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The Number of Cycles in a Connected Graph

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Abstract The numbers of vertices, edges and cycles of a graph G are denoted by $|V(G)|$, $|E(G)|$ and $f(G)$, respectively. Let $F(k) = \{f(G); G$ is a connected graph without loops such that $|E(G)| - |V(G)| = k\}$, $n(k) = \min F(k)$ and $N(k) = \max F(k)$. In this paper, the following results are proved:

(1) $n(k) = k + 1$;

(2) $N(k) \leqslant 2^{k+1} - 1$;

(3) For every integer $k \geqslant 1$, $N(k) \geqslant 2^k + k(k-1) + 1$ and the equality holds when $1 \leqslant k \leqslant 4$;

(4) For every integer $k \geqslant 1$,

$$N(k) \geqslant 2^k + 2^{\frac{k+3}{2}} + (k-1)(2^{\frac{k-1}{2}} + \frac{k-1}{8} - \frac{7}{4}) - 3$$

if k is odd and $N(k) \geqslant 2^k + k(2^{\frac{k}{2}} + \frac{k}{8} - \frac{7}{4}) + 1$ if k is even.

Key words connected graph; cycle; number of cycles

1 Introduction

The numbers of vertices, edges and cycles of a graph G are denoted by $|V(G)|$, $|E(G)|$ and $f(G)$, respectively. Let $F'(k) = \{f(G); G$ is a Hamilton graph without loops such that $|E(G)| - |V(G)| = k\}$, $m(k) = \min F'(k)$ and $M(k) = \max F'(k)$.

In 1983, Yap and Teo raised the following questions[2]:

(1) Is it true that $m(k) = \frac{(k+1)(k+2)}{2}$?

(2) Is it true that $M(k) = 2^k + k$?

We have answered the first question in the affirmative and the second question in the negative. Some results on upper and lower bounds of $M(k)$ have been obtained in [3].

In this paper we extend the questions of Yap and Teo to connected graphs without loops.

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Let $F(k)=\{f(G);G$ is a connected graph without loops such that $|E(G)|-|V(G)|=k\}$, $n(k)=\min F(k)$ and $N(k)=\max F(k)$. The main results are stated in the abstract above.

In this paper, we consider finite undirected connected graphs without loops. All notations and definitions not given here can be found in [1].

2 Determining $n(k)$

Let G be any connected graph without loops such that $|E(G)|-|V(G)|=k$ and T be a spanning tree of G. Let $A=E(G)-E(T)$. Then $|A|=k+1$. For $S\subseteq A$, $S\neq\varnothing$, let C_S' be the set of cycles containing all edges in S and containing no edges in $A-S$. The number of cycles in C_S' is denoted by $g'(S)$. Cleary $g'(S)=1$ if $S\subseteq A$ and $|S|=1$. Thus

$$f(G)=\sum_{S\subseteq A}g'(S)\geqslant\sum_{S\subseteq A,\ |S|=1}g'(S)=k+1,$$

and hence

$$n(k)\geqslant k+1. \tag{1}$$

Let G^* be a connected graph without loops such that it has $k+1$ blocks each of which is a 2-cycles. Clearly G^* is such that $|E(G^*)|-|V(G^*)|=k$ and G^* has exactly $k+1$ cycles. By definition of $n(k)$, we obtain that

$$n(k)\leqslant k+1. \tag{2}$$

Combining (1) and (2), we obtain immediately the following

Theorem 2.1 $n(k)=k+1$.

3 On upper bounds of $N(k)$

We first prove the following important lemma.

Lemma 3.1 For any $S\subseteq A$, $g'(S)\leqslant 1$.

Proof Consider the edge-induced subgraph $G[S]$ of G induced by S. If there exists a vertex v in $G[S]$ such that $d_{G[S]}(v)\geqslant 3$, then clearly $g'(S)=0$; If $G[S]$ contains one cycle of length less than $|S|$, then $g'(S)=0$; If $G[S]$ is a cycle, then $g'(S)=1$. Therefore we may assume that $G[S]$ contains no cycle and $d_{G[S]}(v)\leqslant 2$ for all $v\in V(G[S])$. Thus each component of $G[S]$ is a path of length at least 1. Let ω be the number of components of $G[s]$ and $v_1,v_2,\cdots,v_{2\omega}$ be the end-vertices of these components (i.e. paths). Suppose that $|C_S'|\geqslant 2$. Let C_1 and C_2 be two cycles in C_S'. For $i=1,2$, let $G_i=C_i-S$ and G_i^* denote the graph obtained from G_i by deleting all isolated vertices. Clearly for $i=1,2$, G_i^* has exactly ω components each of which is a path of length at least 1. It is easily seen that the set of end-vertices of these paths is $Q=\{v_1,v_2,\cdots,v_{2\omega}\}$, and we have the following fact: If P is a component (i.e. path) of G_1^* and the end-vertices of P are v_{i_1} and v_{i_2}, then clearly there is exactly one component (i.e. path) in G_2^* such that its end-vertices are also v_{i_1} and v_{i_2}.

Let G^* denote the set of common components of G_1^* and G_2^*, and let $H_i=G_i^*-G^*$, $i=$

1,2. Clearly each component of $H_i (i = 1,2)$ is a path and the number of components of H_1 is equal to that of H_2. Also the set of end-vertices of components of H_1 is equal to that of H_2 and is denoted by Q'. It is easily seen that $Q' \subseteq Q$. Let $Q' = \{v'_1, v'_2, \cdots, v'_r\}$, $H' = H_1 \cup H_2$. Clearly for each $i \in \{1,2,\cdots,r\}$, H' has exactly two paths P_{1i} and P_{2i} such that $P_{1i} \in H_1, P_{2i} \in H_2$ and $P_{1i} \cap P_{2i} = \{v'_i\}$. Let v'_i be the origin of P_{1i} and P_{2i}, and let u_i be the last vertex common to P_{1i} and P_{2i}. (we do not exclude the possibility that $u_i = v_i$). Since $H' \subseteq T$, the (v'_i, u_i)-section of P_{1i} and the (v'_i, u_i)-section of P_{2i} must be the same path. Let P_i denote the (v'_i, u_i)-path. Since any two paths in G_i^* are vertex-disjoint, P_i and any path in $H' - \{P_{1i}, P_{2i}\}$ are also vertex-disjoint.

Let $H^* = H' - \bigcup_{i=1}^{r} E(P_i)$ and H denote the graph obtained from H^* by deleting all isolated vertices. Thus $d_H(v) = 2$ for all $v \in V(H)$, and hence H contain a cycle. Since $H \subseteq T$, T contains a cycle, a contradiction. Therefore $|C_S'| \leqslant 1$, i.e, $g'(S) \leqslant 1$.

We now prove the following

Theorem 3.2 $N(k) \leqslant 2^{k+1} - 1$.

Proof Let G be any connected graph without loops such that $|E(G)| - |V(G)| = k$ and T be a spanning tree of G, then $|A| = k + 1$. Using Lemma 3.1, we have

$$f(G) = \sum_{S \subseteq A} g'(S) = \sum_{i=1}^{k+1} \sum_{S \subseteq A,\ |S|=i} g'(S) \leqslant$$

$$\binom{k+1}{1} + \binom{k+1}{2} + \cdots + \binom{k+1}{k+1} = 2^{k+1} - 1,$$

and hence $N(k) \leqslant 2^{k+1} - 1$.

4 On lower bounds of $N(k)$

To obtain lower bounds of $N(k)$, we first form three Hamilton graphs without loops such that $|E(G)| - |V(G)| = k$ and then count the number of cycles in each of them.

Each of the three graphs is formed from a Hamilton cycle C by adding k edges joining k pairs of distinet vertices of C. We assume that the k edges of G are drawn in the interior of C and we call these edges the bridges of G.

Let $B = E(G) - E(C)$. Then B is the set of bridges of G. For $S \subseteq B$, let C_S be the set of cycles containing all bridges in S and containing no bridges in $B - S$. The number of cycles in C_S is denoted by $g(S)$. We define $C_\varnothing = \{C\}$, and hence $g(\varnothing) = 1$.

Two bridges b_1 and b_2 are said to be skew if there are four distinct vertices u, v, u' and v' such that u and v are end-vertives of b_1, u' and v' are end-vertices of b'_2 and the four vertices appear in the cyclic order u, u', v, v' on C.

Two bridges are said to be parallel if they are not skew.

Two skew bredges b_1 and b_2 of S are said to be a pair of companionate skew bridges (CSBs) of S if there are four distinct vertices u, v, u' and v' such that $b_1 = uv$, $b_2 = u'v'$ and the four vertices appear in the cyclic order u, u', v, v' on C, and there is no bridge b in $S - \{b_1, b_2\}$ such that one of its end-vertices lies on $C[u, u'] \cup C[v, v']$ or $C[u', v] \cup C[v', u]$.

Two parallel bridges b_1 and b_2 of S are said to be a pair of companionate parallel bridges (CPBs) of S if there are four distinct vertices u, u', v', v such that $b_1 = uv$, $b_2 = u'v'$ and the four vertices appear in the cyclic order u, u', v', v on C, and there is no bridge b in $S - \{b_1, b_2\}$ such that one of its end-vertices lies on $C[u, u'] \cup C[v', v]$.

We now form three Hamilton graphs denoted by H_1, H_2 and H_3 (See Fig 1), respectively. The Hamilton cycle of H_i is denoted by $C(H_i)$ and the set of bridges of H_i is denoted by $B(H_i)$ for $i = 1, 2, 3$. The three graphs are such that

$$C(H_1) = a_1 a_2 \cdots a_{2k} a_1,$$
$$B(H_1) = \{a_i a_{k+i} \mid i = 1, 2, \cdots, k\};$$
$$C(H_2) = a_1 b_1 a_2 b_2 \cdots a_{2m} b_{2m} a_1,$$
$$B(H_2) = \{a_i b_{m+i}, b_i a_{m+i} \mid i = 1, 2, \cdots, m\}, \text{ where } k = 2m;$$
$$C(H_3) = a_1 b_1 a_2 b_2 \cdots a_m b_m u a_{m+1} b_{m+1} \cdots a_{2m} b_{2m} v a_1,$$
$$B(H_3) = \{a_i b_{m+i}, b_i a_{m+i} \mid i = 1, 2, \cdots, m\} \cup \{uv\}, \text{ where } k = 2m + 1.$$

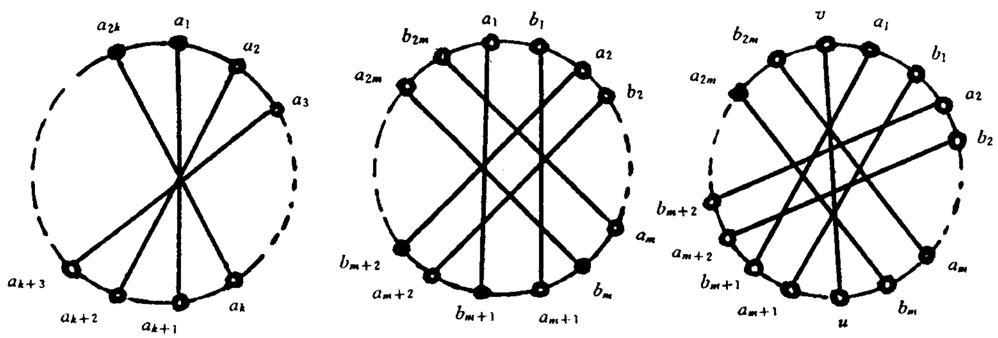

Fig 1 Three Hamilton graphs H_1, H_2 and H_3

To count the number $f(H_i)$ of cycles in H_i for $i = 1, 2, 3$, we first prove the following four lemmas. In the coming discussion, let $B \in \{B(H_1), B(H_2), B(H_3)\}$ and let $C[v_i, v_j]$ denote the (v_i, v_j)-path which follows the clockwise orientation of C.

Lemma 4. 1 If $S \subseteq B$ and $|S|$ is odd, then $g(S) = 2$.

Proof Let $|S| = t$ and $|S|$ contains exactly r pairs of CPBs. By induction on r and t, for $r = 0$ and $t = 1$, the Lemma clearly holds. Let $r = 0$ and $t \geqslant 3$, then the t bridges in S have $2t$ end vertices which appear in the clockwise order $v_1, v_2, \cdots, v_{2t} = v_0$ on C (See Fig 2).

Let $P_i = C[v_i, v_{i+1}]$, $i = 0, 1, \cdots, 2t - 1$. If $C_S \neq \varnothing$ and $C' \in C_S$, then C' must contain exactly one of two consecutive paths P_i and P_{i+1} for $i = 0, 1, \cdots, 2t - 1$. (For otherwise either C' does not contain bridges which are incident with v_{i+1} or C' contains the vertes v_{i+1} which is incident with at least three edges of C', a contradiction). It follows that either $\bigcup_{i=1}^{t} P_{2i-1} \subset C'$ or $\bigcup_{i=0}^{t-1} P_{2i} \subset C'$. Let $G_1 = S \cup (\bigcup_{i=1}^{t} P_{2i-1})$ and $G_2 = S \cup (\bigcup_{i=0}^{t-1} P_{2i})$ be two subgraphs of G, then clearly

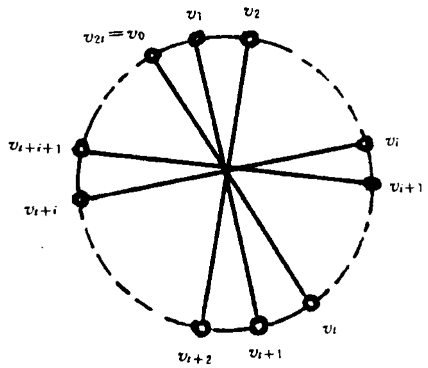

Fig 2 $r=0$

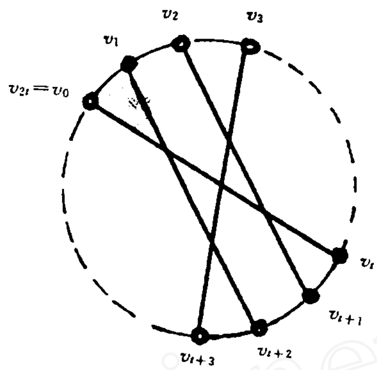

Fig 3 $r \neq 0$

C' can only be either G_1 (if G_1 is a cycle) or G_2 (if G_2 is a cycle), and hence $g(S) = |C_S| \leqslant 2$. If $t = 3$, then clearly both G_1 and G_2 are cycles, and hence $C_S = \{G_1, G_2\}$, i. e., $g(S) = 2$. Assume that the lemma holds for $r = 0$ and $t = j$, i. e., $g(S) = 2$. Consider $r = 0$ and $t = j + 2$. Let $b_1 = v_1v_{t+1}$, $b_2 = v_2v_{t+2}$ and $S^* = S - \{b_1, b_2\}$. By the induction hypothesis, $g(S^*) = 2$, i. e., $C_{S^*} = \{G_1^*, G_2^*\}$, where

$$G_1^* = S^* \cup \Big(\bigcup_{i=2}^{\frac{t-1}{2}} P_{2i-1}\Big) \cup C[v_t, v_{t+3}] \cup \Big(\bigcup_{i=\frac{t+5}{2}} P_{2i-1}\Big)$$

and

$$G_2^* = S^* \cup C[v_{2t}, v_3] \cup \Big(\bigcup_{i=2}^{\frac{t-1}{2}} P_{2i}\Big) \cup \Big(\bigcup_{i=\frac{t+3}{2}}^{t-1} P_{2i}\Big).$$

Clearly $C[v_1, v_2] \subset C[v_{2t}, v_3] \subset G_2^*$, $C[V_{t+1}, v_{t+2}] \subset C[v_t, v_{t+3}] \subset G_1^*$.

Replacing by the path $C[v_{2t}, v_1] \cup v_1v_{t+1} \cup C[v_{t+1}, v_{t+2}] \cup v_{t+2}v_2 \cup C[v_2, v_3]$ the path $C[v_{2t}, v_3]$ on G_2^* results in a cycle G_2 containing all bridges in S and containing no bridges in $B - S$. Similary, replacing by the path $C[v_t, v_{t+1}] \cup v_{t+1}v_1 \cup C[v_1, v_2] \cup v_2v_{t+2} \cup C[v_{t+2}, v_{t+3}]$ the path $C[v_t, v_{t+3}]$ on G_1^* results in a cycle G_1 containing all bridges in S and containing no bridges in $B - S$. Hence $g(S) = 2$. Therefore the lemma holds for $r = 0$ and $t \geqslant 1$.

Assume that the lemma holds for $r = j$. Consider $r = j + 1$. Let $b_1 = v_1v_{t+2}$ and $b_2 = v_2v_{t+1}$ be a pair of CPBs (See Fig 3). Let $S^* = S - \{b_1, b_2\}$. By the induction hypothesis, $g(S^*) = 2$, i. e., $C_{S^*} = \{G_1^*, G_2^*\}$; where G_i^* $(i = 1, 2)$ is similarly formed as above and

$$C[v_1, v_2] \subset C[v_{2t}, v_3] \subset G_2^*, \quad C[v_{t+1}, v_{t+2}] \subset C[v_t, v_{t+3}] \subset G_1^*.$$

Replacing by the path $C[v_{2t}, v_1] \cup v_1v_{t+2} \cup C[v_{t+1}, v_{t+2}] \cup v_{t+1}v_2 \cup C[v_2, v_3]$ the path $C[v_{2t}, v_3]$ on G_2^* results in a cycle G_2 containing all bridges in S and containing no bridges in $B - S$. Similarly, replacing by the path $C[v_t, v_{t+1}] \cup v_{t+1}v_2 \cup C[v_1, v_2] \cup v_1v_{t+2} \cup C[v_{t+2}, v_{t+3}]$ the path $C[v_t, v_{t+3}]$ on G_1^* results in a cycle G_1 containing all bridges in S and containing no bridges in $B - S$. Therefore $g(S) = 2$. The lemma follows by the principle of induction. □

Lemma 4.2 Let $S \subseteq B$ and $|S| = t \geqslant 4$. If t is even and S contains exactly $\frac{t}{2} - 1$ pairs of CPBs, then $g(S) = 1$ if S contains a pair of CSBs and $g(S) = 0$ otherwise.

Proof We first suppose that S contains a pair of CSBs and prove $g(S)=1$ by induction on t. If $t=4$, then let u_1u_2 and w_1w_2 be the pair of CSBs of S and let v_1v_4 and v_2v_3 be the pair of CPBs (See Fig 4). Clearly

$G_1 = C[w_1,v_1] \cup v_1v_4 \cup C[v_4,u_1] \cup u_1u_2 \cup C[v_2,u_2] \cup v_2v_3 \cup C[w_2,v_3] \cup w_2w_1$

is a cycle and

$$G_2 = S \cup C[u_1,w_1] \cup C[v_1,v_2] \cup C[u_2,w_2] \cup C[v_3,v_4]$$

is not a cycle. Thus $g(S)=1$ and $C_S=\{G_1\}$. Assume that the lemma holds for $t=j$ ($j \geqslant 4$ is even). Consider $t=j+2$. Let $b=v_0v_j$ and $b'=v_{2j-1}v_{j-1}$ be the pair of CSBs of S and let $b_1=u_1w_1$ and $b_2=u_2w_2$, $b_3=v_1v_{j+2}$ and $b_4=v_2v_{j+1}$ be two pairs of CPBs. The end-vertices of these bridges appear in the clockwise order v_{2j-1}, v_0, u_1, u_2, v_1, v_2, $\cdots$, v_{j-1}, v_j, w_2, w_1, v_{j+1}, v_{j+2}, $\cdots$ on C (See Fig 5).

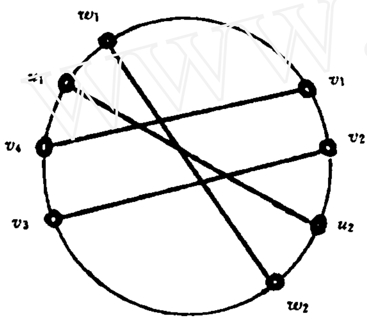

Fig 4 S contains a pair of CSBs and $t=4$

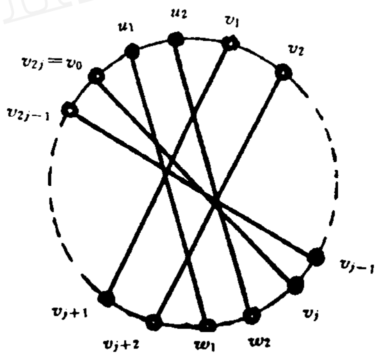

Fig 5 S contains a pair of CSBs and $t>4$

Let $P_i=C[v_i,v_{i+1}]$, $i=0,1,\cdots,2j-1$. Clearly $C[u_1,u_2]\subset P_0$, $C[w_2,w_1]\subset P_j$. Let $S^*=S-\{b_1,b_2\}$. *By the induction hypothesis*, $g(S^*)=1$. *Let* $G_1^*=S^*\cup(\bigcup_{i=0}^{j-1}P_{2i})$. Then clearly G_1^* is a cycle and hence $C_{S^*}=\{G_1^*\}$. Replacing by

$$C[u_2,v_1]\cup u_2w_2\cup C[v_j,w_2]\cup v_jv_0\cup C[v_0,u_1]\cup u_1w_1\cup C[w_1,v_{j+1}]$$

the path $C[v_0,v_1]\cup v_0v_j\cup C[v_j,v_{j+1}$ on G_1^* results in a cycle. G_1 containing all bridges in S and containing no bridge in $B-S$. Clearly $C_S=\{G_1\}$ and hence $g(S)=1$.

We now suppose that S does not contain a pair of CSBs. Form G_1 and G_2 by the similar method of Lemma 4.1 and by induction on t. It is easily seen that neither G_1 nor G_2 is a cycle, and hence $g(S)=0$. □

Lemma 4.3 Let $S\subseteq B$, $|S|=t$ and let S contain exactly r pairs of CPBs, then

$$g(S)=\begin{cases}2, & \text{if } r=0 \text{ and } t=2;\\ 1, & \text{if } r=1,\ t=2 \text{ or } r=2,\ t=4;\\ 0, & \text{if } r=0 \text{ and } t=4.\end{cases}$$

Proof This follows from a simple observation. □

Lemma 4.4 Let $S\subseteq B$, $|S|=t\geqslant 6$. If t is even and S contains exactly $r\neq \frac{t}{2}-1$ pairs of CPBs, then $g(S) = 0$.

Proof　Form G_1 and G_2 by the similar method of Lemma 4.1 and by induction on t. It is easily proved that neither G_1 nor G_2 is a cycle, and hence $g(S) = 0$. □

We now count the number $f(H_i)$ of cycles in H_i for $i = 1, 2, 3$.

Proposition 4.5　$f(H_1) = 2^k + k(k-1) + 1$.

Proof　By Lemmas 4.1～4.2, for any $S \subseteq B$, $g(S) = 0$ if $|S| \geqslant 4$ is even and $g(S) = 2$ otherwise. It follows that

$$f(H_1) = \sum_{S\subseteq B} g(S) = \sum_{S\subseteq B,\ |S| \text{ is odd}} g(S) + \sum_{S\subseteq B,\ |S|=2} g(S) + g(\varnothing) =$$

$$2\binom{k}{1} + 2\binom{k}{3} + \cdots + 2\binom{k}{2[\frac{k+1}{2}]-1} + 2\binom{k}{2} + 1 =$$

$$2^k + k(k-1) + 1.$$ □

Proposition 4.6　$f(H_2) = 2^k + k(2^{\frac{k}{2}} + \frac{k}{8} - \frac{7}{4}) + 1$.

Proof　By Lemma 4.1, $\sum_{S\subseteq B,\ |S| \text{ is odd}} g(S) = 2\binom{k}{1} + 2\binom{k}{3} + \cdots + 2\binom{k}{k-1} = 2^k$. To count $\sum_{S\subseteq B,\ |S|=2} g(S)$, we consider two cases.

Case 1　S consists of a pair of CSBs. In this case the two bridges in such S may be obtained as follows: we first take any two pairs from m pairs of CPBs and then take any one bridge from each of the two chosen pairs of CPBs. Clearly there are exactly $4\binom{m}{2}$ such S's.

Case 2　S consists of a pair of CPBs. Clearly there are exactly m such S's. By Lemma 4.3, we obtain

$$\sum_{S\subseteq B,\ |S|=2} g(S) = 2 \times 4\binom{m}{2} + m = 4m\binom{m-1}{1} + m.$$

To count $\sum_{S\subseteq B,\ |S|=4} g(S)$, by Lemmas 4.2 and 4.3, we consider only the following two cases.

Case 1　S consists of one pair of CPBs and one pair of CSBs. In this case, such S may be obtained as follows: We first take any three pairs from m pairs of CPBs and take any two pairs from the three chosen pairs of CPBs, from each of which we take any one bridge. Thus S consists of the two chosen bridges and the remaining pair of CPBs that we have chosen. Since $\binom{m}{3} \times \binom{3}{2} \times 4 = 4m\binom{m-1}{2}$, there are exactly $4m\binom{m-1}{2}$ such S's.

Case 2　S consists of two pairs of CPBs. Clearly there are exactly $\binom{m}{2}$ such S's.

Using Lemmas 4.2 and 4.3, we have

$$\sum_{S\subseteq B,\ |S|=4} g(S) = 4m\binom{m-1}{2} + \binom{m}{2}.$$

To count $\sum_{S\subseteq B,\ |S|=2i} g(S)$ for $i = 3, 4, \cdots, m-1$, by Lemmas 4.3 and 4.4, we only need to count the number of subsets of B each of which consists of exactly one pair of CSBs and $(i-1)$ pairs of CPBs. Such subset S is obtained as follows: we first take any $i+1$ pairs from m pairs of CPBs and then take any two consecutive pairs from the $i+1$ chosen pairs of CPBs, from each of which we take any one bridge. Thus S consists of two chosen bridges and the $(i-1)$ remaining pairs of CPBs that we have chosen. Since $\binom{m}{i+1}\times\binom{i+1}{1}\times 4 = 4m\binom{m-1}{i}$, there are exactly $4m\binom{m-1}{i}$ such S's. Thus

$$\sum_{S\subseteq B,\ |S| \text{ is even}} g(S) = 4m\left(\binom{m-1}{1}+\binom{m-1}{2}+\cdots+\binom{m-1}{m-1}\right)+m+\binom{m}{2}+1 =$$

$$4m(2^{m-1}-1)+m+\frac{1}{2}m(m-1)+1 =$$

$$m(2^{m+1}+\frac{m}{2}-\frac{7}{2})+1 = k(2^{\frac{k}{2}}+\frac{k}{8}-\frac{7}{4})+1.$$

Therefore
$$f(H_2) = 2^k + k(2^{\frac{k}{2}}+\frac{k}{8}-\frac{7}{4})+1.$$

Proposition 4.7 $f(H_3) = 2^k + 2^{\frac{k+3}{2}} + (k-1)(2^{\frac{k-1}{2}}+\frac{k-1}{8}-\frac{7}{4})-3.$

Proof Let $b = uv$. A simlilar discussion to that for proposition 4.6 yields that

$$\sum_{S\subseteq B,\ |S| \text{ is odd}} g(S) = 2^k,$$

$$\sum_{S\subseteq B,\ |S| \text{ is even}} g(S) = m(2^{m+1}+\frac{m}{2}-\frac{7}{2})+1 = (k-1)(2^{\frac{k-1}{2}}+\frac{k-1}{8}-\frac{7}{2})+1.$$

By Lemma 4.3, it is easily seen that

$$\sum_{S\subseteq B,\ b\in S,\ |S|=2i} g(S) = 4\times\binom{m}{1}.$$

To count $\sum_{S\subseteq B,\ b\in S,\ |S|=2i} g(S)$ for $i=2,3,\cdots,m$ by Lemmas 4.2, 4.3 and 4.4, we only need to count the number of subsets of B each of which consists of one pair of CSBs containing the bridge b and $(i-1)$ pairs of CPBs. Such subset is denoted by S. We now take any i pairs from m pairs of CPBs. We may assume that the end-vertices of the chosen $2i$ bridges and $b=uv$ appear in the clockwise order $v, a_{j_1}, b_{j_1}, \cdots, a_{j_i}, b_{j_i}, u, a_{m+j_1}, b_{m+j_1}, \cdots, a_{m+j_i}, b_{m+j_i}$ on C. Let S' denote the set consisting of the $2i+1$ bridges. If $S\subseteq S'$, then clearly S must be one of the following four sets:

$$S'-\{a_{j_1}b_{m+j_1}\},\ S'-\{b_{j_1}a_{m+j_1}\},\ S'-\{a_{j_i}b_{m+j_i}\},\ S'-\{b_{j_i}a_{m+j_i}\}.$$

Thus there are exactly $4\binom{m}{i}$ such S's, and hence

$$\sum_{S\subseteq B,\ b\in S,\ |S|\geqslant 2 \text{ is even}} g(S) = 4\left(\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{m}\right) =$$

$$4(2^m-1) = 2^{m+2}-4 = 2^{\frac{k+3}{2}}-4.$$

Therefore

$$f(H_3) = 2^k + 2^{\frac{k+3}{2}} + (k-1)(2^{\frac{k-1}{2}}+\frac{k-1}{8}-\frac{7}{4})-3. \qquad \square$$

Theorem 4.8 For each integer $k \geqslant 1$, $N(k) \geqslant 2^k + k(k-1) + 1$.

Proof This follows directly from Proposition 4.5 and the definition of $N(k)$.

Theorem 4.9 For each integer $k \geqslant 1$,

$$N(k) \geqslant 2^k + 2^{\frac{k+3}{2}} + (k-1)(2^{\frac{k-1}{2}} + \frac{k-1}{8} - \frac{7}{4}) - 3,$$

if k is odd and $N(k) \geqslant 2^k + k(2^{\frac{k}{2}} + \frac{k}{8} - \frac{7}{4}) + 1$ if k is even.

Proof This follows directly from Propositions 4.6 and 4.7. □

It is easy to verify that $2^{k+1} - 1 = 2^k + k(k-1) + 1$ for $k = 1, 2, 3$. Furthermore we have proved that $N(k) = 2^k + k(k-1) + 1$ for $k = 4$ (This proof is not difficult and therefore is left for the reader). Thus the equality in Theorem 4.8 holds for $k = 1, 2, 3, 4$.

It is also easy to verify that the lower bound in Theorem 4.9 is better than that in Theorem 4.8 for each integer $k \geqslant 5$.

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连通图中圈数

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提　要 用 $|V(G)|$、$|E(G)|$ 和 $f(G)$ 分别表示图 G 的顶点数、边数和圈数. 设 $F(k) = \{f(G); G$ 是满足 $|E(G)| - |V(G)| = k$ 的无环连通图$\}$, $n(k) = \min F(k)$ 和 $N(k) = \max F(k)$. 证明了下述结果:

(1) $n(k) = k + 1$;

(2) $N(k) \leqslant 2^{k+1} - 1$;

(3) 对每个整数 $k \geqslant 1$, $N(k) \geqslant 2^k + k(k-1) + 1$ 且当 $1 \leqslant k \leqslant 4$ 时等式成立;

(4) 对每个整数 $k \geqslant 1$ 是奇数时, $N(k) \geqslant 2^k + 2^{\frac{k+3}{2}} + (k-1)(2^{\frac{k-1}{2}} + \frac{k-1}{8} - \frac{7}{4}) - 3$; 当 $k \geqslant 2$ 是偶数时, $N(k) \geqslant 2^k + k(2^{\frac{k}{2}} + \frac{k}{8} - \frac{7}{4}) + 1$.

关键词 连通图; 圈; 圈数

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