A Model for Surface Wave Propagation across a Shearing Current

WILLIAM D. MCKEE

School of Mathematics, University of New South Wales, Sydney, New South Wales, Australia

30 January 1995 and 5 June 1995

ABSTRACT

A new model is derived for the propagation of surface waves across a shearing current, which is applicable when the wavelength is comparable to the lateral length scale over which the current changes. When applied to wave propagation across a jet-type current, this model predicts a pronounced minimum in the reflection coefficient when the water depth is comparable with the wavelength.

1. Introduction

For some time, the present author has been interested in the problem of the propagation of water waves across a horizontally sheared current. Even the full linear problem is intractable, so various simplifications have been introduced over the years. At one extreme, current changes have been modeled as one or more vortex sheets (see Evans 1975; McKee and Tesoriero 1987; Smith 1983). At the other extreme, the currents have been assumed to be slowly varying on the scale of a wavelength, which leads to WKB-type solutions or extensions thereof (see McKee 1987; Mei 1983). A good early review of the whole subject is given in Peregrine (1976). The present work introduces a third approach to the problem that can be thought of as extending the second approach to cases in which the current is not necessarily slowly varying.

2. The basic equations

As in McKee (1987), we consider wave motion in an inviscid fluid of constant density ρ. The x and z axes are taken horizontal with the y axis vertically down. The undisturbed free surface is at $y = 0$, and the bottom at $y = H$. In order to concentrate on the effects of the current, it will be assumed that H is constant. The basic current is a shear flow $(0, 0, W(x))$. Wavy perturbations to this basic state are now considered in which all perturbation quantities are proportional to $\exp i(nz - \omega t)$ where $\omega > 0$. If W varies with x on a length scale L, we scale x and z with L, y, and H with g/ω^2, W with g/ω, n with ω^2/g and the perturbation pressure due to the waves with $\rho g a$ where a is a typical free-surface amplitude due to the waves. This scaling uses velocity, wavenumber, and depth scales appropriate to waves in deeper water. As shown in McKee (1987), the dimensionless pressure perturbation p, for linear theory, satisfies the equation

$$(p_x/\Omega^2)_x + \epsilon^2(p_y/\Omega^2)_y - n^2\epsilon^2 p/\Omega^2 = 0, \quad (1)$$

subject to the boundary conditions

$$p_y + \Omega^2 p = 0 \quad \text{at} \quad y = 0 \quad (2)$$

and

$$p_y = 0 \quad \text{at} \quad y = H, \quad (3)$$

where all variables are now dimensionless; subscripts indicate partial differentiation,

$$\Omega(x) = 1 - nW(x), \quad \text{and} \quad \epsilon = \omega^2 L/g.$$

The parameter ϵ essentially measures how rapidly the current changes on the scale of the waves—smaller values of ϵ signify a more rapidly changing current. As discussed in McKee (1987), values of ϵ of order 100 or more are to be expected in most oceanographical situations because of the large lateral length scales over which most ocean currents vary. Nearer shore, smaller values of ϵ might occur since river outflows or tidal currents between islands, for example, can vary over much smaller lateral length scales.

Let $k(x)$ be the unique positive root of

$$\Omega^2(x) = k(x)\tanh(k(x)H) \quad (4)$$

and define $\Phi(y; x)$ by

$$\Phi(y; x) = \frac{\cosh(k(x)(y - H))}{\cosh(k(x)H)}. \quad (5)$$

These definitions imply that $\Phi(y; x)$ is a local eigenfunction for surface wave propagation, which satisfies

$$\Phi_{yy} = k^2\Phi \quad (6)$$

Corresponding author address: Dr. William D. McKee, School of Mathematics, University of New South Wales, Sydney 2052, Australia.
E-mail: billmc@solution.maths.unsw.edu.au

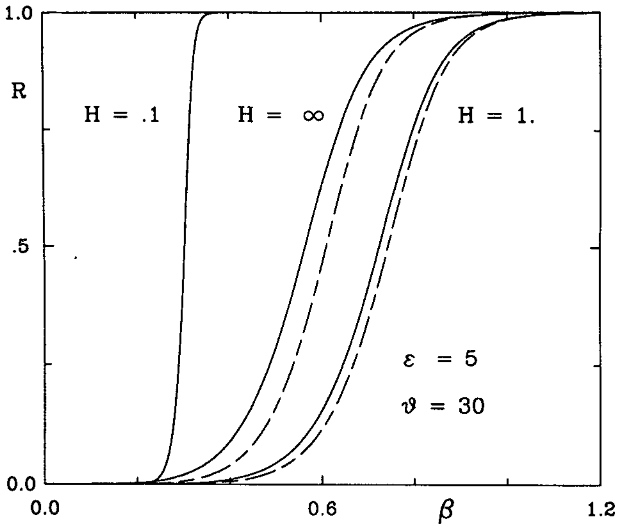

FIG. 1. The magnitude of the reflection coefficient as a function of the dimensionless current strength for a jet current for three different depths. The results for the mild-shear equation are shown by a dashed line, and for the extended method by a solid line. The angle of incidence is $\vartheta = 30°$.

subject to

$$\Phi_y + \Omega^2\Phi = 0 \quad \text{at} \quad y = 0 \tag{7}$$

and

$$\Phi_y = 0 \quad \text{at} \quad y = H. \tag{8}$$

3. The approximate equation

To derive the approximate equation, we multiply (1) by $\Phi\Omega^2$ and integrate from $y = 0$ to $y = H$. After integration by parts and use of the boundary conditions and the definition of Φ, we find

$$\Omega^2 \int_0^H (p_x/\Omega^2)_x \Phi dy + \epsilon^2(k^2 - n^2) \int_0^H p\Phi dy = 0. \tag{9}$$

Thus far, no extra approximations have been made. If we now assume that the evanescent modes make no contribution and, hence, that

$$p = \eta(x)\Phi(y; x), \tag{10}$$

we find that η, which can be interpreted as the dimensionless free-surface elevation, satisfies

$$\Omega^2 \frac{d}{dx}\left(\frac{d\eta}{dx}\Omega^{-2}\int_0^H \Phi^2 dy\right) + \epsilon^2(k^2 - n^2)\eta \int_0^H \Phi^2 dy = \Omega^2 \mathcal{R}(x)\eta, \tag{11}$$

where

$$\mathcal{R}(x) = -\int_0^H \Phi(\Phi_x/\Omega^2(x))_x dy. \tag{12}$$

It was further argued in McKee (1987) that Φ varies on the scale of the current but η varies on the scale of the waves. Hence, for large ϵ the x derivatives of η are formally $O(\epsilon)$, whereas $\mathcal{R}(x)$ is formally $O(1)$. Neglecting $\mathcal{R}(x)$ entirely thus gives the following analog of the mild-slope equation, called the mild-shear equation, which was derived in McKee (1987):

$$\frac{d}{dx}\left(\Psi(x)\frac{d\eta}{dx}\right) + \epsilon^2(k^2(x) - n^2)\Psi(x)\eta = 0. \tag{13}$$

In this equation,

$$\Psi(x) = \Omega^{-2}(x)\int_0^H \Phi^2(y; x)dy.$$

However, for any value of ϵ, we may evaluate $\mathcal{R}(x)$ explicitly since Ω is independent of y. Our proposal is therefore to modify the mild-shear equation (13) to include the $\mathcal{R}(x)\eta$ term on the right and thus remove the restriction that ϵ is large. This is analogous to the strategy adopted by Massel (1993) for the somewhat similar problem of propagation over varying bottom topography. For arbitrary depth, the expression for $\mathcal{R}(x)$ is quite lengthy and is relegated to the appendix. When $H = \infty$, things are much simpler, and we find that $k = \Omega^2$, $\Phi = \exp(-ky)$, and

$$\mathcal{R}(x) = -\frac{1}{2}(3n^2(W')^2 + n\Omega W'')\Omega^{-6}(x), \tag{14}$$

where the prime denotes differentiation with respect to x.

In the situation where W varies from one constant value at $x = -\infty$ to another constant value at $x = +\infty$, it was shown in McKee (1987) for (13) that wave ac-

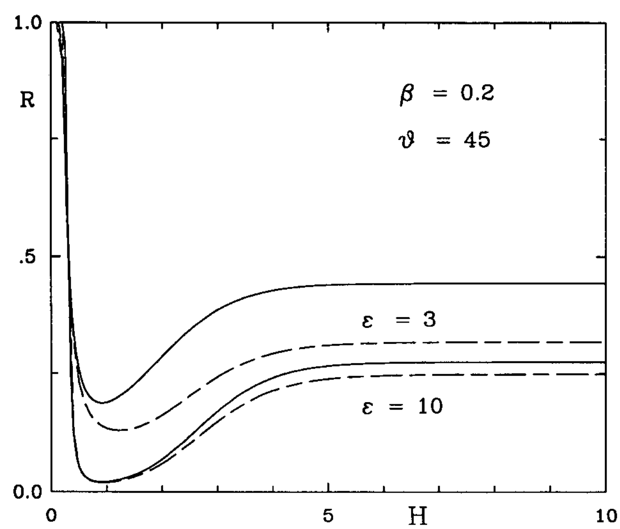

FIG. 2. The magnitude of the reflection coefficient as a function of the dimensionless depth for a jet current. The results for the mild-shear equation are shown by a dashed line, and for the extended method by a solid line for two different values of ϵ. The angle of incidence is $\vartheta = 45°$.

tion flux is conserved, provided there are no critical layers. This conservation law depends only on the Wronskian of two solutions of (13)—that is, upon Ψ and so is independent of the right-hand side of (13). Thus, it will hold also when the right-hand side is replaced by $\mathcal{R}(x)\eta$.

4. Some numerical results

This section will present some results of numerical solutions of both the mild-shear equation (13) and the extension proposed above, in which $\mathcal{R}(x)\eta$ is added to the right-hand side. Figure 1 shows the magnitude of the reflection coefficient as a function of the dimensionless current strength β for a dimensionless jet current $W(x) = \beta \exp(-x^2)$ for three different depths when the angle of incidence is 30° and $\epsilon = 5$. Only positive values of β need be considered. These correspond to waves entering a following current. For waves entering an adverse current (negative β) the reflection is very weak. The results for the mild-shear equation are shown by a dashed line and those for its modification by a solid line. For $H = 0 \cdot 1$, the two are virtually indistinguishable. In fact, both forms reduce to standard linear shallow water theory as $H \rightarrow 0$. For larger values of ϵ, the two methods agree more closely. The most noteworthy feature of these results is that the reflection coefficient is not a monotone function of the depth H. This point is further brought out in Fig. 2, which shows the reflection coefficient as a function of the dimensionless depth for a jet current of the same form when the angle of incidence is 45° and the dimensionless current strength is $\beta = 0 \cdot 2$. As expected, the smaller ϵ is, the greater the difference between the two solutions, but both predict a minimum in the reflection coefficient around $H = 1$. That $R \rightarrow 1$ as $H \rightarrow 0$ is a consequence of the scaling used. As the depth tends to zero, the phase speed of the waves also tends to zero, so any finite following current will be effectively infinitely fast as far as the waves are concerned, leading to total reflection.

5. Discussion

The mild-shear equation of McKee (1987) has been extended to apply to situations in which the waves are not necessarily short compared with the length scale over which the current changes. As shown by (14), this new method incorporates explicit information about the first and second derivatives of the current, in contrast to the mild-shear equation. Probably the most noteworthy feature of the results is the pronounced minimum in the reflection coefficient as a function of the depth.

Acknowledgments. Much of this work was performed at the Department of Mathematics, Monash University while the author was on study leave from the University of New South Wales. He thanks both institutions for making this possible and James Stynes for setting him an inspiring example for many years.

APPENDIX

Finite Depth Case

In this appendix, we present the form of $\mathcal{R}(x)$ in water of finite depth:

$$\mathcal{R}(x) = -C_1(I_1 - I_2 H \tanh kH) - C_2 k'(I_3 - H^2 I_2),$$

where the prime denotes differentiation with respect to x. In these expressions,

$$I_1 = \frac{2kH \cosh 2kH - \sinh 2kH}{8k^2}$$

$$I_2 = \frac{2kH + \sinh 2kH}{4k}$$

$$I_3 = \{3[1 + 2(kH)^2] \sinh 2kH + 2kH[2(kH)^2 - 3 \cosh 2kH]\}/24k^3$$

and

$$C_1 = 2B^{-2}\Omega^{-2}(-nB\Omega W'' - n^2 B(W')^2 + n\Omega W'B')$$

$$C_2 = -2n\Omega^{-1}W'B^{-1},$$

where

$$B = kH + \frac{1}{2}\sinh 2kH$$

and so

$$B' = 2Hk' \cosh^2 kH,$$

where

$$k' = \frac{-2n\Omega W'}{\tanh kH + kH \operatorname{sech}^2 kH}.$$

REFERENCES

Evans, D. V., 1975: The transmission of deep-water waves across a vortex sheet. *J. Fluid Mech.*, **68**, 389–401.
Massel, S. T., 1993: Extended refraction–diffraction equation for surface waves. *Coastal Eng.*, **19**, 97–126.
McKee, W. D., 1987: Water wave propagation across a shearing current. *Wave Motion*, **9**, 209–215.
——, and F. Tesoriero, 1987: Reflection of water waves from a vertical vortex sheet in water of finite depth. *J. Aust. Math. Soc. Ser. B*, **29**, 127–141.
Mei, C. C., 1983: *The Applied Dynamics of Ocean Surface Waves.* Wiley, 740 pp.
Peregrine, D. H., 1976: Interaction of waves and currents. *Adv. Appl. Mech.*, **16**, 9–17.
Smith, J., 1983: On surface gravity waves crossing weak current jets. *J. Fluid Mech.*, **134**, 277–299.