming Kelvin wave for the second strait. The solutions of the single strait model for each strait can then combine and form a linear system from which the sea level of, and volume flux through, each strait can be determined. One should nonetheless keep in mind that the far-field asymptotic behavior of the Green's function solution in the semi-infinite oceans, namely the representation of a Kelvin wave, is valid only at distances two or more times the Kelvin-wave scale $[\max((gh)^{1/2}/f, (gh)^{1/2}/\omega)]$ away from the mouth of the strait. Another extreme case is studied in detail in RC, namely that of straits packed in a distance much less than the scale of a Kelvin wave so that the phases of the incoming Kelvin wave in one ocean can be considered as uniform.

7. Summary

A linear, frictionless model of the fluctuating flow through straits of nonuniform depth has been considered here. The depth of the strait changes in the along-strait direction; the continuity of the depth between the strait and open oceans avoids the step-like topography used in other studies. A key assumption of the model is the cross-strait geostrophic balance, which requires $\omega v/fu \ll 1$. Thus the present theory applies to long straits, where the strait length and the length scale of the depth variation are much greater than the width of the strait, or to low-frequency, meteorologically-forced flows, where $\omega \ll f$. The model is mathematically tractable and dynamically sound, applicable to many of the strait flow problems of the real oceans.

The nondimensional parameter $\sigma = fWh_x/2h\omega$ is found to be a key parameter in determining the importance of the depth variation of the strait. The parameter σ can be considered to be the aspect ratio based on the nondimensional width and length of the strait, i.e., $\sigma = W'/L'_h$, where W' is the ratio of the width of the strait W to the width of Kelvin waves (Rossby radius of deformation) $(gh)^{1/2}/f$, and L'_h is the ratio of the length scale of the depth variation $2h/h_x$ to the wavelength of Kelvin waves in the strait $(gh)^{1/2}/\omega$. If $\sigma < 0.6$, the variation of the strait depth has almost no effect on the flux, and the strait can be considered to be a strait of uniform depth. If σ increases, the effect of the depth variation is to shorten the length of the strait, thus allowing more flux through the strait. At the value of about $\sigma = \pi/2$, the strait is almost invisible to the open oceans as far as the flux is concerned, and the solution is nearly the same as when the two oceans are separated by a wall and connected by a gap of the wall.

For the Strait of Gibraltar, the width and length of the strait are about 12 and 100 km, respectively. The change of strait depth over the length of the strait is about 500 m, of the same order as the depth itself, so $L_h \sim L$. For the tidal motion, $\sigma \approx 0.1$; thus the effect of the depth variation is negligible. While for meteorologically forced motions, σ can be around 1, and the variation of the strait depth should be taken into account.

The mechanism of the geostrophic control on the flux through the strait, that is, the control of the flux at the limit of $\omega \rightarrow 0$, is studied by considering the energy balance. The analysis clearly shows that the limit of the flux is determined by the limit of the energy that the two outgoing Kelvin waves can carry: the flux through the strait can not be greater than the geostrophic limit, otherwise it will geostrophically set up such big alternating cross-strait slopes beyond the ends of the strait, which will in turn generate in the open oceans such big Kelvin waves, that they would carry away more energy than the strait system can get from the incoming Kelvin waves.

The present model verifies the conjecture of TG that an effective length L_e should be added to the actual length of the strait in the flux formula. This effective length is due to the diffraction near the mouths of the strait and is about 1 to 3 times the width of the strait. It is also confirmed that the amplitude of the incoming Kelvin wave can be assigned to the two corners of the strait which are on the side of the upstream Kelvin waves, as suggested by TG; the corners of the strait, however, should be considered to be the corners of the virtual strait—the strait with the length $L + L_e$.

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