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Intuitionistic (S,T)-Fuzzy M-Subsemigroups of an M-Semigroup

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Abstract: Intuitionistic fuzzy sets are generalized fuzzy sets which were first introduced by Atanassov in 1986. In this paper, we introduce the concept of intuitionistic fuzzy M-subsemigroups of an M-semigroup M with respect to an s-norm S and a t-norm T on intuitionistic fuzzy sets and study their properties. In particular, intuitionistic (S,T)-direct products of M-semigroups are considered and some recent results of fuzzy M-subsemigroups of M-semigroups obtained by Zhan and Tan[21] are extended and generalized to intuitionistic (S,T)-fuzzy M-subsemigroups over M-semigroups.

Key words: M-semigroup; (imaginable) intuitionistic fuzzy M-subsemigroup; intuitionistic (S,T)-direct product.
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1. Introduction

Direct product of semigroups is an important topic in the theory of semigroups. The structure of prime ideals in the direct product of semigroups was first studied by M.Petrich in 1962 [13]. Moreover, R.Plemmas have further studied the structure of maximal ideals in the direct product of two semigroups[9]. On the other hand, Tamura, Markel and Latimer[16−17] have discussed the M-groupoids as the direct product of a right zero semigroup and a groupoid with identity. It was shown by Warner in[18] that an M-semigroup is isomorphic to the direct product of a right zero semigroup and a semigroup with two-sided identity. In 1965, Zadeh has initiated the fuzzy set theory which turned out to be far reaching implications. A detailed study on fuzzy subgoupoids and fuzzy subgroups has been done by Rosenfeld[15]. In particular, Kuroki[8−10] has contributed many results in fuzzy semigroups. Recently, AL.Narayanan and AR.Meenakshi[12] have defined the concept of fuzzy M-subsemigroups of an M-semigroup and extended the characterization theorem of an M-semigroup to a fuzzy M-subsemigroup. The notion of intuitionistic fuzzy sets was introduced by Atanassov[1]. The concept of intuitionistic fuzzy sets was applied by Biswas to the theory of groups and he studied the intuitionistic fuzzy subgroups of semigroups. Kim et al.[6] introduced the notion of intuitionistic fuzzy ideals of semigroups and obtained some interesting results. Also, Dudek, Davvaz and Jun have introduced the intuitionistic sub-hyperquasigroup in a hyperquasigroup and investigated some properties of such hyperquasigroups[5].

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In this paper, we consider the intuitionistic (S,T)-fuzzy M-subsemigroups of an M-semigroup M defined on intuitionistic fuzzy sets. By using fuzzy M-subsemigroups and the new operations defined on the intuitionistic fuzzy sets, we are able to extend some results of fuzzy M-subsemigroups to intuitionistic (S,T)-fuzzy M-subsemigroups of an M-semigroup. In particular, the properties of intuitionistic (S,T)-direct products of an M-semigroup M are described. The fuzzy subsemigroups (ideals) theory of semigroups and other fuzzy algebraic systems have been recently widely studied by many authors.

For notations and terminologies not given in this paper, the readers are referred to [1–3], [12–14], [18–19] and [21–22].

2. Preliminaries

In this section, we first give some definitions and theorems to be used in the paper.

Definition 2.1[12] *Let X_1 and X_2 be semigroups. Then we define the direct product of X_1 and X_2 as the set $X_1 \times X_2$ of all pairs (x,y) of elements x of X_1 and y of X_2 with the coordinate multiplication $(x,y)(x_1,y_1) = (xx_1, yy_1)$, for $x, x_1 \in X_1$ and $y, y_1 \in X_2$.*

Definition 2.2[7] *A semigroup X is said to be a right (left) zero semigroup if for all $x, y \in X, xy = y$ $(xy = x)$.*

Definition 2.3[12] *Let X be a semigroup. If there exists an element $e \in X$ such that $xe = ex = x$ for every $x \in X$, then X is called a monoid.*

Definition 2.4[17] *A semigroup M is called an M-semigroup if the following two conditions are satisfied:*

(i) There exists at least one left identity $e \in M$ such that $ex = x$, for all $x \in M$;

(ii) For every $x \in M$, there is a unique left identity, say e_x such $xe_x = x$, that is, e_x is a two-sided identity for x.

We now call an M-semigroup a left M-semigroup if we consider only the left identities. In the same manner, we can define a right M-semigroup. Throughout this paper, an M-semigroup always means a left M-semigroup unless otherwise specified.

Lemma 2.5[18] *An M-semigroup M is isomorphic to the direct product of a right zero semigroup R and a semigroup X with two sided identity. That is, $M \cong R \times X$, where the right zero semigroup R is the set of all left identities of M and $X = Me$ for a left identity $e \in M$.*

Lemma 2.6[18] *The direct product of a right zero semigroup R and a semigroup X with two sided identity is an M-semigroup.*

Lemma 2.7[11] *An M-semigroup M can be always decomposed into the union of some isomorphic M-subsemigroups of M.*

Recall that a mapping $\mu : X \to [0,1]$, where X is an arbitrary non-empty set, is called a fuzzy set in X. The complement of μ, denoted by the $\overline{\mu}$, is the fuzzy set in X defined by

$\overline{\mu}(x) = 1 - \mu(x)$. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$ is called an upper α-level cut of μ and the set $L(\mu;\alpha) = \{x \in X \mid \mu(x) \leq \alpha\}$ is called a lower α-level cut of μ.

Definition 2.8[7] *(i) The fuzzy meet of two fuzzy sets μ and ν of X, denoted $\mu \cap \nu$ of X, is a fuzzy set of X defined by $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$;*

(ii) The fuzzy union of two fuzzy sets μ and ν of X, denoted $\mu \cup \nu$ of X, is a fuzzy set of X defined by $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$.

Definition 2.9[12] *Let M be an M-semigroup. A fuzzy subset $\mu : M \to [0,1]$ is a fuzzy M-subsemigroup of M if the following conditions are satisfied:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$;

(ii) $\mu(e) = 1$ for every left identity $e \in M$.

For the sake of completeness, we now review some basic definitions.

Definition 2.10[19] *By a t-norm T, we mean a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions: (T1) $T(x,1) = x$; (T2) $T(x,y) \leq T(x,z)$ if $y \leq z$; (T3) $T(x,y) = T(y,x)$; (T4) $T(x, T(y,z)) = T(T(x,y), z)$, for all $x, y, z \in [0,1]$.*

For a t-norm T on $[0,1]$, we write it by $\Delta_T = \{\alpha \in [0,1] | T(\alpha,\alpha) = \alpha\}$. It is clear that every t-norm T has the following property: $T(\alpha,\beta) \leq \min\{\alpha,\beta\}$ for all $\alpha,\beta \in [0,1]$.

Definition 2.11[19] *By an s-norm S, we mean a function $S : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions: (S1) $S(x,0) = x$; (S2) $S(x,y) \leq S(x,z)$ if $y \leq z$; (S3) $S(x,y) = S(y,x)$; (S4) $S(x, S(y,z)) = S(S(x,y), z)$, for all $x, y, z \in [0,1]$.*

Let S be an s-norm on $[0,1]$. Then we write $\Delta_S = \{\alpha \in [0,1] | S(\alpha,\alpha) = \alpha\}$. It is clear that for every s-norm S, $S(\alpha,\beta) \geq \max\{\alpha,\beta\}$, for all $\alpha,\beta \in [0,1]$.

Definition 2.12 *Let T (resp. S) be a t-norm (resp. s-norm). Then we say that the fuzzy set μ in X satisfies the imaginable property if $\mathrm{Im}\mu \subseteq \Delta_T$ (resp. $\mathrm{Im}\mu \subseteq \Delta_S$).*

As an important generalization of the notion of fuzzy sets in X, Atanassov[1−2] introduced the concept of intuitionistic fuzzy sets defined on a non-empty set X which are the objects having the form $A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$, where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively, and the following inequality $0 \leq \alpha_A(x) + \beta_A(x) \leq 1, \forall x \in X$ is satisfied.

For the sake of simplicity, we use $A = (\alpha_A, \beta_A)$ to denote the intuitionistic fuzzy set A, that is, A=$IFSA = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$.

For every two intuitionistic fuzzy sets $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in X, we define[2] the following:

(i) $A \subseteq B$ if and only if $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \geq \beta_B(x)$, for all $x \in X$,

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,

(iii) $\overline{A} = \{(x, \beta_A(x), \alpha_A(x)) | x \in X\}$,

(iv) $A \bigcap B = \{(x, \alpha_A(x) \wedge \alpha_B(x), \beta_A(x) \vee \beta_B(x)) | x \in X\}$,

(v) $A\bigcup B=\{(x,\alpha_A(x)\vee\alpha_B(x),\beta_A(x)\wedge\beta_B(x))|x\in X\}$.

3. Intuitionistic (S,T)-fuzzy M-subsemigroups

In what follows, let M be an M-semigroup unless otherwise specified. We first consider the intuitionistic fuzzy M-subsemigroups in M-semigroups as follows.

Definition 3.1 *An intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ in M is called an intuitionistic fuzzy M-subsemigroup of M if it satisfies the following conditions:*

(IF1) $\alpha_A(xy)\geq\min\{\alpha_A(x),\alpha_A(y)\}$ and $\beta_A(xy)\leq\max\{\beta_A(x),\beta_A(y)\}$, for all $x,y\in M$;

(IF2) $\alpha_A(e)=1$ and $\beta_A(e)=0$, for every left identity $e\in M$.

Example 3.2 Let $M=\{e,f,a,b\}$ be an M-semigroup with the following Cayley table:

$\cdot$	e	f	a	b
e	e	f	a	b
f	e	f	a	b
a	a	b	e	f
b	a	b	e	f

Define an intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ in M by

$$\alpha_A(x)=\begin{cases}1, & \text{if } x=e,f\\ \alpha_1, & \text{otherwise}\end{cases},\quad \beta_A(x)=\begin{cases}0, & \text{if } x=e,f\\ \alpha_2, & \text{otherwise}\end{cases},$$

where $0\leq\alpha_1<1$, $0<\alpha_2\leq 1$ and $0<\alpha_1+\alpha_2\leq 1$.

By routine calculation, we see that $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy M-subsemigroup of M.

Definition 3.3 *An intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ in M is called an intuitionistic fuzzy M-subsemigroup of M with respect to t-norm T and s-norm S (briefly, intuitionistic (S,T)-fuzzy M-subsemigroup of M) if it satisfies the condition (IF2) and (ISTF1) $\alpha_A(xy)\geq T(\alpha_A(x),\alpha_A(y))$ and $\beta_A(xy)\leq S(\beta_A(x),\beta_A(y))$, for all $x,y\in M$.*

Definition 3.4 *An intuitionistic (S,T)-fuzzy M-subsemigroup $A=(\alpha_A,\beta_A)$ of M is said to be imaginable if α_A and β_A satisfy the imaginable property.*

Example 3.5 Let M be an M-semigroup as in Example 3.2. Define an intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ in M by

$$\alpha_A(x)=\begin{cases}1, & \text{if } x=e,f\\ 0, & \text{otherwise}\end{cases},\quad \beta_A(x)=\begin{cases}0, & \text{if } x=e,f\\ 1, & \text{otherwise}\end{cases}.$$

Let T be a t-norm defined by $T(\alpha,\beta)=\max\{\alpha+\beta-1,0\}$ and S an s-norm defined by $S(\alpha,\beta)=\min\{\alpha+\beta,1\}$ for all $\alpha,\beta\in[0,1]$. By routine calculation, we can check that $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M.

Proposition 3.6 *Every imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M is an intuitionistic fuzzy M-subsemigroup of M.*

Proof Assume that $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M. Then we have

$$\alpha_A(xy)\geq T(\alpha_A(x),\alpha_A(y)) \text{ and } \beta_A(xy)\leq S(\beta_A(x),\beta_A(y)) \text{ for all } x,y\in M.$$

Since $A=(\alpha_A,\beta_A)$ is imaginable, we have

$$\begin{aligned}\min\{\alpha_A(x),\alpha_A(y)\}&=T(\min\{\alpha_A(x),\alpha_A(y)\},\min\{\alpha_A(x),\alpha_A(y)\})\leq T(\alpha_A(x),\alpha_A(y))\\&\leq\min\{\alpha_A(x),\alpha_A(y)\}.\end{aligned}$$

That is, $T(\alpha_A(x),\alpha_A(y))=\min\{\alpha_A(x),\alpha_A(y)\}$. On the other hand, we have

$$\begin{aligned}\max\{\beta_A(x),\beta_A(y)\}&=S(\max\{\beta_A(x),\beta_A(y)\},\max\{\beta_A(x),\beta_A(y)\})\\&\geq S(\beta_A(x),\beta_A(y))\geq\max\{\beta_A(x),\beta_A(y)\}.\end{aligned}$$

That is, $S(\beta_A(x),\beta_A(y))=\max\{\beta_A(x),\beta_A(y)\}$.

It follows that $\alpha_A(xy)\geq T(\alpha_A(x),\alpha_A(y))=\min\{\alpha_A(x),\alpha_A(y)\}$, $\beta_A(xy)\leq S(\beta_A(x),\beta_A(y))=\max\{\beta_A(x),\beta_A(y)\}$. Therefore, $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy M-subsemigroup of M.

The following is a characterization theorem of intuitionistic (S,T)-fuzzy M-subsemigroups of an M-semigroup.

Theorem 3.7 *Let T and S be, respectively, t-norm and s-norm. If $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup in M, then the level sets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are M-subsemigroups of M, for every $\alpha\in[0,1]$, whenever $U(\alpha_A;\alpha)\neq\emptyset\neq L(\beta_A;\alpha)$.*

Proof Assume that $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M. Let $x,y\in U(\alpha_A;\alpha)$. Then $\alpha_A(x)\geq\alpha$ and $\alpha_A(y)\geq\alpha$. It follows that $\alpha_A(xy)\geq T(\alpha_A(x),\alpha_A(y))\geq T(\alpha,\alpha)=\alpha$, and thus $xy\in U(\alpha_A;\alpha)$. Hence $U(\alpha_A;\alpha)$ is a subsemigroup of M. For any left identity $e\in M$, we have $\alpha_A(e)=1$, and hence $e\in U(\alpha_A;\alpha)$. Let $x\in U(\alpha_A;\alpha)$. Then $ex\in U(\alpha_A;\alpha)\subseteq M$. This leads to $ex=x$, and so $U(\alpha_A;\alpha)$ is an M-subsemigroup of M. Similarly, we can show that $L(\beta_A;\alpha)$ is an M-subsemigroup of M. □

The following example illustrates that there exist t-norm T and s-norm S such that an intuitionistic fuzzy M-subsemigroup of M may not be an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M.

Example 3.8 Let $M=\{e,f,a,b\}$ be an M-semigroup with the following Cayley table:

$\cdot$	e	f	a	b
e	e	f	a	b
f	f	e	e	f
a	a	e	e	a
b	b	b	b	e

Define an intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ by

$$\alpha_A(x)=\begin{cases}1, & \text{if } x=e,\\ 0.6, & \text{if } x=f,a,\\ 0.3, & \text{if } x=b,\end{cases}\qquad \beta_A(x)=\begin{cases}0, & \text{if } x=e,\\ 0.2, & \text{if } x=f,a,\\ 0.5, & \text{if } x=b.\end{cases}$$

Then, $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy M-subsemigroup of M.

Let $\gamma\in(0,1)$ and define the binary operation T_γ and S_γ on (0,1) as follows:

$$T_\gamma(\alpha,\beta)=\begin{cases}\alpha\wedge\beta, & \text{if } \max\{\alpha,\beta\}=1,\\ 0, & \text{if } \max\{\alpha,\beta\}<1, \alpha+\beta\le 1+\gamma,\\ \gamma, & \text{otherwise;}\end{cases}$$

$$S_\gamma(\alpha,\beta)=\begin{cases}\alpha\vee\beta, & \text{if } \min\{\alpha,\beta\}=0,\\ 1, & \text{if } \min\{\alpha,\beta\}>0, \alpha+\beta\ge 1+\gamma,\\ \gamma, & \text{otherwise,}\end{cases}$$

for all $\alpha,\beta\in[0,1]$. Then T_γ and S_γ are, respectively, the t-norm and s-norm on [0,1].

Hence $T_\gamma(\alpha_A(f),\alpha_A(f))=T_\gamma(0.6,0.6)=\gamma\neq\alpha_A(f)$ and $S_\gamma(\beta_A(f),\beta_A(f))=S_\gamma(0.2,0.2)=\gamma\neq\beta_A(f)$ whenever $\gamma<0.2$, and so $\alpha_A(f)\notin\Delta_T$ and $\beta_A(f)\notin\Delta_S$, i.e., $\mathrm{Im}\alpha_A\subsetneq\Delta_{T_\gamma}$ and $\mathrm{Im}\beta_A\subsetneq\Delta_{S_\gamma}$ whenever $\gamma<0.2$. Thus, $A=(\alpha_A,\beta_A)$ is not an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M whenever $\gamma<0.2$.

The following Lemma is obvious, and we omit the proof.

Lemma 3.9 *Let $A=(\alpha_A,\beta_A)$ be an intuitionistic fuzzy set in M. If each level subsets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are M-subsemigroups of M, for every $\alpha\in[0,1]$, whenever $U(\alpha_A;\alpha)\neq\emptyset\neq L(\beta_A;\alpha)$, then $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy M-subsemigroup of M.*

Now, we consider the converse of Theorem 3.7.

Theorem 3.10 *Let T and S be, respectively, t-norm and s-norm, and let $A=(\alpha_A,\beta_A)$ be an imaginable intuitionistic fuzzy set in M. If each level subsets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are M-subsemigroups of M, for every $\alpha\in[0,1]$, whenever $U(\alpha_A;\alpha)\neq\emptyset\neq L(\beta_A;\alpha)$, then $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M.*

Proof Assume that each level subsets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are M-subsemigroups of M. Then by Lemma 3.10, $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy M-subsemigroup of M, and so

$$\alpha_A(xy)\ge\min\{\alpha_A(x),\alpha_A(y)\}=T(\alpha_A(x),\alpha_A(y))$$

and $\beta_A(xy)\le\max\{\beta_A(x),\beta_A(y)\}=S(\beta_A(x),\beta_A(y))$ for all $x,y\in M$. Hence $A=(\alpha_A,\beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of M. □

Theorem 3.11 *Let T and S be, respectively, t-norm and s-norm. If $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, then for all $x\in M$, we have*

$$\alpha_A(x)=\sup\{\alpha\in[0,1]|x\in U(\alpha_A;\alpha)\},\quad \beta_A(x)=\inf\{\alpha\in[0,1]|x\in L(\beta_A;\alpha)\}.$$

Proof Suppose that $\delta=\sup\{\alpha\in[0,1]|x\in U(\alpha_A;\alpha)\}$ and let $\varepsilon>0$ be given. Then, we can see that $\delta-\varepsilon<\alpha$, for some $\alpha\in[0,1]$ such that $x\in U(\alpha_A;\alpha)$. This means that $\delta-\varepsilon<\alpha_A(x)$ and hence $\delta<\alpha_A(x)$, because ε is arbitrary.

We now show that $\alpha_A(x)<\delta$. If $\alpha_A(x)=\beta$, then we have $x\in U(\alpha_A;\beta)$ so that $\beta\in\{\alpha\in[0,1]|x\in U(\alpha_A;\alpha)\}$. Thus, $\alpha_A(x)=\delta\le\sup\{\alpha\in[0,1]|x\in U(\alpha_A;\alpha)\}=\delta$. Consequently, we can deduce that $\alpha_A(x)=\delta=\sup\{\alpha\in[0,1]|x\in U(\alpha_A;\alpha)\}$.

Now, let $\eta = \inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$. Then, we have $\inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\} < \eta + \varepsilon$, for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$, for some $\alpha \in [0,1]$ with $x \in L(\beta_A;\alpha)$. Since $\beta_A(x) \le \alpha$ and ε is arbitrary, we obtain $\beta_A(x) \le \eta$.

In order to prove that $\beta_A(x) \ge \eta$, we let $\beta_A(x) = \xi$. Then we have $x \in L(\beta_A;\xi)$ and so $\xi \in \{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$. Hence, $\inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\} \le \xi$, i.e., $\eta \le \xi = \beta_A(x)$. Consequently, we obtain $\beta_A(x) = \eta = \inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$. □

Definition 3.12 *A mapping $f : M \to M'$ of M-semigroups is called a homomorphism if*

(i) $f(xy) = f(x)f(y)$, for all $x, y \in M$;

(ii) f maps all left identities of M into the same left identity of M'.

Definition 3.13 *Let $f : M \to M'$ be a mapping of M-semigroups. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy set in M', then the inverse image of A under f, denoted by $f^{-1}(A)$, is an intuitionistic fuzzy set in M, defined by $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$.*

Theorem 3.14 *Let T and S be, respectively, t-norm and s-norm, and let $f : M \to M'$ be a homomorphism of M-semigroups. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M', then the inverse image $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$ of A under f is an intuitionistic (S,T)-fuzzy M-subsemigroup of M.*

Proof Assume that $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M' and $x, y \in M$. Then we have

(ISTF1)

$$\begin{aligned} f^{-1}(\alpha_A)(xy) &= \alpha_A(f(xy)) = \alpha_A(f(x)f(y)) \ge T(\alpha_A(f(x)), \alpha_A(f(y))) \\ &= T(f^{-1}(\alpha_A)(x), f^{-1}(\alpha_A)(y)), \\ f^{-1}(\beta_A)(xy) &= \beta_A(f(xy)) = \beta_A(f(x)f(y)) \le S(\beta_A(f(x)), \beta_A(f(y))) \\ &= S(f^{-1}(\beta_A)(x), f^{-1}(\beta_A)(y)). \end{aligned}$$

(IF2)

$$f^{-1}(\alpha_A)(e) = \alpha_A(f(e)) = \alpha_A(e') = 1, \quad f^{-1}(\beta_A)(e) = \beta_A(f(e)) = \beta_A(e') = 0.$$

This completes the proof. □

Definition 3.15 *Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy set on M, and let f be a mapping defined on M. The intuitionistic fuzzy set $A^f = (\alpha_A^f, \beta_A^f)$ in $f(M)$ is defined by*

$$\alpha_A^f(y) = \sup_{x \in f^{-1}(y)} \alpha_A(x), \quad \beta_A^f(y) = \inf_{x \in f^{-1}(y)} \beta_A(x),$$

for all $y \in f(M)$. We now call α_A^f the image of A under f.

Definition 3.16 *A t-norm T (resp. s-norm) on $[0,1]$ is called a continuous t-norm if T (resp. S) is a continuous function from $[0,1] \times [0,1] \to [0,1]$ with respect to the usual topology.*

We observe that the functions "min" and "max" are always continuous t-norm and s-norm, respectively.

Theorem 3.17 *Let T and S be continuous t-norm and s-norm, respectively, and let f be a homomorphism on M. If $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, then the image of A under f is an intuitionistic (S,T)-fuzzy M-subsemigroup of $f(M)$.*

Proof First, we show that α_A^f is a T-fuzzy M-subsemigroup of $f(M)$. Let $A_1=f^{-1}(y_1), A_2=f^{-1}(y_2)$ and $A_{12}=f^{-1}(y_1y_2)$, where $y_1,y_2\in f(M)$. Consider the set

$$A_1A_2=\{x\in M|x=a_1a_2 \ \text{ for some } \ a_1\in A_1 \ \text{ and } \ a_2\in A_2\}.$$

If $x\in A_1A_2$, then $x=x_1x_2$ for some $x_1\in A_1$ and $x_2\in A_2$ so that we have $f(x)=f(x_1x_2)=f(x_1)f(x_2)=y_1y_2$, that is , $x\in f^{-1}(y_1y_2)=A_{12}$. Thus $A_1A_2\subseteq A_{12}$. It hence follows that

$$\begin{aligned}\alpha_A^f(y_1y_2)&=\sup\{\alpha_A(x)|x\in f^{-1}(y_1y_2)\}=\sup\{\alpha_A(x)|x\in A_{12}\}\\&\geq\sup\{\alpha_A(x)|x\in A_1A_2\}=\sup\{\alpha_A(x_1x_2)|x_1\in A_1,x_2\in A_2\}\\&\geq\sup\{T(\alpha_A(x_1),\alpha_A(x_2))|x_1\in A_1,x_2\in A_2\}.\end{aligned}$$

Since T is continuous, for every $\varepsilon>0$, we see that if $\sup\{\alpha_A(x_1)|x_1\in A_1\}-x_1^*\leq\delta$ and $\sup\{\alpha_A(x_2)|x_2\in A_2\}-x_2^*\leq\delta$, then

$$T(\sup\{\alpha_A(x_1)|x_1\in A_1\},\sup\{\alpha_A(x_2)|x_2\in A_2\})-T(x_1^*,x_2^*)\leq\varepsilon.$$

Choose $a_1\in A_1$ and $a_2\in A_2$, such that

$$\sup\{\alpha_A(x_1)|x_1\in A_1\}-\alpha_A(a_1)\leq\delta,\quad \sup\{\alpha_A(x_2)|x_2\in A_2\}-\alpha_A(a_2)\leq\delta.$$

Then we have $T(\sup\{\alpha_A(x_1)|x_1\in A_1\},\sup\{\alpha_A(x_2)|x_2\in A_2\})-T(\alpha_A(a_1),\alpha_A(a_2))\leq\varepsilon$. Consequently, we have

$$\alpha_A^f(y_1y_2)\geq\sup\{T(\alpha_A(x_1),\alpha_A(x_2))|x_1\in A_1,x_2\in A_2\}\geq T(\sup\{\alpha_A(x_1)|x_1\in A_1\},$$

$$\sup\{\alpha_A(x_2)|x_2\in A_2\})=T(\alpha_A^f(y_1),\alpha_A^f(y_2)),\alpha_A^f(e')=\sup\{\alpha_A(e)|e\in f^{-1}(e')\}=1.$$

This shows that α_A^f is a T-fuzzy M-subsemigroup of $f(M)$. Similarly, we can show that β_A^f is an S-fuzzy M-subsemigroup of $f(M)$. Therefore, $A^f=(\alpha_A^f,\beta_A^f)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of $f(M)$. □

4. Intuitionistic (S,T)-direct products

Hereafter, we use R and X to denote the right zero semigroup and semigroup with two sided identity respectively, unless otherwise stated.

Any element $x\in M\cong R\times X$ can be written as the product (binary operation in M) of two elements: One element in R is called the two sided identity(right identity) of x, denoted by e_x, and the other element is xe in $X(=Me)$. That is , $x=xe_x=x(ee_x)=(xe)e_x$. Hence, we can express an element x in M by the element (e_x,xe) in $R\times X$.

Definition 4.1 *Let T and S be, respectively, t-norm and s-norm. The intuitionistic (S,T)-direct product of A and B, denoted by $A \times B$, is an intuitionistic fuzzy set in M, defined by*

$$A \times B = (\alpha_A, \beta_A) \times (\alpha_B, \beta_B) = (\alpha_A \times \alpha_B, \beta_A \times \beta_B),$$

where $(\alpha_A \times \alpha_B)(x,y) = T(\alpha_A(x), \alpha_B(y))$ and $(\beta_A \times \beta_B)(x,y) = S(\beta_A(x), \beta_B(y))$ for all $x, y \in M$.

Definition 4.2 *An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in M is called an intuitionistic fuzzy subsemigroup of M with respect to t-norm T and s-norm S (briefly, intuitionistic (S,T)-fuzzy subsemigroup of M) if it satisfies only the condition (ISTF1).*

Clearly, every intuitionistic (S,T)-fuzzy M-subsemigroup is an intuitionistic (S,T)-fuzzy subsemigroup.

Theorem 4.3 *Let T and S be, respectively, t-norm and s-norm. If $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of $M \cong R \times X$ such that $\mathrm{Im}\alpha_A = \{1, \alpha_1\}$ and $\mathrm{Im}\beta_A = \{0, \alpha_2\}$, where $0 \le \alpha_1 < 1, 0 < \alpha_2 \le 1$ and $0 < \alpha_1 + \alpha_2 \le 1$. Then $A = (\alpha_A, \beta_A)$ can be written as a union of intuitionistic (S,T)-fuzzy subsemigroups.*

Proof Let $M_\alpha = \{x \in M | \alpha_A(x) = 1 \text{ and } \beta_A(x) = 0\}$. Then we have

$$\alpha_A(x) = \begin{cases} 1, & \text{if } x \in M_\alpha, \\ 0, & \text{otherwise} \end{cases}, \quad \beta_A(x) = \begin{cases} 0, & \text{if } x \in M_\alpha, \\ 1, & \text{otherwise} \end{cases}.$$

We claim that M_α is an M-subsemigroup of M. In fact, if we choose $x, y \in M_\alpha$, then we have $\alpha_A(x) = \alpha_A(y) = 1$ and $\beta_A(x) = \beta_A(y) = 0$. Since $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, we have

$$\alpha_A(xy) \ge T(\alpha_A(x), \alpha_A(y)) = T(1,1) = 1, \quad \beta_A(xy) \le S(\beta_A(x), \beta_A(y)) = S(0,0) = 0.$$

Hence $\alpha_A(xy) = 1$ and $\beta_A(xy) = 0$, so that $xy \in M_\alpha$ and hence M_α is a subsemigroup of M. By the definition of intuitionistic (S,T)-fuzzy M-subsemigroup, we have $\alpha_A(e) = 1$ and $\beta_A(e) = 0$ for every left identity $e \in R$. Hence $e \in M_\alpha$. Since M_α is a subgroup, we can choose $e, x \in M_\alpha$, such that $ex \in M_\alpha$. Clearly $ex \in M$. This leads to $ex = x$, and so M_α is an M-subsemigroup of M. By applying Lemma 2.7, there exist M-subsemigroups $M_\alpha e$, $M_\alpha f$, $\ldots$ of M_α, such that $M_\alpha = M_\alpha e \bigcup M_\alpha f \bigcup \cdots$, where $e, f, \ldots \in R$.

For each $e \in R$, define the intuitionistic fuzzy sets $A_e = (\alpha_{A_e}, \beta_{A_e})$ on M by

$$\alpha_{A_e}(x) = \begin{cases} 1, & \text{if } x \in M_\alpha e, \\ \alpha_1, & \text{otherwise} \end{cases}, \quad \beta_{A_e}(x) = \begin{cases} 0, & \text{if } x \in M_\alpha e, \\ \alpha_2, & \text{otherwise} \end{cases}.$$

Then it is easy to prove that every $A_e = (\alpha_{A_e}, \beta_{A_e})$ is an intuitionistic (S,T)-fuzzy subsemigroup of M. Now it is straightforward to verify that $\alpha_A = \alpha_{A_e} \bigcup \alpha_{A_f} \bigcup \cdots$ and $\beta_A = \beta_{A_e} \bigcap \beta_{A_f} \bigcap \cdots$. The proof is now completed. □

Example 4.4 Let M be an M-semigroup as in Example 3.2. Let $M_\alpha = \{e, f\}, M_\alpha e = \{e\}$ and $M_\alpha f = \{f\}$. Then $M_\alpha = M_\alpha e \bigcup M_\alpha f$. Moreover, we have

$$\alpha_{A_e}(e) = 1, \alpha_{A_e}(f) = \alpha_{A_e}(a) = \alpha_{A_e}(b) = \alpha_1; \; \alpha_{A_f}(f) = 1, \alpha_{A_f}(e) = \alpha_{A_f}(a) = \alpha_{A_f}(b) = \alpha_1.$$

$$\beta_{A_e}(e) = 0, \beta_{A_e}(f) = \beta_{A_e}(a) = \beta_{A_e}(b) = \alpha_2;\ \beta_{A_f}(f) = 0, \beta_{A_f}(e) = \beta_{A_f}(a) = \beta_{A_f}(b) = \alpha_2.$$

Clearly, $\alpha_A = \alpha_{A_e} \bigcup \alpha_{A_f}$ and $\beta_A = \beta_{A_e} \bigcap \beta_{A_f}$.

Now, we generalize Theorem 4.3.

Corollary 4.5 *Let T and S be, respectively, t-norm and s-norm. If $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy M-subsemigroup of $M \cong R \times X$ such that $\mathrm{Im}\alpha_A = \{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\mathrm{Im}\beta_A = \{0, \beta_1, \beta_2, \ldots, \beta_n\}$, where $1 > \alpha_1 > \cdots > \alpha_n \geq 0$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_n \leq 1$. Then $A = (\alpha_A, \beta_A)$ can be written as a union of intuitionistic (S,T)-fuzzy subsemigroups.*

Lemma 4.6[19] *Let T and S be, respectively, t-norm and s-norm. Then for all $\alpha, \beta, \delta, \gamma \in [0,1]$, we have $T(T(\alpha,\beta), T(\gamma,\delta)) = T(T(\alpha,\gamma), T(\beta,\delta))$; $S(S(\alpha,\beta), S(\gamma,\delta)) = S(S(\alpha,\gamma), S(\beta,\delta))$.*

By using the above lemma, we obtain the following lemma.

Lemma 4.7 *Let T and S be, respectively, t-norm and s-norm. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of R and $B = (\alpha_B, \beta_B)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of X, then for any $e, f \in R$ and $x_1, x_2 \in X$, we have*

$$T(T(\alpha_A(e), \alpha_A(f)), T(\alpha_B(x_1), \alpha_B(x_2))) = T(T(\alpha_A(e), \alpha_B(x_1)), T(\alpha_A(f), \alpha_B(x_2)))$$

$$S(S(\beta_A(e), \beta_A(f)), S(\beta_B(x_1), \beta_B(x_2))) = S(S(\beta_A(e), \beta_B(x_1)), S(\beta_A(f), \beta_B(x_2))).$$

Theorem 4.8 *Let T and S be, respectively, t-norm and s-norm, and let $M \cong R \times X$ be an M-semigroup. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of R and $B = (\alpha_B, \beta_B)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of S, then $A \times B$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, where $A \times B$ is defined by*

$$A \times B = (\alpha_A, \beta_A) \times (\alpha_B, \beta_B) = (\alpha_A \times \alpha_B, \beta_A \times \beta_B),$$

where $(\alpha_A \times \alpha_B)(a,b) = T(\alpha_A(a), \alpha_B(b))$ and $(\beta_A \times \beta_B)(a,b) = S(\beta_A(a), \beta_B(b))$, for all $a \in R$ and $b \in X$.

Proof Let $x = (e, x_1)$ and $y = (f, x_2)$ in $R \times X \cong M$, where $e, f \in R$ and $x_1, x_2 \in X$. Then we have

(ISTF1)

$$\begin{aligned}(\alpha_A \times \alpha_B)(xy) &= (\alpha_A \times \alpha_B)((e,x_1)(f,x_2)) = (\alpha_A \times \alpha_B)(ef, x_1x_2)\\ &= T(\alpha_A(ef), \alpha_B(x_1x_2)) \geq T(T(\alpha_A(e), \alpha_A(f)), T(\alpha_B(x_1), \alpha_B(x_2)))\\ &= T(T(\alpha_A(e), \alpha_B(x_1)), T(\alpha_A(f), \alpha_B(x_2)))\\ &= T((\alpha_A \times \alpha_B)(e, x_1), (\alpha_A \times \alpha_B)(f, x_2))\\ &= T((\alpha_A \times \alpha_B)(x), (\alpha_A \times \alpha_B)(y)).\end{aligned}$$

Similarly, $(\beta_A \times \beta_B)(xy) \leq S((\beta_A \times \beta_B)(x), (\beta_A \times \beta_B)(y))$.

(IF2) Let e_1 be any left identity of R and e the two sided identity of X. Then, we have

$$(\alpha_A \times \alpha_B)(e_1, e) = T(\alpha_A(e_1), \alpha_B(e)) = T(1,1) = 1$$

and $(\beta_A \times \beta_B)(e_1, e) = S(\beta_A(e_1), \beta_B(e)) = S(0,0) = 0$.

It follows that $A \times B$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M. □

Lemma 4.9 *Let T and S be, respectively, t-norm and s-norm. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of $M \cong R \times X$, then there exists a homomorphic image A^φ on R, which is also an intuitionistic (S,T)-fuzzy M-subsemigroup of R.*

Proof Let $x \in M$. Denote the unique (right) identity of x by e_x. Then $xe_x = x$. Consider the map $\varphi : M \to R$ such that $\varphi(x) = e_x$ for all $x \in M$. Since $(xy)e_{xy} = xy = x(ye_y) = (xy)e_y$, we have $e_{xy} = e_y$. Since e_x is a left identity of e_y, we have $e_{xy} = e_y = e_x e_y$. This shows that $\varphi(xy) = e_{xy} = e_x e_y = \varphi(x)\varphi(y)$. Clearly, φ is an onto homomorphism.

Define an intuitionistic fuzzy subset $B = (\alpha_B, \beta_B)$ on R by

$$\alpha_B(e_x) = \sup_{x\in\varphi^{-1}(e_x)} \alpha_A(x) = \alpha_A(e_x) = 1, \quad \beta_B(e_x) = \inf_{x\in\varphi^{-1}(e_x)} \beta_A(x) = \beta_A(e_x) = 0.$$

For any $e_x, e_y \in R$, by the definition of $B = (\alpha_B, \beta_B)$ and that φ is a homomorphism from M onto R, we have

(ISTF1)

$$\alpha_B(e_x e_y) = \alpha_A^\varphi(e_x e_y) = \alpha_A^\varphi(e_{xy}) = 1 = \alpha_A(e_x e_y),$$

$$\beta_B(e_x e_y) = \beta_A^\varphi(e_x e_y) = \beta_A^\varphi(e_{xy}) = 0 = \beta_A(e_x e_y).$$

Since $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, we have

$$\alpha_A(e_x e_y) \geq T(\alpha_A(e_x), \alpha_A(e_y)), \beta_A(e_x e_y) \leq S(\beta_A(e_x), \beta_A(e_y)),$$

and thus

$$\alpha_B(e_x e_y) = \alpha_A(e_x e_y) \geq T(\alpha_A(e_x), \alpha_A(e_y)) = T(\alpha_B(e_x), \alpha_B(e_y)),$$

$$\beta_B(e_x e_y) = \beta_A(e_x e_y) \leq S(\beta_A(e_x), \beta_A(e_y)) = S(\beta_B(e_x), \beta_B(e_y)).$$

(IF2) $\alpha_B(e_x) = \alpha_A^\varphi(e_x) = \max \alpha_A(e_x) = 1$ and $\beta_B(e_x) = \beta_A^\varphi(e_x) = \min \beta_A(e_x) = 0$.
This shows that $B = A^\varphi$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of R. □

Lemma 4.10 *Let T and S be, respectively, t-norm and s-norm. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of $M \cong R \times X$, then there exists a homomorphic image A on X which is also an intuitionistic (S,T)-fuzzy M-subsemigroup of X.*

Proof Consider the map $\psi : M \to X(= Me)$ such that $\psi(x) = xe$ for all $x \in M$. Then $\psi(xy) = xye = x(ey)e = (xe)(ye) = \psi(x)\psi(y)$. Clearly, ψ is an onto homomorphism.

Define an intuitionistic fuzzy set $B = (\alpha_B, \beta_B)$ on X by

$$\alpha_B(xe) = \sup_{x\in\psi^{-1}(xe)} \alpha_A(x) = \alpha_A(xe), \quad \beta_B(xe) = \inf_{x\in\psi^{-1}(xe)} \beta_A(x) = \beta_A(xe).$$

Putting $r = xe, z = ye$ in $X(= Me)$. Then by the definition of B and that ψ is a homomorphism from M onto X, we have

(ISTF1)

$$\alpha_B(rz)=\alpha_A^{\psi}(rz)=\alpha_A^{\psi}(xeye)=\alpha_A^{\psi}(xye)=\alpha_A(xye)=\alpha_A(rz),$$
$$\beta_B(rz)=\beta_A^{\psi}(rz)=\beta_A^{\psi}(xeye)=\beta_A^{\psi}(xye)=\beta_A(xye)=\beta_A(rz).$$

Since $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, we have

$$\alpha_B(rz)=\alpha_A(rz)\geq T(\alpha_A(r),\alpha_A(z))=T(\alpha_B(r),\alpha_B(z)),$$
$$\beta_B(rz)=\beta_A(rz)\leq S(\beta_A(r),\beta_A(z))=S(\beta_B(r),\beta_B(z)).$$

(IF2)

$$\alpha_B(fe)=\alpha_A^{\psi}(fe)=\alpha_A^{\psi}(e)=\alpha_A(ee)=\alpha_A(e)=1,$$
$$\beta_B(fe)=\beta_A^{\psi}(fe)=\beta_A^{\psi}(e)=\beta_A(ee)=\beta_A(e)=0.$$

Therefore, $B=A^{\psi}$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of X. □

Now, we give the following main result.

Theorem 4.11 *Let T and S be, respectively, t-norm and s-norm. If $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of $M\cong R\times X$, then $A=(\alpha_A,\beta_A)$ can be written as the intuitionistic (S,T)-direct product of intuitionistic (S,T)-fuzzy M-subsemigroup of R and $X(=Me)$, respectively.*

Proof Since $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, by Lemmas 4.9 and 4.10, there exist A^{φ} and A^{ψ} which are intuitionistic (S,T)-fuzzy M-subsemigroups of R and $X(=Me)$, respectively. Again since $A=(\alpha_A,\beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, for each $x\in M$, we have

$$\begin{aligned}\alpha_A(x)&=\alpha_A(xe_x)=\alpha_A(xee_x)\geq T(\alpha_A(e_x),\alpha_A(xe))=T(\alpha_A^{\varphi}(e_x),\alpha_A^{\psi}(xe))\\&=(\alpha_A^{\varphi}\times\alpha_A^{\psi})(e_x,xe)=(\alpha_A^{\varphi}\times\alpha_A^{\psi})(x),\\\beta_A(x)&=\beta_A(xe_x)=\beta_A(xee_x)\leq S(\beta_A(e_x),\beta_A(xe))=S(\beta_A^{\varphi}(e_x),\beta_A^{\psi}(xe))\\&=(\beta_A^{\varphi}\times\beta_A^{\psi})(e_x,xe)=(\beta_A^{\varphi}\times\beta_A^{\psi})(x).\end{aligned}$$

Thus $\alpha_A\supseteq\alpha_A^{\varphi}\times\alpha_A^{\psi}$ and $\beta_A\subseteq\beta_A^{\varphi}\times\beta_A^{\psi}$.

Now, let $y\in M$, then we have $y=(e_y,ye)\in R\times X$. Hence we can deduce that

$$\begin{aligned}(\alpha_A^{\varphi}\times\alpha_A^{\psi})(y)&=(\alpha_A^{\varphi}\times\alpha_A^{\psi})(e_y,ye)=T(\alpha_A^{\varphi}(e_y),\alpha_A^{\psi}(ye))\\&=T(\alpha_A(e_y),\alpha_A(ye))=T(1,\alpha_A(ye))\\&=\alpha_A(ye)\geq T(\alpha_A(y),\alpha_A(e))=T(\alpha_A(y),1)=\alpha_A(y),\\(\beta_A^{\varphi}\times\beta_A^{\psi})(y)&=(\beta_A^{\varphi}\times\beta_A^{\psi})(e_y,ye)=S(\beta_A^{\varphi}(e_y),\beta_A^{\psi}(ye))\\&=S(\beta_A(e_y),\beta_A(ye))=S(0,\beta_A(ye))=\beta_A(ye)\leq S(\beta_A(y),\beta_A(e))\\&=S(\beta_A(y),0)=\beta_A(y).\end{aligned}$$

This leads to $\alpha_A\subseteq\alpha_A^{\varphi}\times\alpha_A^{\psi}$ and $\beta_A\supseteq\beta_A^{\varphi}\times\beta_A^{\psi}$.

Therefore, $\alpha_A = \alpha_A^{\varphi} \times \alpha_A^{\psi}$ and $\beta_A = \beta_A^{\varphi} \times \beta_A^{\psi}$, that is, $A = A^{\varphi} \times A^{\psi}$. □

Theorem 4.12 *Let T and S be, respectively, t-norm and s-norm. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic (S,T)-fuzzy M-subsemigroup of $M \cong R \times X$. For $e \in R$, let the intuitionistic fuzzy set $A_e^* = (\alpha_{A_e}^*, \beta_{A_e}^*)$ be defined on M by $\alpha_{A_e}^*(x) = \alpha_A(xe)$ and $\beta_{A_e}^*(x) = \beta_A(xe)$ for all $x \in M$. Then we have*

(i) A_e^* is an intuitionistic (S,T)-fuzzy M-subsemigroup of M;

(ii) $\bigcap\limits_{e\in R} A_e^*$ is also an intuitionistic (S,T)-fuzzy M-subsemigroup of M.

Proof (i) Since $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy M-subsemigroup of M, we have:

(ISTF1) For all $x, y \in M$,

$$\alpha_{A_e}^*(xy) = \alpha_A(xye) = \alpha_A(xeye) \geq T(\alpha_A(xe), \alpha_A(ye)) = T(\alpha_{A_e}^*(x), \alpha_{A_e}^*(y)),$$
$$\beta_{A_e}^*(xy) = \beta_A(xye) = \beta_A(xeye) \leq S(\beta_A(xe), \beta_A(ye)) = S(\beta_{A_e}^*(x), \beta_{A_e}^*(y)).$$

(IF2) For every $f \in R$, $\alpha_{A_e}^*(f) = \alpha_A(fe) = \alpha_A(e) = 1$ and $\beta_{A_e}^*(f) = \beta_A(fe) = \beta_A(e) = 0$. Hence A_e^* is an intuitionistic (S,T)-fuzzy M-subsemigroup of M.

(ii) For any $x, y \in M$, we have:

(ISTF1)

$$\begin{aligned}\bigcap_{e\in R} \alpha_{A_e}^*(xy) &= \min\{\alpha_{A_e}^*(xy)|e \in R\} \geq \min\{T(\alpha_{A_e}^*(x), \alpha_{A_e}^*(y))|e \in R\} \\ &\geq T(\min\{\alpha_{A_e}^*(x)|e \in R\}, \min\{\alpha_{A_e}^*(y)|e \in R\}) \\ &= T(\bigcap_{e\in R} \alpha_{A_e}^*(x), \bigcap_{e\in R} \alpha_{A_e}^*(y)),\end{aligned}$$
$$\begin{aligned}\bigcup_{e\in R} \beta_{A_e}^*(xy) &= \max\{\beta_{A_e}^*(xy)|e \in R\} \leq \max\{S(\beta_{A_e}^*(x), \beta_{A_e}^*(y))|e \in R\} \\ &\leq S(\max\{\beta_{A_e}^*(x)|e \in R\}, \max\{\beta_{A_e}^*(y)|e \in R\}) \\ &= S(\bigcup_{e\in R} \beta_{A_e}^*(x), \bigcup_{e\in R} \beta_{A_e}^*(y)).\end{aligned}$$

(IF2) For any $f \in R$, we have

$$\bigcap_{e\in R} \alpha_{A_e}^*(f) = \min\{\alpha_{A_e}^*(f)|e \in R\} = \min\{\alpha_A(fe)|e \in R\} = \min\{\alpha_A(e)|e \in R\} = 1,$$
$$\bigcup_{e\in R} \beta_{A_e}^*(f) = \max\{\beta_{A_e}^*(f)|e \in R\} = \max\{\beta_A(fe)|e \in R\} = \max\{\beta_A(e)|e \in R\} = 0.$$

Therefore, $\bigcap_{e\in R} A_e^*$ is also an intuitionistic (S,T)-fuzzy M-subsemigroup of M. □

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References:

[1] ATANASSOV K T. *Intuitionistic fuzzy sets* [J]. Fuzzy Sets and Systems, 1986, **20**(1): 87–96.

[2] ATANASSOV K. T. *New operations defined over the intuitionistic fuzzy sets* [J]. Fuzzy Sets and Systems, 1994, **61**(2): 137–142.
[3] BISWAS R. *Intuitionistic fuzzy subgroups* [J]. Math. Forum., 1989, **10**: 37–46.
[4] CLIFFORD A H, PRESTON G.B. *The Algebraic Theory of Semigroups* [M]. American Mathematical Society, Providence, R.I. 1961.
[5] DUDEK W A, DAVVAZ B, JUN Y B. *On intuitionistic fuzzy sub-quasigroups of quasigroups*[J]. Inform. Sci., 2005, **170**: 251–262.
[6] KIM K H, JUN Y B. *Intuitionistic fuzzy ideals of semigroups* [J]. Indian J. Pure Appl. Math., 2002, **33**(4): 443–449.
[7] KLIR G J, BOYUAN. *Fuzzy Sets and Fuzzy Logic Theory and Applications*[M]. Pretice-Hall Inc, Englewood Cliff, N.J., USA, 1995.
[8] KUROKI N. *Fuzzy bi-ideals in semigroups*[J]. Comment. Math. Univ. St, Paul., 1980, **28**(1): 17–21.
[9] KUROKI N. *Fuzzy semiprime ideals in semigroups* [J]. Fuzzy Sets and Systems, 1982, **8**(1): 71–79.
[10] KUROKI N. *On fuzzy semigroups* [J]. Inform. Sci., 1991, **53**: 203–236.
[11] LAKSHMANAN L. *Certain studies in the structure of an algebraic semigroups*[D]. Ph.D. Thesis, Bangalor Univ, 1983.
[12] NARAYANAN A L, MEENAKSHI A R. *On fuzzy sub M-semigroups of an M-semigroup* [J]. J. Fuzzy Math., 2003, **11**(1): 41–52.
[13] PETRICH M. *Prime ideals of the carteisan product of two semigroups* [J]. Czech. Math. J., 1962, **12**: 150–152.
[14] PLEMMAS R. *Maximal ideals in the direct product of two semigroups* [J]. Czech. Math. J., 1967, **17**: 257–260.
[15] ROSENFELD A. *Fuzzy groups* [J]. J. Math. Anal. Appl., 1971, **35**: 512–517.
[16] TAMURA T, MARKEL R B, LATIMAR J F. *The direct product of right singular semigroups and certain groupoids* [J]. Proc. Amer. Math. Soc., 1963, **14**: 118–123.
[17] TAMURA T, MARKEL R B, LATIMAR J F. *Correction and supplement to the direct product of right singular semigroups and certain groupoid* [J]. Proc. Amer. Math. Soc., 1965, **16**: 841.
[18] WARNER R J. *The direct product of right zero semigroups and certain groups* [J]. Amer. Math. Monthly, 1967, **74**: 160–164.
[19] YU Y, MORDESON J N, CHENG S C. *Elements of L-Algebras* [M]. Lecture Notes in Fuzzy Math, and Computer Sciences, Creighton Univ, Omaha, Nebraska, 68178, USA(1994).
[20] ZADEH L A. *Fuzzy sets* [J]. Inform. and Control, 1965, **8**: 338–353.
[21] ZHAN Jian-ming, TAN Zhi-song. *Properties of fuzzy M-semigroups with t-norms* [J]. J. Math. Res. Exposition, 2006, **26**(1): 67–76. (in Chinese)
[22] ZHAN Jian-ming, SHUM K P. *On intuitionistic (S,T)-fuzzy left h-ideals in hemirings* [J]. J. Fuzzy Math., 2006, **14**(4): 929–945.

M- 半群的直觉 (S,T)- 模糊 M- 子半群的性质

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摘要: K.T. Atanassov 在 1986 年首先引入直觉模糊集的概念．本文利用 s- 范数和 t- 范数，引入 M- 半群的直觉 (S,T)- 模糊 M- 子半群的概念，刻画其性质和特征．我们再进一步给出了直觉 (S,T)- 直积的概念，并由此探讨了一些有意义的相关结论．

关键词: M- 半群； (虚) 直觉 (S,T)- 模糊 M- 子半群；直觉 (S,T)- 直积．