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Complementarity Problems for Multivalued Non-Monotone Operators in Banach Spaces

GUO Wei-ping

(Department of Applied Mathematics, University of Science and Technology of Suzhou, Jiangsu 215009, China)

(E-mail: guoweiping18@yahoo.com.cn)

Abstract: We utilize Park's maximal element theorem in H-space to prove the existence theorems of solutions of the complementarity problems for multivalued non-monotone operators in Banach spaces.

Key words: H-space; Banach space; multivalued non-monotone operator; complementarity problem.
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1. Introduction and preliminaries

The complementarity problems theory for single valued operators was applied to many realistic problems for mathematical programming, join circuit, economics and transportation equilibrium[1−4]. Hence it is important to generalize the complementarity problems from single valued operators to multivalued operators. Recently, we discuss the complementarity problems for multivalued monotone operator in [5]. In this paper we utilize Park's maximal element theorem to discuss the complementarity problems for multivalued non-monotone operators in Banach spaces.

Throughout this paper, we always assume that E is a real Banach space, E^* denotes the conjugate space of E, 2^{E^*} denotes the family of all nonempty subsets of E^*, $\langle\cdot,\cdot\rangle$ denotes the pairing between E^* and E. Let $K\subset E$ be a convex cone, we denote by K^* the conjugate cone of K, i.e.,

$$K^*=\{u\in E^*:\langle u,x\rangle\geq 0,\forall x\in K\}.$$

Let $T:K\to 2^{E^*}$ be a multivalued operator. The so called the complementarity problem of T is to find points $\bar{x}\in K$ and $\bar{u}\in T\bar{x}$ such that

$$T\bar{x}\subset K^* \text{ and } \langle\bar{u},\bar{x}\rangle=0.$$

An operator $T:D\subset E\to 2^{E^*}$ is called semi-monotone[6] if for any $x,y\in D$ we have

$$\inf_{v\in Ty}\langle v,x-y\rangle\leq\inf_{u\in Tx}\langle u,x-y\rangle.$$

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It is clear from the definition that if T is monotone, then T is semi-monotone; but the converse is not true in general (see Example 2 in [6]).

In order to prove our main results, we first give the following lemmas.

Lemma 1 (Park's maximal element theorem[7]) *Let (X, D, Γ) be an H-space and $S : D \to 2^X$, $T : X \to 2^X$ multifunctions such that*

(1) for each $x \in D, Sx \subset Tx$ and Sx is compactly open; and

(2) for each $y \in X, T^{-1}y$ is H-convex.

Suppose that there exists a nonempty compact subset K of X such that either

(i) $X \backslash K \subset S(M)$, for some nonempty finite subset M of D; or

(ii) for each nonempty finite subset N of D, there exists a compact H-subspace L_N of X containing N such that

$$L_N \backslash K \subset S(L_N \cap D).$$

Then either there exists a $y_0 \in K$ such that $S^{-1}y_0 = \emptyset$ or there exists an $x_0 \in X$ such that $x_0 \in Tx_0$.

Lemma 2[8] *Let E be a real normed linear space, and X be a nonempty subset of E and $T : X \to 2^{E^*}$ be an upper semicontinuous multimap such that each Tx is (norm-) compact. Then for each $y \in E$, the real valued function $g_y : X \to R$ defined by*

$$g_y(x) = \inf_{w \in Tx} \langle w, x - y\rangle \text{ for each } x \in X$$

is lower semicontinuous.

We denote by co$A$ convex bull of the set A and denote by $\overline{\text{co}}A$ convex closed bull of the set A.

2. Complementarity problems for non-monotone operators

In this section, we utilize Park's maximal element theorem to discuss the complementarity problems for multivalued non-monotone operators in Banach spaces.

Theorem 1 *Let E be a Banach space and $K \subset E$ be a closed convex cone. Suppose that $T : K \to 2^{E^*}$ is upper semicontinuous from the norm topology in K to the norm topology in E^* and each Tx is norm compact; If there exist two nonempty compact subsets Q and Ω in K, for each $x \in K \backslash Q$ there exists a $y \in \Omega$ such that*

$$\inf_{u \in Tx} \langle u, x - y\rangle > 0$$

and for each fixed $x \in Q$ we have

$$\inf_{u \in Tx} \langle u, y - x\rangle \geq 0 \text{ for all } y \in K.$$

Then there exist $\bar{x} \in Q \subset K$ and $\bar{u} \in T\bar{x}$ such that $T\bar{x} \subset K^*$ and $\langle \bar{u}, \bar{x}\rangle = 0$.

Proof For any finite subset A of E, let $\Gamma_A =$ co$A$. It is easy to know that $(E, \{\Gamma_A\})$ is an

H-space in accordance with the norm topology in E and K is an H-convex set in $(E, \{\Gamma_A\})$. Consequently, $(K, \{\Gamma_A\}) = (K, \{\Gamma_A \cap K\})$ is also an H-space[7,9].

Define a multivalued mapping $G : K \to 2^K$ by

$$G(y) = \left\{ x \in K : \inf_{u \in Tx} \langle u, x - y \rangle > 0 \right\} \text{ for each } y \in K.$$

It follows from Lemma 2 that $G(y)$ is an open set, and it is a compactly open set for each $y \in K$. Now we prove that

$$G^{-1}(x) = \left\{ y \in K : \inf_{u \in Tx} \langle u, x - y \rangle > 0 \right\}$$

is an H-convex. Suppose that $G^{-1}(x) \neq \emptyset$ for each $x \in K$. Let $y_1, y_2 \in G^{-1}(x)$ and $0 \leq \alpha \leq 1$. Then $\widehat{y} = \alpha y_1 + (1-\alpha) y_2 \in K$ and

$$\inf_{u \in Tx} \langle u, x - \widehat{y} \rangle \geq \alpha \inf_{u \in Tx} \langle u, x - y_1 \rangle + (1-\alpha) \inf_{u \in Tx} \langle u, x - y_2 \rangle > 0.$$

The above formula shows that $G^{-1}(x)$ is a nonempty convex set, so $G^{-1}(x)$ is a nonempty H-convex set. Taking a point $x_* \in K \backslash Q$, for any finite subset N of K, let

$$L_N = \overline{\mathrm{co}} \left(\{x_*\} \cup N \cup Q \cup \Omega \right).$$

Since Q and Ω are compact sets in Banach space, L_N is a compact convex subset in K and $L_N \supset N$, and this implies that L_N is compact H-convex. Hence $(L_N, \{\Gamma_A\}) = (L_N, \{\Gamma_A \cap L_N\})$ is a compact H-subspace of $(K, \{\Gamma_A\})$. Since $x_* \in K \backslash Q$, so $L_N \backslash Q \neq \emptyset$, thus $x \in K \backslash Q$ for any $x \in L_N \backslash Q$. It follows from the condition of Theorem 1 that there exists $y \in \Omega$ such that $\inf_{u \in Tx} \langle u, x - y \rangle > 0$. Thus $x \in G(y)$, and

$$L_N \backslash Q \subset \bigcup_{y \in \Omega} G(y) \subset \bigcup_{y \in L_N} G(y) = G(L_N \cap K).$$

Let $K = D = X$ and $G = S = T$. Then by Lemma 1, there exists $\bar{y} \in K$ such that $\bar{y} \in G(\bar{y})$, consequently $\inf_{u \in T\bar{y}} \langle u, \bar{y} - \bar{y} \rangle > 0$. This is a contradiction. Hence there exists an $\bar{x} \in Q \subset K$ such that $G^{-1}(\bar{x}) = \emptyset$, that is,

$$\inf_{u \in T\bar{x}} \langle u, \bar{x} - y \rangle \leq 0 \text{ for each } y \in K.$$

We denote by θ the zero vector of E, then $\theta \in K$. By the above formula we have

$$\inf_{u \in T\bar{x}} \langle u, \bar{x} \rangle = \inf_{u \in T\bar{x}} \langle u, \bar{x} - \theta \rangle \leq 0.$$

Since K is convex and $\bar{x} \in Q \subset K$ we know that $2\bar{x} \in K$. By the condition of Theorem 1 we have

$$\inf_{u \in T\bar{x}} \langle u, \bar{x} \rangle = \inf_{u \in T\bar{x}} \langle u, 2\bar{x} - \bar{x} \rangle \geq 0.$$

Combining the above two inequalities, we have $\inf_{u \in T\bar{x}} \langle u, \bar{x} \rangle = 0$. Note that the real valued function $u \mapsto \langle u, \bar{x} \rangle$ is continuous on the compact set $T\bar{x}$. Therefore, there exists a $\bar{u} \in T\bar{x}$ such that

$$\langle \bar{u}, \bar{x} \rangle = \inf_{u \in T\bar{x}} \langle u, \bar{x} \rangle = 0.$$

Finally, we prove that $T\bar{x} \subset K^*$. In fact, for any $u \in T\bar{x}$ and $y \in K$, by the condition of Theorem 1 we have

$$\langle u, y\rangle \geq \inf_{u\in T\bar{x}}\langle u, y\rangle = \inf_{u\in T\bar{x}}\langle u, y\rangle - \inf_{u\in T\bar{x}}\langle u, \bar{x}\rangle \geq \inf_{u\in T\bar{x}}\langle u, y-\bar{x}\rangle \geq 0.$$

The proof is complete.

Corollary 1 *Let E be a Banach space and $K \subset E$ be a closed convex cone. Suppose that $T : K \to 2^{E^*}$ is upper semicontinuous from the norm topology in K to the norm topology in E^* and each Tx is norm compact. If there exist two nonempty totally bounded subsets Q and Ω in K, satisfying for each $x \in K\backslash\overline{\text{co}}Q$ there exists a $y \in \overline{\text{co}}\Omega$*

$$\inf_{u\in Tx}\langle u, x-y\rangle > 0$$

and for each fixed $x \in \overline{\text{co}}Q$ we have

$$\inf_{u\in Tx}\langle u, y-x\rangle \geq 0, \text{ for all } y \in K.$$

Then there exist $\bar{x} \in \overline{\text{co}}Q \subset K$ and $\bar{u} \in T\bar{x}$ such that $T\bar{x} \subset K^*$ and $\langle \bar{u}, \bar{x}\rangle = 0$.

Proof Since Q and Ω are totally bounded sets in Banach space, $\overline{\text{co}}Q$ and $\overline{\text{co}}\Omega$ are totally bounded complete sets, which shows that $\overline{\text{co}}Q$ and $\overline{\text{co}}\Omega$ are compact sets. It follows from Theorem1 that Corollary 1 is true.

Now we discuss the complementarity problems for multivalued semi-monotone operators in real Banach spaces.

Theorem 2 *Let E be a Banach space and $K \subset E$ be a closed convex cone. Suppose that $T : K \to 2^{E^*}$ is upper semicontinuous from the norm topology in K to the norm topology in E^* and semi-monotone, and each Tx is norm compact. If there exist two nonempty compact subsets Q and Ω in K, satisfying for each $x \in K\backslash Q$ there exists a $y \in \Omega$*

$$\inf_{v\in Ty}\langle v, x-y\rangle > 0$$

and for each fixed $x \in Q$ we have

$$\inf_{u\in Tx}\langle u, y-x\rangle \geq 0 \text{ for all } y \in K.$$

Then there exist $\bar{x} \in Q \subset K$ and $\bar{u} \in T\bar{x}$ such that $T\bar{x} \subset K^*$ and $\langle \bar{u}, \bar{x}\rangle = 0$.

Proof Using the condition of Theorem 2 and noting that T is semi-monotone, for each $x \in K\backslash Q$ there exists a $y \in \Omega$ such that

$$\inf_{u\in Tx}\langle u, x-y\rangle \geq \inf_{v\in Ty}\langle v, x-y\rangle > 0.$$

It follows from Theorem1 that Theorem 2 is true.

Corollary 2 *Let E be a Banach space and $K \subset E$ be a closed convex cone. Suppose that*

$T: K \to 2^{E^*}$ is upper semicontinuous from the norm topology in K to the norm topology in E^* and semi-monotone, and each Tx is norm compact. If there exist two nonempty totally bounded subsets Q and Ω in K, satisfying for each $x \in K \backslash \overline{\text{co}}Q$ there exists a $y \in \overline{\text{co}}\Omega$

$$\inf_{v \in Ty} \langle v, x-y \rangle > 0$$

and for each fixed $x \in \overline{\text{co}}Q$ we have

$$\inf_{u \in Tx} \langle u, y-x \rangle \geq 0 \text{ for all } y \in K.$$

Then there exist $\bar{x} \in \overline{\text{co}}Q \subset K$ and $\bar{u} \in T\bar{x}$ such that $T\bar{x} \subset K^*$ and $\langle \bar{u}, \bar{x} \rangle = 0$.

Remark Theorems 1, 2 and Corollarys 1, 2 extend some main results in [2,4,10] to multivalued non-monotone operators.

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Banach 空间中多值非单调算子的相补问题

郭 伟 平

(苏州科技学院应用数学系，江苏 苏州 215009)

摘要: 应用 H- 空间中的 Park 极大元定理，在 Banach 空间中证明了多值非单调算子的相补问题的解的存在性定理.

关键词: H- 空间； Banach 空间；多值非单调算子；相补问题.