A New Coastal Wave Model. Part V: Five-Wave Interactions

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ABSTRACT

The authors study the action flux associated with three-dimensional wave–wave interactions of ocean surface waves. Over deep water, two-dimensional wave–wave interactions are dominant: the three-dimensional five-wave interactions are two orders of magnitude smaller than the two-dimensional four-wave interactions. However, the five-wave interactions become increasingly important as the water depth decreases. Because of the effects of finite depth, three-dimensional five-wave interactions, involving steep finite-amplitude waves, dominate over two-dimensional four-wave interactions. Specifically, when the water depth h is less than 10 m, or nondimensionalizing with the spectral peak wavenumber K_p when $K_p h \leq 3.6$ and nonlinearity, $\epsilon = Ka(3 + \tanh^2 Kh)/4 \tanh^3 Kh \geq 0.3$, the five-wave interactions completely dominate. Results are consistent with the instability study by McLean.

1. Introduction

The study of weak nonlinear wave–wave interactions was initiated by Phillips (1960) in a seminal paper showing that four gravity waves in deep water could interact resonantly. His idea was extended theoretically by Hasselmann (1962) to the spectral formulation for a random wave field and confirmed experimentally by Longuet-Higgins and Smith (1966) and McGoldrick et al. (1966). As a result, resonant wave–wave interactions have become the centerpiece of modern wave dynamics theory and applied wave prediction modeling.

Both Phillips (1960) and Hasselmann (1962) used perturbation analysis to investigate wave–wave interactions. In order to simulate the wave energy evolution to the third order, the perturbation expansions had to be carried to the fifth order, as shown by Hasselmann (1962). This high order of perturbation expansion severely limited the range of validity of the expansion. To alleviate this limitation, Zakharov (1968, 1991) proposed a Hamiltonian approach, which resulted in an expression for the nonlinear wave–wave interactions similar to that of Hasselmann (1962), although simpler, with a higher degree of symmetry and a wider range of validity. In fact, Crawford et al. (1981) showed quantitatively that, for deep water waves, the discrepancy between exact solutions, as obtained by Longuet-Higgins (1978), and the Zakharov (1968) expression, is within 10% for a wave steepness of 0.3. The Hasselmann (1962) expansion would give a similar discrepancy for a wave steepness of only 0.06. However, there were shortcomings in the Hamiltonian structure derived by Zakharov (1968) and Crawford et al. (1980). These were resolved by Krasitskii (1994), who used canonical transformations to generalize the continuous case from discrete mechanics, automatically leading to reduced equations with Hamiltonian structure and with nonresonant terms eliminated.

Four-wave interactions are primarily unidirectional. That is, the nonlinear transfer is confined mostly to the direction of the mean wave propagation. However, as demonstrated by Martin and Yuen (1980), the interaction equation allows leakage of energy to higher modes outside its range of validity. It thus becomes inconsistent. Moreover, Crawford et al. (1981) pointed out that if one compares the experimental data (Lake et al. 1977) and theory, one can see the need for a higher-order approximation to study finite-amplitude wave–wave in-

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teractions in deep water. They also pointed out that instability, for a three-dimensional disturbance on a uniform wave train, does not possess a high wavenumber cutoff. Finally, in addition to four-wave interactions, Su (1982a,b) showed observationally that there are three-dimensional instabilities in deep water wave trains.

Inspired by Su's (1982a,b) observations, McLean (1982) studied the instabilities of finite-amplitude gravity waves in both deep water and water of finite depth. He used a global method to study the instabilities, with variables extending over a set of orthogonal functions, which include the higher-order modes. McLean's (1982) deep water results showed that the three-dimensional instabilities were a consequence of five-wave interactions. In shallow water, he found that when the wave steepness Ka exceeded 0.29, the growth rate for the five-wave interactions was higher than that of four-wave interactions. The existence of three-dimensional instabilities, dominating over four-wave interactions in shallow water, is consistent with field observations by Su et al. (1982a,b) for the formation of three-dimensional crescent-shaped breakers under growing seas.

The study by McLean (1982) represents a pioneering investigation in water wave instability. However, it gives only a qualitative understanding of nonlinear, steep, finite-amplitude wave–wave interactions. A quantitative understanding of these interactions requires an investigation of higher modes. Recently Dyachenko et al. (1995) conducted an analytic study of the five-wave interactions confined to one direction over deep water. They found that there was a tendency for wave energy to undergo wide angular spreading with respect to the wave spectral propagation direction.

The primary goal of this paper is to evaluate the nonlinear spectral transfer due to five-wave interactions. We are concerned with angular wave energy spreading in deep and shallow water. We give the explicit formulas for three-dimensional five-wave interactions. We also numerically compute the nonlinear transfer due to five-wave interactions and compare it with the corresponding four-wave interactions. Our results indicate that the five-wave interactions contribute strongly in shallow water, enhancing the angular spreading of spectral energy, particularly when the wave amplitude is large.

The ultimate goal of this paper is to investigate new mechanics: the three-dimensional wave–wave interactions. However, in the course of this investigation, when the five-wave interactions are shown to be important, one also needs to consider six-wave, seven-wave, . . . interactions to obtain the completely accurate solution. Unfortunately, there is no analytical method that can include all this series of interactions. Therefore, for a qualitative analytical study of three-dimensional wave–wave interactions, we only study five-wave interactions. In the next paper, we will use a numerical global method that will include all the four-wave, five-wave, six-wave, . . . interactions. However, the disadvantage of this numerical method is that one cannot separate each wave–wave interaction. One only can obtain the sum total. It is therefore difficult to use this numerical solution to discuss the new mechanics, for example, from the role of four-wave and five-wave interactions individually.

Compared to Krasitskii (1994), we use the so-called classical method in this study, following Zauderer (1983), Lin (1990), and Lin et al. (1988). This involves the application of a solvability condition and an adjoint operator, mapping the n-dimensional system into an $(n - 1)$-dimensional system in order to eliminate nonresonant terms. We use the classical method because the adjoint operator is easy to find and avoids truncation errors. Thus, the procedure may be simpler than that of Krasitskii (1994). Of course, our final results should be equivalent to those obtained by Krasitskii (1994).

2. Mathematical model

The basic equations, boundary conditions and the perturbation analysis are presented in this section. This approach follows similar analyses in Lin and Perrie (1997).

a. Basic equations

1) CONTINUITY EQUATION

$$\nabla_H^2\Phi + \frac{\partial^2\Phi}{\partial Z^2} = 0, \quad \text{for } -h \le Z \le \eta, \tag{1}$$

where

$$\boldsymbol{\nabla}_H = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

2) BOUNDARY CONDITIONS

(i) At the free surface: $Z = \eta$

a) Kinematics:

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}_H\Phi\cdot\boldsymbol{\nabla}_H\eta = \frac{\partial\Phi}{\partial Z}, \tag{2}$$

b) Dynamics:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}_H\Phi)^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial Z}\right)^2 + g\eta = 0. \tag{3}$$

(ii) At the bottom: $Z = -h$

$$\frac{\partial\Phi}{\partial Z} = 0. \tag{4}$$

Variables denoted as t, H, h, g, η, and Φ are time, the horizontal coordinate, the local depth of the ocean, the gravity acceleration, the free surface elevation, and the potential function.

b. Small perturbation analysis

Expanding all functions as power series in ϵ, which is a small parameter, we obtain

$$\Phi = \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots$$
$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots$$
$$\omega = \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \quad (5a)$$

where ω is the frequency, and the upper subscript number is the order of the small parameter. Based on first- and second-order analyses, which are not presented here, Lin and Perrie (1997) suggested that the perturbation expansion parameter should be given by

$$\epsilon = aK(3 + \tanh^2 Kh)/4 \tanh^3(Kh),$$

where K and a are the wavenumber and (first order) wave amplitude. This is a *natural* choice because, in the first- and second-order analysis, $aK(3 + \tanh^2(Kh))/4 \tanh^3(Kh)$ was found to equal the ratio of the second-order wave amplitude to the first-order wave amplitude a_2/a_1.

As mentioned in the introduction, Hasselmann (1962) obtained the nonlinear energy transfer rate for finite depth by using a perturbation method. Zakharov (1968, 1991) obtained the nonlinear energy transfer rate for deep water by using a Hamiltonian representation. Both methods only considered linear dispersion. We use Zakharov's approach because it is algebraically simpler, it lends itself to computation that is orders of magnitude faster, and it can be considered accurate for steeper waves ($aK \leq 0.3$) compared to Hasselmann's (1962) approach.

To apply Zakharov's (1968, 1991) analysis, we must extend it to finite water depth and include consideration of the nonlinear dispersion relationship. Following Lin and Perrie (1997), we substitute the expansions of Eq. (5a) into Eqs. (2) and (3) at $z = \eta$. After some rearranging, we obtain the usual first- and second-order analyses, as well as the more general relations,

$$\frac{\partial\eta_n}{\partial t} + \nabla_H\Phi_{n-1}\cdot\nabla_H\eta_1 + \cdots + \nabla_H\Phi_1\cdot\nabla_H\eta_{n-1} = 0, \quad (5b)$$

$$\frac{\partial\Phi_n}{\partial t} + \frac{1}{2}(\nabla_H\Phi_{n-1}\cdot\nabla_H\Phi_1 + \cdots + \nabla_H\Phi_1\cdot\nabla_H\Phi_{n-1}) + g\eta_n = 0, \quad (5c)$$

where $n = 3, 4, \cdots$. To obtain the nonlinear dispersion term, we assume the free surface elevation has the following form:

$$\Phi_n = A_n \cosh nKh\{\exp[in(\omega t + \mathbf{K}\cdot\mathbf{r})] - \exp[-in(\omega t + \mathbf{K}\cdot\mathbf{r})]\}$$
$$\eta_n = a_n\{\exp[in(\omega t + \mathbf{K}\cdot\mathbf{r})] - \exp[-in(\omega t + \mathbf{K}\cdot\mathbf{r})]\}. \quad (5d)$$

Using the first- and second-order analysis, we then substitute expressions for Φ_1, η_1, Φ_2, η_2, Φ_n, η_n, and the corresponding dispersion relations at $z = \eta$, following Whitham (1974), we obtain dispersion term:

$$\omega_3 = \omega_1\frac{9 - 10\tanh^2 Kh + 9\tanh^4 Kh}{8\tanh^4 Kh}K^2a^2,$$
$$\omega_4 = 0. \quad (5e)$$

The linear first-order dispersion relation is, of course,

$$\omega_1^2 = gK\tanh(Kh). \quad (5f)$$

Equation (5e) is derived in detail in Lin and Perrie (1997).

Following Zakharov (1968, 1991) and Lin and Perrie (1997), we introduce the transformation

$$\psi(\mathbf{r}, t) = \Phi(\mathbf{r}, z, t)|_{z=\eta}$$
$$\frac{\partial\psi}{\partial t} = \frac{\partial\Phi}{\partial t} + \frac{\partial\eta}{\partial t}\frac{\partial\Phi}{\partial z}\bigg|_{z=\eta}. \quad (5g)$$

Assuming a gently varying bottom, such that in one wavelength, $h_{(x,y)} = h_0 + \delta h_{(x,y)}$ and Kh_0 is approximately constant, where $\delta h < \epsilon^3 h_0$, it follows from Eq. (5f) that ω_1 as a function only of wavenumber K and water depth h. If we also assume that the wave amplitude a is constant over one wavelength, then from Eq. (5e) it follows that ω_n is a function of K and h. Therefore, to third-order approximation the frequency should also be truncated at third order. With these general assumptions we can make the Fourier representation:

$$\eta_{(\mathbf{r},t)} = \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\frac{|K|^{1/2}}{\omega_{(K)}^{1/2}}\{b_{(\mathbf{K},t)}\exp[i(\mathbf{K}\cdot\mathbf{r})] + b^*_{(\mathbf{K},t)} \times \exp[-i(\mathbf{K}\cdot\mathbf{r})]\}\,d\mathbf{K} \quad (5h)$$

$$\psi_{(\mathbf{r},t)} = \frac{-i}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\frac{\omega_{(K)}^{1/2}}{|K|^{1/2}}\{b_{(\mathbf{K},t)}\exp[i(\mathbf{K}\cdot\mathbf{r})] - b^*_{(\mathbf{K},t)} \times \exp[-i(\mathbf{K}\cdot\mathbf{r})]\}\,d\mathbf{K}. \quad (5i)$$

Following Zakharov (1968) and Yuen and Lake (1982), we define a complex variable,

$$b(\mathbf{K}, t) = \left(\frac{\omega}{2|\mathbf{K}|}\right)^{1/2}\hat{\eta}(\mathbf{K}, t) + i\left(\frac{|\mathbf{K}|}{2\omega}\right)^{1/2}\hat{\Phi}(\mathbf{K}, t),$$

where $\hat{\eta}$ and $\hat{\Phi}$ are the Fourier coefficients of $\mathbf{K}$ for $\eta(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, z = \eta, t)$, respectively. Expanding b as power series in ϵ, we obtain

$$b = \epsilon B + \epsilon^2 B^{(2)} + \cdots. \quad (5j)$$

Substituting the expansions of Eqs. (5a) and (5j) into Eqs. (1)–(4), we arrive at the second-order equations, as in Lin and Perrie (1997). Fourier transforming and integrating these equations, we obtain

$$B^{(2)}_{(\mathbf{K}_i,t)} = -\iint_{-\infty}^{\infty}\left\{V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2)}B_{(\mathbf{K}_1)}B_{(\mathbf{K}_2)}\delta(\mathbf{K}_i - \mathbf{K}_1 - \mathbf{K}_2)\frac{\exp[i(\omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t]}{\omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} + 2V^{(-)}_{(\mathbf{K}_1,\mathbf{K}_i,\mathbf{K}_2)}B^*_{(\mathbf{K}_1)}B_{(\mathbf{K}_2)}\delta(\mathbf{K}_i + \mathbf{K}_1 - \mathbf{K}_2)\frac{\exp[i(\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t]}{\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} + V^{(+)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2)}B^*_{(\mathbf{K}_1)}B^*_{(\mathbf{K}_2)}\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2)\frac{\exp[i(\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})t]}{\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}}\right\} d\mathbf{K}_1\, d\mathbf{K}_2, \quad (6)$$

where $V^{(-)}$ and $V^{(+)}$ are functions of wave amplitude, wavenumber, and water depth, as described in the appendix. These expressions differ from Zakharov (1968) and Crawford et al. (1980) because of the finite depth assumption and the nonlinear dispersion relationship. For the remainder of the paper, we abbreviate wavenumber dependence by subscripts, for example, $B^{(2)}_{(\mathbf{K}_i,t)} = B^{(2)}_i$ or $V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2)} = V^{(-)}_{i,1,2}$.

c. Third-order equations and analysis

Substituting the expansions of Eqs. (5i) and (5j) into Eq. (1)–(4), we obtain the third-order equations

$$\begin{aligned}
&\nabla^2_H\Phi^{(3)} + \frac{\partial^2\Phi^{(3)}}{\partial z^2} = 0, && -h < z < 0\\
&\frac{\partial\eta^{(3)}}{\partial t} - \frac{\partial\Phi^{(3)}}{\partial z} = -\nabla_H\Phi^{(1)}\cdot\nabla_H\eta^{(2)} - \nabla_H\Phi^{(2)}\cdot\nabla_H\eta^{(1)}, && \text{at } z = \eta\\
&g\eta^{(3)} + \frac{\partial\Phi^{(3)}}{\partial t} = -\left[\nabla_H\Phi^{(1)}\cdot\nabla_H\Phi^{(2)} + \left(\frac{\partial\Phi^{(1)}}{\partial z}\right)\left(\frac{\partial\Phi^{(2)}}{\partial z}\right)\right], && \text{at } z = \eta\\
&\frac{\Phi^{(3)}}{\partial z} = 0, && \text{at } z = -h.
\end{aligned} \quad (7)$$

Fourier transforming and integrating as in the previous section, we arrive at the third-order representation for $B^{(3)}_{(\mathbf{K}_i,t)}$,

$$B^{(3)}_i = \iiint_{-\infty}^{\infty}\left\{T^{(1)}_{i,1,2,3}B_1B_2B_3\delta(\mathbf{K}_i - \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3)\frac{\exp[i(\omega_i - \omega_1 - \omega_2 - \omega_3)t]}{\omega_i - \omega_1 - \omega_2 - \omega_3} + T^{(2)}_{i,1,2,3}B^*_1B_2B_3\delta(\mathbf{K}_i + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3)\frac{\exp[i(\omega_i + \omega_1 - \omega_2 - \omega_3)t]}{\omega_i + \omega_1 - \omega_2 - \omega_3} + T^{(3)}_{i,1,2,3}B^*_1B^*_2B_3\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3)\frac{\exp[i(\omega_i + \omega_1 + \omega_2 - \omega_3)t]}{\omega_i + \omega_1 + \omega_2 - \omega_3} + T^{(4)}_{i,1,2,3}B^*_1B^*_2B^*_3\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)\frac{\exp[i(\omega_i + \omega_1 + \omega_2 + \omega_3)t]}{\omega_i + \omega_1 + \omega_2 + \omega_3}\right\} d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3, \quad (8)$$

where $T^{(1)}_{i,1,2,3,4}$, $T^{(2)}_{i,1,2,3,4}$, $T^{(3)}_{i,1,2,3,4}$, and $T^{(4)}_{i,1,2,3,4}$ are given in the appendix.

d. Fourth-order equations and analysis

To eliminate the nonresonant terms in our model at this level, we must invoke the solvability condition (Zauderer 1983)

$$\left\langle \begin{pmatrix}\Phi^* \\ \eta^*\end{pmatrix} xL \begin{pmatrix}\Phi \\ \eta\end{pmatrix} - \begin{pmatrix}\Phi \\ \eta\end{pmatrix} xL^* \begin{pmatrix}\Phi^* \\ \eta^*\end{pmatrix} \right\rangle = 0, \tag{9}$$

where the L operator is

$$L = \begin{pmatrix} \nabla^2 & 0 \\ -\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial t} & g\eta \\ \dfrac{\partial}{\partial z} & 0 \end{pmatrix} \tag{10}$$

and its adjoint is

$$L^* = \begin{pmatrix} \nabla^2 & 0 \\ -\dfrac{\partial}{\partial z} & -\dfrac{\partial}{\partial t} \\ -\dfrac{\partial}{\partial t} & g\eta \\ \dfrac{\partial}{\partial z} & 0 \end{pmatrix}. \tag{11}$$

Therefore, the solvability condition, in this case, is

$$\begin{aligned}
&\left\langle \phi^*, \frac{\partial \Phi^{(1)}}{\partial t}\right\rangle + \left\langle \eta^*, \frac{\partial \eta^{(1)}}{\partial t}\right\rangle \\
&\quad = \left\langle \Phi^*, \left[-(\nabla_H \Phi^{(3)})\cdot(\nabla_H \Phi^{(1)}) - \left(\frac{\partial \Phi^{(3)}}{\partial z}\right)\left(\frac{\partial \Phi^{(1)}}{\partial z}\right)\right]\right\rangle \\
&\qquad + \left\langle \Phi^*, \frac{1}{2}\left[-(\nabla_H \Phi^{(2)})\cdot(\nabla_H \Phi^{(2)}) - \left(\frac{\partial \Phi^{(2)}}{\partial z}\right)\left(\frac{\partial \Phi^{(2)}}{\partial z}\right)\right]\right\rangle \\
&\qquad + \langle \eta^*, [-\nabla_H \Phi^{(3)}\cdot\nabla_H \eta^{(1)} - \nabla_H \Phi^{(1)}\cdot\nabla_H \eta^{(3)} \\
&\qquad\qquad - \nabla_H \Phi^{(2)}\cdot\nabla_H \eta^{(2)}]\rangle.
\end{aligned} \tag{12}$$

Once more, we Fourier transform, following Eqs. (5.8)–(5.9), and obtain

$$\begin{aligned}
i\frac{\partial B_i}{\partial t} = &- \iint_{-\infty}^{\infty} \{V^{(-)}_{i,1,2}(B^{(2)}_1 B^{(2)}_2 + B_1 B^{(3)}_2 + B^{(3)}_1 B_2)\delta(\mathbf{K}_i - \mathbf{K}_1 - \mathbf{K}_2)\exp[i(\omega_i - \omega_1 - \omega_2)t] \\
&\quad + 2V^{(-)}_{1,i,2}(B^{(2)*}_1 B^{(2)}_2 + B^*_1 B^{(3)}_2 + B^{(3)*}_1 B_2)\delta(\mathbf{K}_1 + \mathbf{K}_i - \mathbf{K}_2)\exp[i(\omega_1 + \omega_i - \omega_2)t] \\
&\quad + V^{(+)}_{i,1,2}(B^{(2)*}_1 B^{(2)*}_2 + B^*_1 B^{(3)*}_2 + B^{(3)*}_1 B^*_2)\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2)\exp[i(\omega_i + \omega_1 + \omega_2)t]\}\, d\mathbf{K}_1\, d\mathbf{K}_2 \\
&+ \iiint_{-\infty}^{\infty} \{W^{(1)}_{i,1,2,3}(B^{(2)}_1 B_2 B_3 + B_1 B^{(2)}_2 B_3 + B_1 B_2 B^{(2)}_3)\delta(\mathbf{K}_i - \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3) \\
&\quad \times \exp[i(\omega_i - \omega_1 - \omega_2 - \omega_2)t] \\
&\quad + W^{(2)}_{i,1,2,3}(B^{(2)*}_1 B_2 B_3 + B^*_1 B^{(2)}_2 B_3 + B^*_1 B_2 B^{(2)}_3)\delta(\mathbf{K}_i + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3) \\
&\quad \times \exp[i(\omega_i + \omega_1 - \omega_2 - \omega_2)t] \\
&\quad + W^{(3)}_{i,1,2,3}(B^{(2)*}_1 B^*_2 B_3 + B^*_1 B^{(2)*}_2 B_3 + B^*_1 B^*_2 B^{(2)}_3)\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3) \\
&\quad \times \exp[i(\omega_i + \omega_1 + \omega_2 - \omega_2)t] \\
&\quad + W^{(4)}_{i,1,2,3}(B^{(2)*}_1 B^*_2 B^*_3 + B^*_1 B^{(2)*}_2 B^*_3 + B^*_1 B^*_2 B^{(2)*}_3)\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3) \\
&\quad \times \exp[i(\omega_i + \omega_1 + \omega_2 + \omega_2)t]\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3 \\
&+ \iiiint_{-\infty}^{\infty} \{R^{(2)}_{i,1,2,3,4} B^*_1 B_2 B_3 B_4 \delta(\mathbf{K}_i + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4)\exp[i(\omega_i + \omega_1 - \omega_2 - \omega_3 - \omega_4)t] \\
&\quad + R^{(3)}_{i,1,2,3,4} B^*_1 B^*_2 B_3 B_4 \delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4) \\
&\quad \times \exp[i(\omega_i + \omega_1 + \omega_2 - \omega_3 - \omega_4)t]\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3\, d\mathbf{K}_4,
\end{aligned} \tag{13}$$

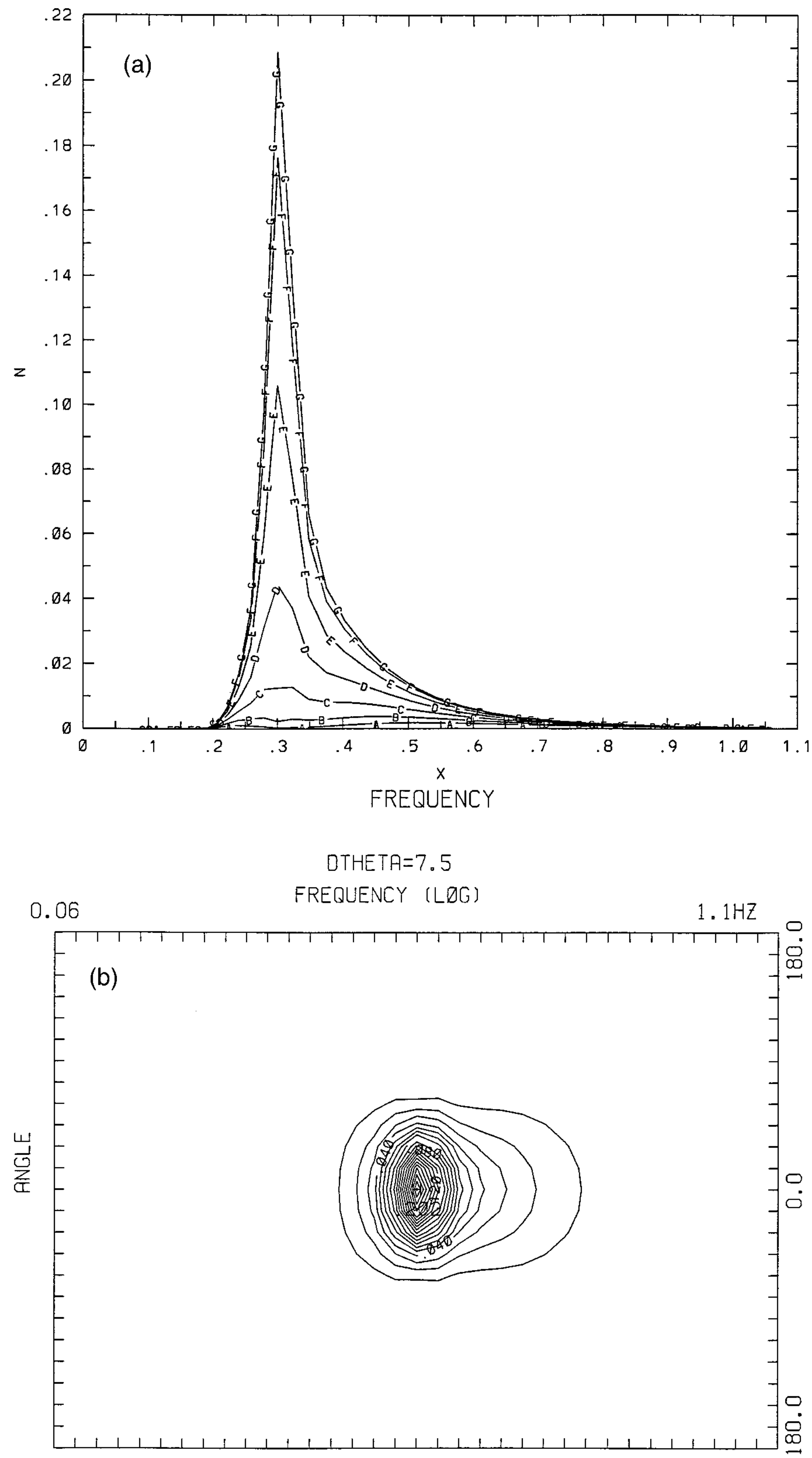


FIG. 1. The reference JONSWAP gravity wave spectrum (a) with Hasselmann–Mistsuyasu directional speading, where lines A, B, C, . . . , G represent the angles 0°, 30°, 60°, . . . , 180°, with 0° toward east and (b) in terms of frequency and direction. In (b) the resolution angle $d\theta$ is indicated by DTHETA, the maximum is 0.22, and the contour interval is 0.01.

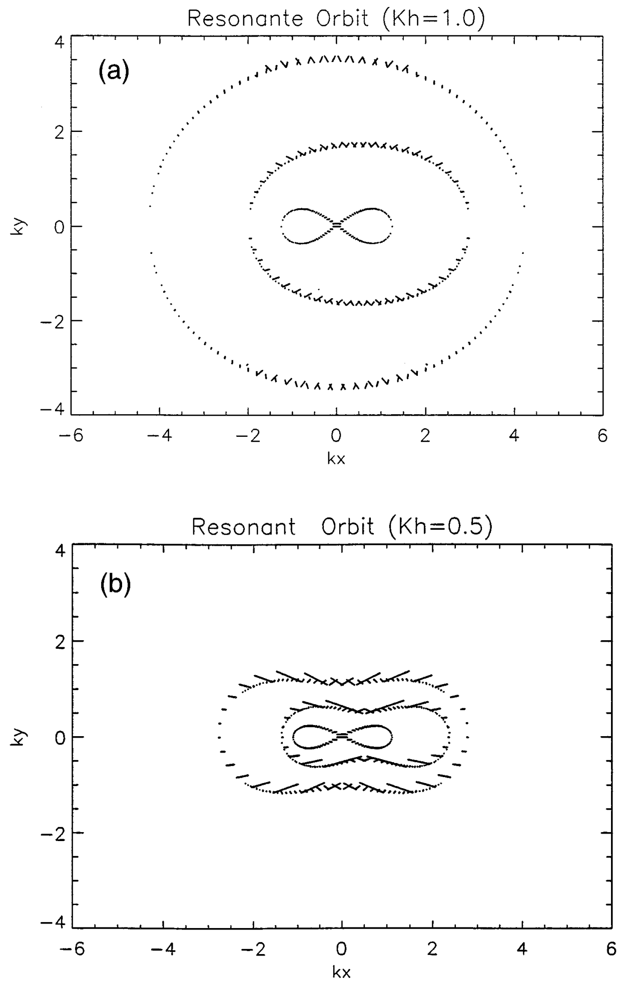


FIG. 2. The resonant orbits of four-wave, five-wave, and six-wave interactions (a) over deep water with $Kh \geq 1$ and (b) over shallow water with $Kh = 0.5$. Wave steepness Ka is assumed to be 0.1. The tick marks represent solutions. When the wave steepness increases, the orbits become thicker and thicker, and the 4-wave resonant orbit will come closer and closer to the 5-wave resonant orbit until they may actually touch.

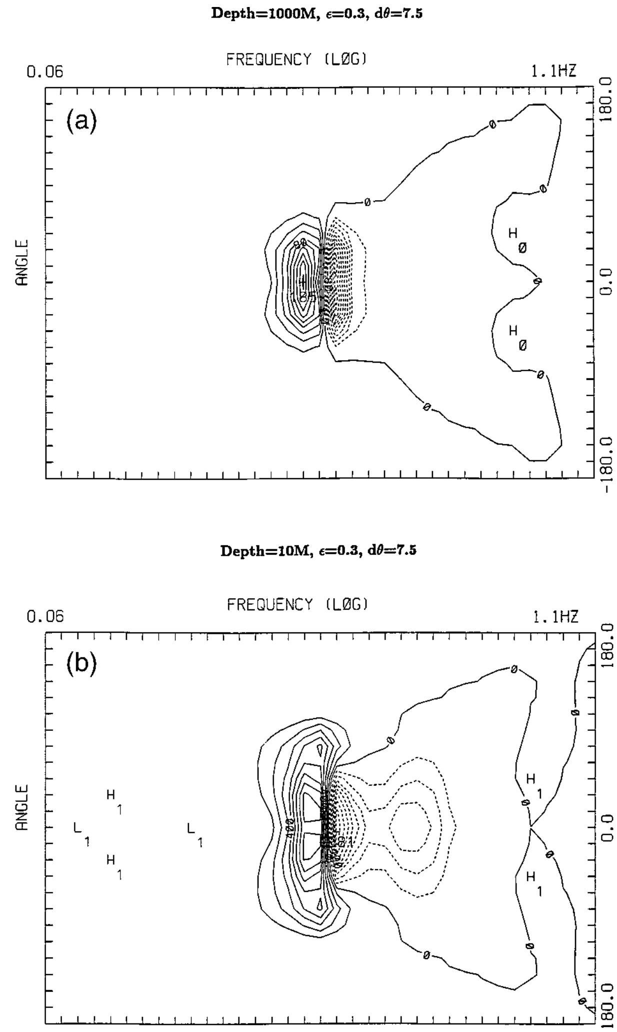


FIG. 3. Total nonlinear transfer (four-wave interactions plus five-wave interactions) when strong nonlinearity, $\epsilon = 0.3$, in terms of frequency and direction (a) over deep water with $h = 1000$ m and (b) in shallow water with $h = 10$ m. Local maxima are represented by H and local minima by L. Positive contours are represented by solid lines, negative contours by dashed lines. In (a), the global maximum is 18.5×10^{-6}, the global minimum is -18.0×10^{-6}, and the contour interval is 2.0×10^{-6}. In (b) the global maximum is 8.0×10^{-6}, global minimum is -8.2×10^{-6}, and the contour interval is 1.0×10^{-6}.

where $W^{(1)}_{i,1,2,3}$, $W^{(2)}_{i,1,2,3}$, $W^{(3)}_{i,1,2,3}$, $W^{(4)}_{i,1,2,3}$, $R^{(2)}_{i,1,2,3,4}$, and $R^{(3)}_{i,1,2,3,4}$ are functions of amplitude, wavenumber, and water depth as given in the appendix.

Their forms differ from analogous expressions given by Zakharov (1968) and Crawford et al. (1980). This is because, although we use solvability condition and adjoint operator to eliminate the resonant terms instead of multiple timescales, and both methods seem to yield consistent expansion terms at each order, the shortcomings of the Zakharov–Crawford approach are due to their derivation, which is not continuous. In this regard their approach is not Hamiltonian, despite the Hamiltonian structure of exact water wave equations. This was realized by Krasitskii (1994), who used a classical method of canonical transformations, generalized to the continuous case. We use solvability conditions to map the three dimensions to two dimensions, to eliminate the nonresonant terms, and avoid the Zakharov–Crawford shortcomings. Therefore, our derivation is continuous and should obtain the same solution as Krasitskii (1994). However, our method is simpler than Krasitskii (1994). Moreover, in our calculation we consider the finite depth assumption and nonlinear dispersion whereby $\omega = \epsilon\omega_1 + \epsilon^3\omega_3 + \epsilon^5\omega_5 + \cdots$ from Eqs. (5a), (5e), and (5f), whereas Krasitskii (1994) uses only $\omega = \omega_1$, which is Eq. (5f), as the dispersion term. These are the differences between this present study and that of Krasitskii (1994).

Substituting Eqs. (6) and (8) into Eq. (12) and following Crawford et al. (1980), we obtain fourth-order amplitude $B^{(4)}$ in terms of B, which is a Zakharov-type equation. Multiplying Eq. (13) by $B^{(4)}$ we obtain the fourth-order action transfer equation for a homogeneous wave field as

$$\begin{aligned}\frac{\partial A_{(K_i)}}{\partial t} = \frac{4\pi}{3}\iiiint_{-\infty}^{\infty} & Q^{(2)}_{i,1,2,3,4}\delta(\mathbf{K}_i + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4)\delta[\omega_{(K_i)} + \omega_{(K_1)} - \omega_{(K_2)} - \omega_{(K_3)} - \omega_{(K_4)}] \\ & \times \{A_{(K_2)}A_{(K_3)}A_{(K_4)}[A_{(K_i)} + A_{(K_1)}] \\ & \quad - A_{(K_i)}A_{(K_1)}[A_{(K_2)}A_{(K_3)} + A_{(K_2)}A_{(K_4)} + A_{(K_3)}A_{(K_4)}]\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3\, d\mathbf{K}_4 \\ - \frac{4\pi}{2}\iiiint_{-\infty}^{\infty} & Q^{(3)}_{i,1,2,3,4}\delta(\mathbf{K}_i + \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4)\delta[\omega_{(K_i)} + \omega_{(K_1)} + \omega_{(K_2)} - \omega_{(K_3)} - \omega_{(K_4)}] \\ & \times \{A_{(K_i)}A_{(K_1)}A_{(K_2)}[A_{(K_4)} + A_{(K_3)}] \\ & \quad - A_{(K_3)}A_{(K_4)}[A_{(K_2)}A_{(K_1)} + A_{(K_i)}A_{(K_1)} + A_{(K_i)}A_{(K_2)}]\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3\, d\mathbf{K}_4, \end{aligned} \tag{14}$$

where $A_{(K_i)}$ is the action energy spectrum. Kernels for the integrals are $Q^{(2)}_{i,1,2,3,4}$ and $Q^{(3)}_{i,1,2,3,4}$, as given in the appendix. Equation (14) is the kinetic transfer equation for the spectrum. It should be equivalent to the results obtained by the canonical transformation method of Krasitskii (1994). However, the nonlinear dispersion relation of this study, $\omega = \epsilon\omega_1 + \epsilon^3\omega_3 + \epsilon^5\omega_5 + \cdots$, may give rise to slightly different results from those obtained by Krasitskii's (1994) dispersion, $\omega = \omega_1$.

3. Action transfer rate

As noted earlier, McLean (1982) pointed out that the three-dimensional wave–wave interactions dominate in shallow water when wave amplitude is large and waves are steep. Moreover, Crawford et al. (1981) suggested that three-dimensional wave–wave interactions on a uniform wave train do not possess a high wavenumber cutoff. To study the nonlinear source function quantitatively, for finite-amplitude steep waves, we have to estimate the action transfer rate due to four-wave (Lin and Perrie 1997a) and five-wave interactions, as given in Eq. (14). For simulation, we use the standard JONSWAP directional spectrum as our initial energy spectrum, as shown in Figs. 1a,b. Frequency and direction are denoted by lines A, B, . . . , G, representing angles 0°, 30°, . . . , 180° (where 0° is pointing to the east, which is the main wave propagation direction).

a. The effects on resonance conditions

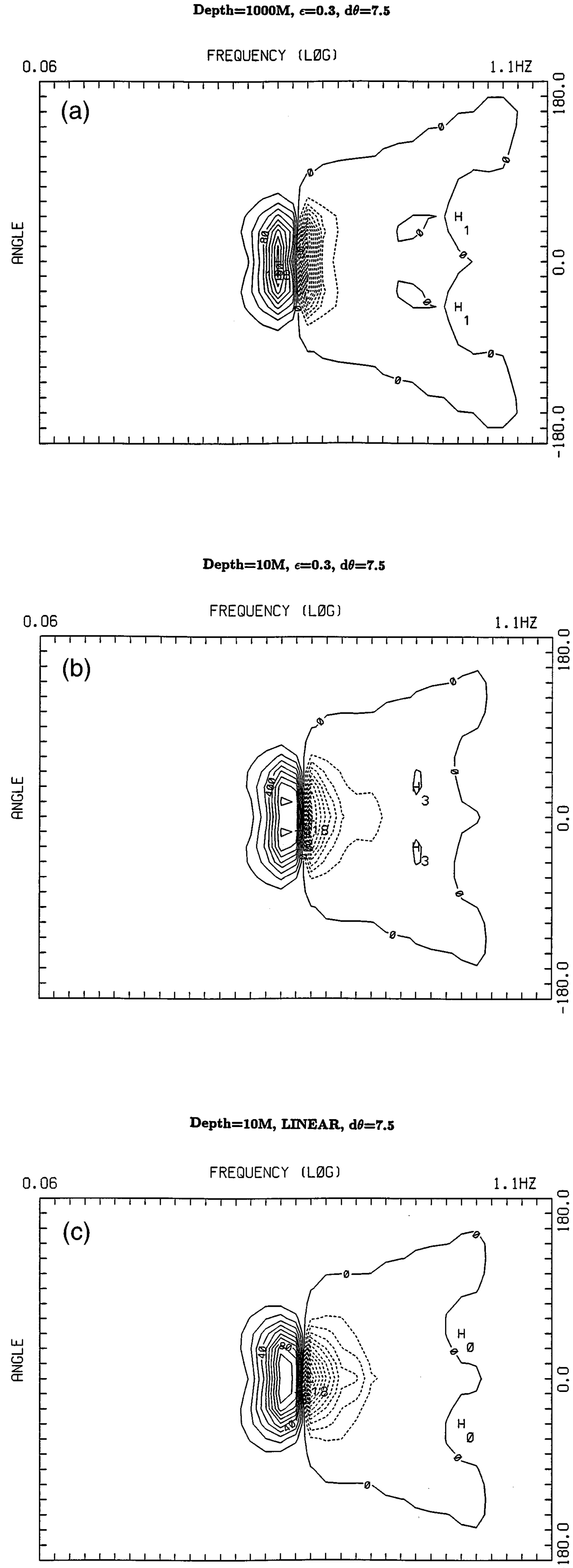
The pioneering study of Phillips (1960) suggested that narrowband instability (four equivalent wave interactions) dominates in deep water. Our results basically support this conclusion even with the nonlinear dispersion relationship and the effects of finite water depth. However, nonlinearity in combination with finite water depth does result in new effects, which are similar to results obtained by McLean (1982).

The famous figure-eight diagram by Phillips (1960), for the trajectory of wavenumbers satisfying the resonance condition for the third-order, binary, four-wave interactions, is given by the innermost orbit of Fig. 2a. This is for deep water, with $Kh = 1.0$. Wavenumber one, K_1, and wavenumber two, K_2, are both set to 1 and are assumed to propagate in the x direction. The horizontal and vertical coordinates represent the x and y components of wavenumber three (K_{x3}, K_{y3}), respectively. The corresponding trajectories of the wavenumbers satisfying the resonance conditions for five-wave and six-wave interactions are the middle and outer orbits, respectively. The wavenumber trajectory for seven-wave interactions, which is not shown, would be outside the six-wave interaction orbit.

As water depth decreases, the orbits become smaller. This is shown in Fig. 2b with $Kh = 0.5$. Moreover, as the wave amplitude become larger, the orbits change from being merely a thin line to becoming an orbital *band.* As the wave amplitude become larger and larger, the band will become wider and wider and eventually the orbits for the four-wave, five-wave, and even the six-wave interactions will overlap.

b. Four-wave and five-wave transfers

The total nonlinear action transfer, including four-wave and five-wave interactions is shown in Figs. 3a, b for strong nonlinearity [steep waves with $aK(3 + \tanh^2 Kh)/4 \tanh^3(Kh) = 0.3$]. Figure 3a is for deep water with depth $h = 1000$ m ($K_p h \rightarrow \infty$). Figure 3b is for shallow water with $h = 10$ m ($K_p h = 3.6$). Figures 3a



and 3b both show that positive nonlinear action transfer occurs toward the lower frequency, while negative transfer occurs toward the higher frequency portion of the spectrum, as usual.

However, Figs. 3a and 3b have differences. Figure 3a shows that the dominant positive action flux is along the original spectral propagation direction. The maximum positive nonlinear transfer is along the 0° direction. The nonlinear transfer remains positive ±60° directions. In contrast, Fig. 3b shows that the positive nonlinear transfer is more widely spread in shallow water. Positive maxima in the nonlinear transfer occur in the ±15° directions. The positive nonlinear transfer extends directionally to between ±105°. Therefore, Fig. 3b shows that the three-dimensional wave–wave interactions are dominant over shallow water for steep finite-amplitude waves ($\epsilon = 0.3$). In deep water, strongly nonlinear steep waves ($\epsilon = 0.3$) correspond to wave–wave interactions, which may be characterized as two-dimensional, as in Fig. 3a. The latter are not distinctively the three-dimensional crescent-shaped breakers under growing seas, observed by Su et al. (1982a,b).

Five-wave interactions are fundamentally different from four-wave interactions, although both are resonant interactions. In four-wave interactions, the strongest interactions occur when all four interacting components are of comparable lengths, as pointed out by Longuet-Higgins (1976) and Phillips (1977). Consequently, the four-wave interactions can only show shallow water effects when $K_p h$ is small. However, for five-wave interactions, shallow water effects are clear, even when $K_p h$ is still quite high. This is because the interaction components are no longer required to have comparable lengths, confined to the neighborhood of the spectral peak. As shown in the interaction locus (Figs. 2a,b), the distribution of the participating components is global. Therefore, even if $K_p h$ is high, some interacting quintets are in shallow water, which makes the shallow water effect obvious. This new feature is central to the way in which five-wave and four-wave interactions differ drastically.

In order to examine the details of the three-dimensional wave–wave interactions we are going to present finite-amplitude four-wave and five-wave interactions separately. Figure 4a shows the nonlinear transfer for steep finite-amplitude waves due to four-wave interac-

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FIG. 4. Nonlinear transfer due to four-wave interactions in terms of frequency and direction: (a) assuming strong nonlinearity with $\epsilon = 0.3$, in deep water, $h = 1000$ m; (b) assuming strong nonlinearity, $\epsilon = 0.3$, in shallow water, $h = 10$ m; and (c) assuming linear dispersion, in shallow water, $h = 10$ m. As in Figs. 3a,b, positive contours are represented by solid lines and negative contours by dashed lines. In Fig. 4a, the global maximum, global minimum, and contour interval are, respectively, 18.8×10^{-6}, -18.0×10^{-6}, and 2.0×10^{-6}; in (b) 8.0×10^{-6}, -8.2×10^{-6}, and 1.0×10^{-6}; and finally, in (c) 10.0×10^{-6}, -8.2×10^{-6}, and 1.0×10^{-6}.

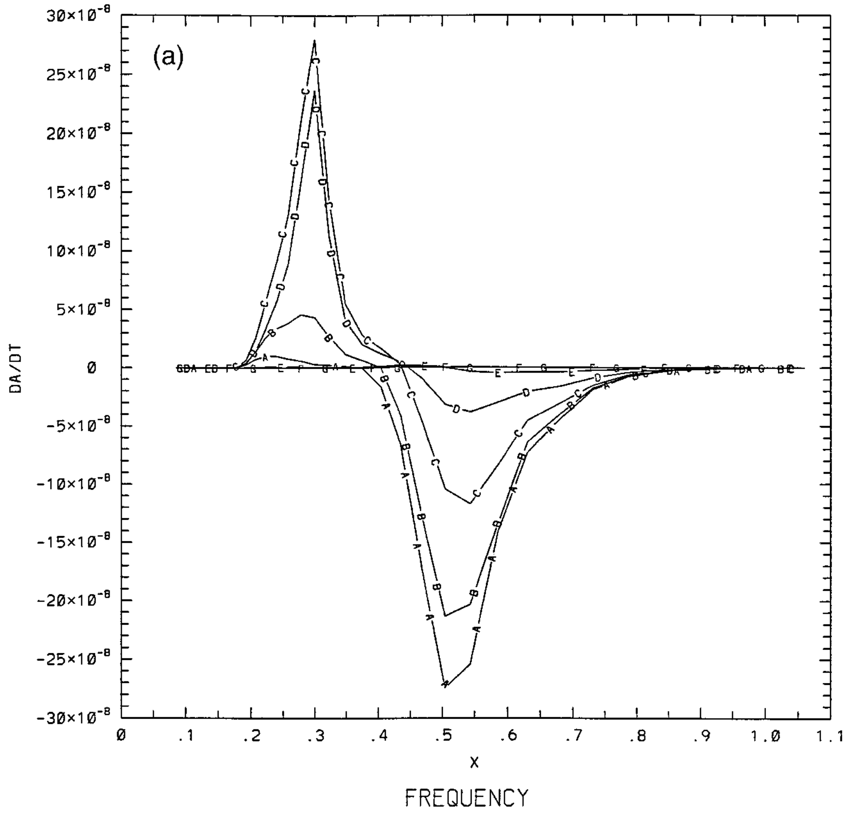
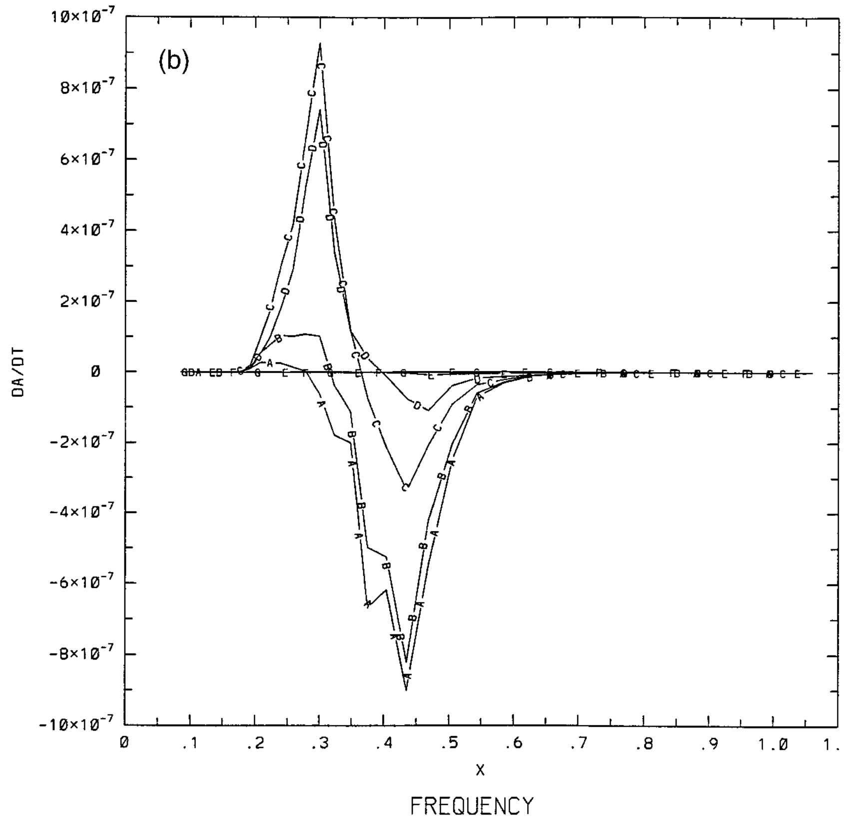


FIG. 5. Nonlinear transfer due to five-wave interactions assuming strong nonlinearity with $\epsilon = 0.3$, as a function of frequency, where lines A, B, C, D, E, F, and G represent the angles 0°, 30°, 60°, 90°, 120°, 150°, and 180° as in Fig. 2a: (a) in deep water with $h = 1000$ m, (b) intermediate depth water with $h = 20$ m, and (c) shallow water with $h = 10$ m.

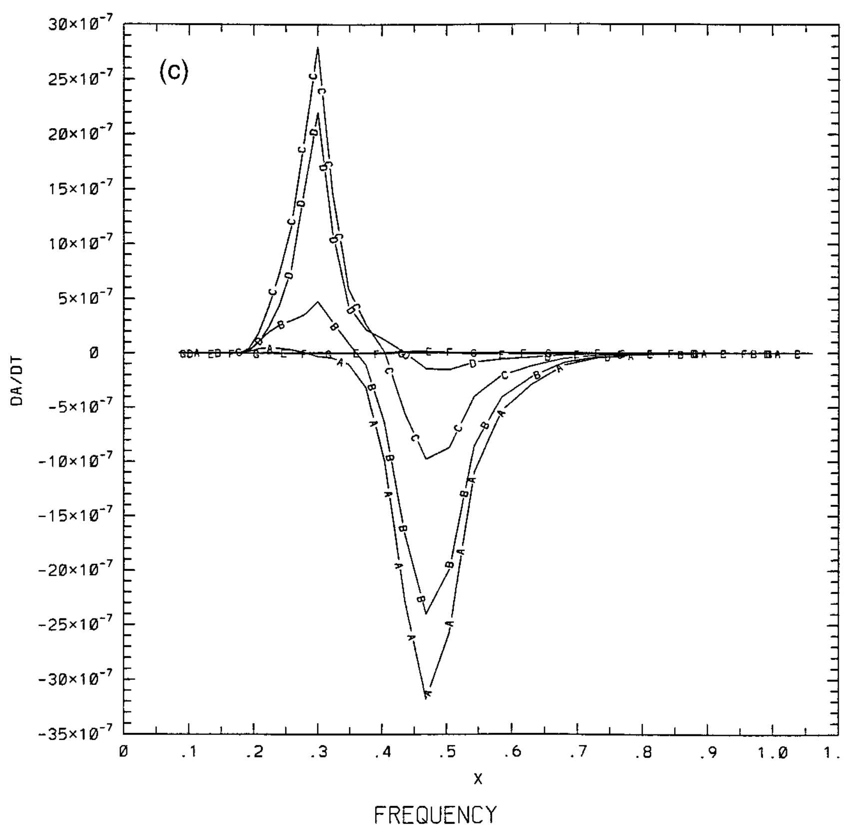
tions over deep water, $h = 1000$ m ($K_p h \rightarrow \infty$), with strong nonlinearity, $\epsilon = 0.3$. This differs from Fig. 3a, which showed both four-wave and five-wave interactions, denoted ''total nonlinear transfer.'' However, nonlinear transfer maximum is still toward lower frequency along the original propagation direction and the positive nonlinear transfer remains between $\pm 60°$ as in Fig. 3a and Lin and Perrie (1997a, 1997b, manuscript submitted to *J. Geophys. Res.*). Although secondary peaks may occur in the high-frequency range at about $\pm 30°$, the high-frequency positive action flux is not important, because it is significantly smaller than the nonlinear transfer occurring at lower frequency. Moreover, the dominant characteristic of the four-wave interactions in Fig. 4a is two-dimensional (in the sense of three-dimensional crescent-shaped breakers in growing seas).

The corresponding nonlinear action transfer in shallow water, assuming strong nonlinearity ($\epsilon = 0.3$), is shown in Fig. 4b. The positive nonlinear action transfer is toward lower frequency with the maxima in positive nonlinear transfer occurring at $\pm 15°$, relative to the spectral propagation direction. Moreover, the positive action transfer extends over a wider directional span on both sides of the spectral propagation direction, as shown in Fig. 4b for shallow water, than for deep water shown in Fig. 4a. Neglecting nonlinear dispersion, Fig. 4c presents the nonlinear transfer for shallow water, as in Fig. 4b, when linear dispersion is assumed. This is again two-dimensional about the central propagation direction, as in Fig. 4a. Therefore, it follows that three-dimensional wave–wave interactions in shallow water are caused by four-wave interactions involving finite-amplitude steep waves.

The nonlinear transfer due to five-wave interactions in deep water, assuming steep finite-amplitude waves is shown in Fig. 5a. As in Figs. 3–4, this assumes $\epsilon = 0.3$ and for deep water, $h = 1000$ m, implying that $K_p h \rightarrow \infty$. Figure 5b assumes intermediate shallow depth, with $h = 20$ m and $K_p h = 7.2$, and Fig. 5c assumes shallow water with $h = 10$ m and $K_p h = 3.6$. Figures 5a, 5b, and 5c show that lower frequencies receive positive action flux and higher frequencies lose action flux as we obtained for four-wave interactions. Moreover, at $h = 10$ m with $K_p h = 3.6$, the nonlinear transfer due to five-wave interactions is of the same order of magnitude as that due to four-wave interactions (the latter

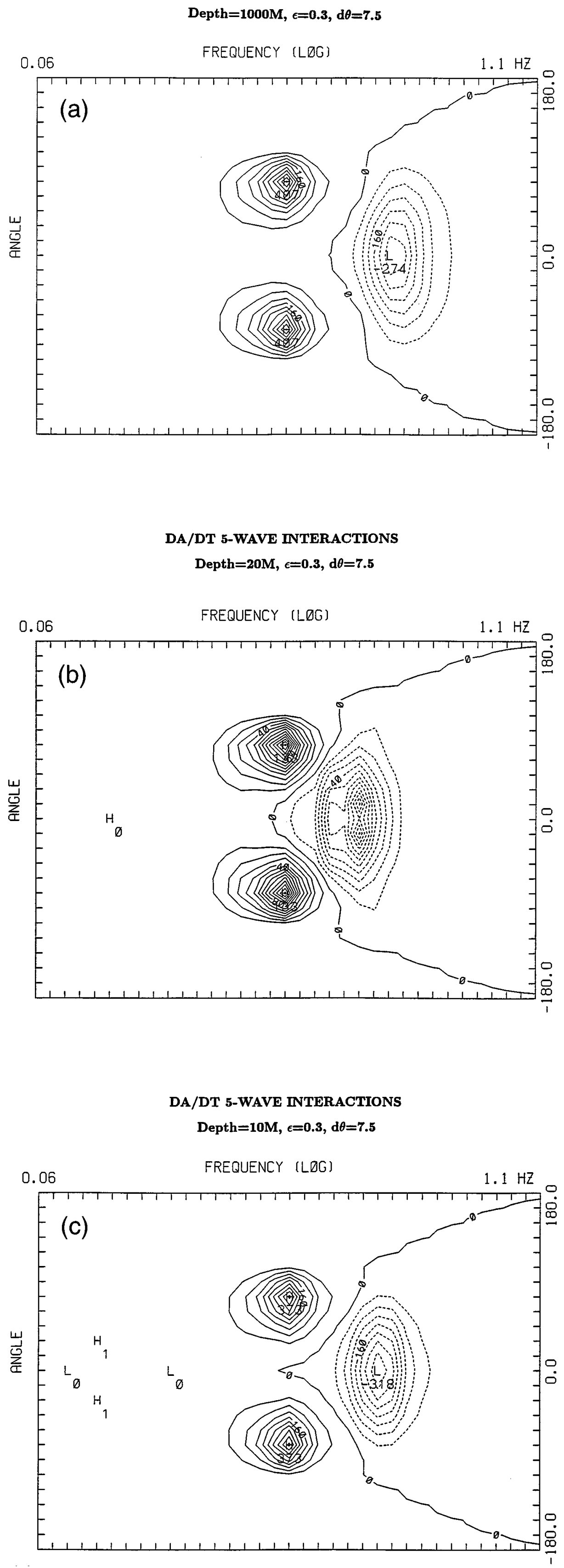


Depth=20M, ϵ=0.3



Depth=10M, ϵ=0.3

Fig. 5. (*Continued*)



are reported by Lin and Perrie 1997a). However, if we compare Figs. 5a–c, we find that the nonlinear transfer increases as water depth decreases and nonlinearity increases. This is opposite to the accepted behavior for four-wave interactions, as reported by Lin and Perrie (1997). Furthermore, Figs. 5a–c show that the maximum positive nonlinear transfers are along the 60° direction (line C has the greatest value).

The exact distribution of the spread of the positive action fluxes due to five-wave interactions is given in Figs. 6a–c using frequency–direction coordinates to present the results of Fig. 5a–c, respectively. This shows that the maxima positive nonlinear transfers are located at ±60° directions, whereas for four-wave interactions, the maximum positive nonlinear transfer occurred in the 0° direction. Moreover, unlike four-wave interactions, the original propagation direction of the wave spectrum (0°) experiences no nonlinear transfer due to five-wave interactions. Five-wave interactions, in all depths of water, involving strongly nonlinear steep waves, are distinctively three-dimensional. In shallow water, they are comparable in magnitude to four-wave interactions, which can also be characterized as being three-dimensional, as shown in Fig. 4b.

4. Conclusions

We have obtained the nonlinear transfer due to four-wave and five-wave interactions. Our results are consistent with the instability study by McLean (1982), which showed that three-dimensional wave–wave interactions involving steep finite-amplitude waves (strong nonlinearity) are dominant in shallow water and two-dimensional wave–wave interactions are dominant in deep water. We note that when wave–wave interactions are referred to as three-dimensional in this discussion, as in the previous section, it is a reference to their association with three-dimensional crescent-shaped breakers in growing seas. In contrast, when we refer to wave–wave interactions as being *two-dimensional,* for example, in deep water, they are associated with two-dimensional propagating wave structures, not crescent-shaped breakers.

In deep water, the nonlinear transfer due to two-dimensional four-wave interactions is about two-orders greater than that due to five-wave interactions, whereas

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FIG. 6. Nonlinear transfer due to five-wave interactions assuming strong nonlinearity with $\epsilon = 0.3$, in terms of frequency and direction domain, (a) in deep water with $h = 1000$ m, (b) intermediate depth water with $h = 20$ m, and (c) shallow water with $h = 10$ m. As in Figs. 3a,b, positive contours are represented by solid lines and negative contours by dashed lines. In (a) the global maximum, global minimum, and contour interval are, respectively, 40.7×10^{-8}, -27.4×10^{-8}, and 4.0×10^{-8}; in (b) 13.3×10^{-7}, -11.0×10^{-7}, and 1.0×10^{-7}; and finally, in (c) 37.3×10^{-7}, -31.3×10^{-7}, and 4.0×10^{-7}.

in shallow water, the nonlinear transfers due to four-wave or five-wave interactions are comparable. Assuming steep finite-amplitude waves, we showed that five-wave interactions create three-dimensional wave–wave interactions in shallow water also in deep water. Moreover, although four-wave interactions involving strongly nonlinear waves cause two-dimensional wave–wave interactions in the deep water, we showed that they cause three-dimensional wave–wave interactions in shallow water.

When the water depth is shallow, for example between 10 m with $K_p h = 3.6$ and 50 m with $K_p h = 18.1$, the maximum of the positive nonlinear transfer by four-wave interactions may be greater than that due to five-wave interactions. However, the maximum positive nonlinear transfer due to five-wave interactions results in a much larger angular spreading of energy, with respect to the original wave spectral propagation direction, than results from four-wave interactions. Unlike four-wave interactions, the maxima positive nonlinear transfer due to five-wave interactions do not occur along the original wave spectral propagation direction.

The nonlinear transfer due to five-wave interactions increases as water depth decreases and nonlinearity increases. When the water depth is less than about 10 m or $K_p h = 3.6$, and nonlinearity $\epsilon \geq 0.3$, five-wave interactions dominate over four-wave interactions. However, four-wave interactions experience exactly the opposite trend. As water depth decreases and nonlinearity ϵ increases, the nonlinear transfer due to four-wave interactions decreases also. Therefore, for extremely shallow water, as in nearshore coastal regions, five-wave interactions is the most important mechanism for the modeling and estimation of spectral wave evolution.

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APPENDIX

The Interaction Coefficients

The third-order interaction coefficients, $V^{(\pm)}$ and W of Eqs. (6) and (13), for example, are given as follows:

$$V^{(\pm)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2)} = \frac{1}{8\pi\sqrt{2}}\Bigg\{[\mathbf{K}_i\cdot\mathbf{K}_1 \pm K_iK_1 \tanh(K_ih)\tanh(K_1h)]\left[\frac{\omega_{(K_i)}\omega_{(K_1)}K_2\tanh(K_2h)}{\omega_{(K_2)}K_iK_1\tanh(K_ih)\tanh(K_1h)}\right]^{1/2}$$
$$+ [\mathbf{K}_i\cdot\mathbf{K}_2 \pm K_iK_2 \tanh(K_ih)\tanh(K_2h)]\left[\frac{\omega_{(K_i)}\omega_{(K_2)}K_1\tanh(K_1h)}{\omega_{(K_1)}K_iK_2\tanh(K_ih)\tanh(K_2h)}\right]^{1/2}$$
$$+ [\mathbf{K}_1\cdot\mathbf{K}_2 + K_1K_2 \tanh(K_1h)\tanh(K_2h)]\left[\frac{\omega_{(K_1)}\omega_{(K_2)}K_i\tanh(K_ih)}{\omega_{(K_i)}K_1K_2\tanh(K_1h)\tanh(K_2h)}\right]^{1/2}\Bigg\}, \quad \text{(A1)}$$

$$W^{(1)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \overline{W}_{(-\mathbf{K}_1,\mathbf{K}_2,-\mathbf{K}_i,\mathbf{K}_3)} - \overline{W}_{(-\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)}, \quad \text{(A2a)}$$

$$W^{(2)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \overline{W}_{(-\mathbf{K}_i,-\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} + \overline{W}_{(\mathbf{K}_2,\mathbf{K}_3,-\mathbf{K}_i,-\mathbf{K}_1)} - \overline{W}_{(\mathbf{K}_2,-\mathbf{K}_1,-\mathbf{K}_i,\mathbf{K}_3)} - \overline{W}_{(-\mathbf{K}_i,\mathbf{K}_2,-\mathbf{K}_1,\mathbf{K}_3)}$$
$$- \overline{W}_{(-\mathbf{K}_i,\mathbf{K}_3,\mathbf{K}_2,-\mathbf{K}_1)} - \overline{W}_{(\mathbf{K}_3,-\mathbf{K}_1,\mathbf{K}_2,-\mathbf{K}_i)}, \quad \text{(A2b)}$$

$$W^{(3)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = 2\overline{W}_{(-\mathbf{K}_i,-\mathbf{K}_1,-\mathbf{K}_2,\mathbf{K}_3)} - \overline{W}_{(-\mathbf{K}_i,\mathbf{K}_3,-\mathbf{K}_1,-\mathbf{K}_2)} + \overline{W}_{(-\mathbf{K}_1,-\mathbf{K}_2,-\mathbf{K}_i,\mathbf{K}_3)} - 2\overline{W}_{(-\mathbf{K}_1,\mathbf{K}_3,-\mathbf{K}_i,-\mathbf{K}_2)}, \quad \text{(A2c)}$$

$$W^{(4)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \overline{W}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} + \overline{W}_{(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_i,\mathbf{K}_3)}, \quad \text{(A2d)}$$

where

$$\overline{W}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \frac{1}{64\pi^2}\left[\frac{\omega_{(K_i)}\omega_{(K_1)}K_iK_1K_2K_3\tanh(K_ih)\tanh(K_1h)\tanh(K_2h)\tanh(K_3h)}{\omega_{(K_2)}\omega_{(K_3)}}\right]^{1/2}$$
$$\times\Bigg\{2\left(\frac{K_i}{\tanh(K_ih)} + \frac{K_1}{\tanh(K_1h)}\right) - |\mathbf{K}_1 + \mathbf{K}_3|\tanh(|\mathbf{K}_1 + \mathbf{K}_3|h)$$
$$- |\mathbf{K}_1 + \mathbf{K}_2|\tanh(|\mathbf{K}_1 + \mathbf{K}_2|h) - |\mathbf{K}_i + \mathbf{K}_3|\tanh(|\mathbf{K}_i + \mathbf{K}_3|h)$$
$$- |\mathbf{K}_i + \mathbf{K}_2|\tanh(|\mathbf{K}_i + \mathbf{K}_2|h)\Bigg\}. \quad \text{(A3)}$$

The coupling coefficients T from Eq. (8) are given by

$$T^{(1)}_{i,1,2,3} = -\frac{V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_i+\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_2+\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_2)}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}} - \frac{V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_i+\mathbf{K}_2,\mathbf{K}_2)}V^{(-)}_{(\mathbf{K}_1+\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_1)}}{\omega_{(\mathbf{K}_1+\mathbf{K}_3)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} - V^{(+)}_{(\mathbf{K}_i,-\mathbf{K}_i-\mathbf{K}_2,\mathbf{K}_2)}V^{(+)}_{(-\mathbf{K}_1-\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_1)}$$
$$\times\left\{\frac{1}{\omega_{(\mathbf{K}_1+\mathbf{K}_3)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_i+\mathbf{K}_2)} + \omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_2)}}\right\} + W^{(1)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)}, \tag{A4a}$$

$$T^{(2)}_{i,1,2,3} = -V^{(-)}_{(\mathbf{K}_3,\mathbf{K}_1,\mathbf{K}_3-\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_2,\mathbf{K}_i-\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_i-\mathbf{K}_2)} - \omega_{(\mathbf{K}_i)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_2,\mathbf{K}_i,\mathbf{K}_2-\mathbf{K}_i)}V^{(-)}_{(\mathbf{K}_1,\mathbf{K}_3,\mathbf{K}_1-\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_2-\mathbf{K}_i)} - \omega_{(\mathbf{K}_2)}}\right\}$$
$$+ V^{(-)}_{(\mathbf{K}_2,\mathbf{K}_1,\mathbf{K}_2-\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_3,\mathbf{K}_i-\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} - \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_3-\mathbf{K}_i)} - \omega_{(\mathbf{K}_i)}}\right\}$$
$$+ V^{(-)}_{(\mathbf{K}_3,\mathbf{K}_i,\mathbf{K}_3-\mathbf{K}_i)}V^{(-)}_{(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_1-\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_3-\mathbf{K}_i)} - \omega_{(\mathbf{K}_3)}}\right\}, \tag{A4b}$$

$$T^{(3)}_{i,1,2,3} = -V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_i+\mathbf{K}_3,\mathbf{K}_3)}V^{(-)}_{(\mathbf{K}_1+\mathbf{K}_2,\mathbf{K}_2,\mathbf{K}_1)}\left\{\frac{1}{\omega_{(\mathbf{K}_i+\mathbf{K}_3)} - \omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,-\mathbf{K}_i+\mathbf{K}_1)}V^{(-)}_{(-\mathbf{K}_2+\mathbf{K}_3,\mathbf{K}_2,\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_3)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_i-\mathbf{K}_1)} + \omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_1)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_i-\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_2-\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_i-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_i)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_1-\mathbf{K}_3,\mathbf{K}_1,\mathbf{K}_3)}V^{(-)}_{(\mathbf{K}_2,\mathbf{K}_i,\mathbf{K}_i-\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_i-\mathbf{K}_2)} + \omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_2)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_i-\mathbf{K}_2,\mathbf{K}_2)}V^{(-)}_{(\mathbf{K}_1-\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_1)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_i-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_i)}}\right\}$$
$$- V^{(+)}_{(\mathbf{K}_i,\mathbf{K}_3,-\mathbf{K}_i-\mathbf{K}_3)}V^{(+)}_{(-\mathbf{K}_1-\mathbf{K}_2,\mathbf{K}_1,\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_i+\mathbf{K}_3)} + \omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_3)}}\right\}$$
$$+ W^{(3)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)}, \tag{A4c}$$

and

$$T^{(4)}_{i,1,2,3} = -\frac{V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_i+\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_2+\mathbf{K}_3,\mathbf{K}_2,\mathbf{K}_3)}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}} - \frac{V^{(-)}_{(\mathbf{K}_i,\mathbf{K}_i+\mathbf{K}_2,\mathbf{K}_2)}V^{(-)}_{(\mathbf{K}_1+\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_1)}}{\omega_{(\mathbf{K}_1+\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_3)}}$$
$$- V^{(+)}_{(\mathbf{K}_i,\mathbf{K}_1,-\mathbf{K}_i-\mathbf{K}_1)}V^{(+)}_{(-\mathbf{K}_2-\mathbf{K}_3,\mathbf{K}_3,\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}_i+\mathbf{K}_1)} + \omega_{(\mathbf{K}_i)} + \omega_{(\mathbf{K}_1)}}\right\}$$
$$+ W^{(4)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)}. \tag{A4d}$$

The coupling coefficients R of Eq. (13) are defined in terms of functionals E,

$$R^{(2)}_{i,1,2,3,4} = E^{(+)}_{i,4,2,3,-1} + E^{(+)}_{i,3,2,-1,4} + E^{(-)}_{i,2,-1,3,4} - E^{(+)}_{i,-1,2,3,4}, \tag{A5a}$$
$$R^{(3)}_{i,1,2,3,4} = E^{(+)}_{i,3,4,-1,-2} - E^{(+)}_{i,-1,3,-2,4} - E^{(+)}_{i,-1,4,3,-2} + E^{(-)}_{i,3,-2,-1,4} + E^{(-)}_{i,4,-2,3,-1} - E^{(-)}_{i-1,-2,3,4}, \tag{A5b}$$

which in turn are specified in terms of coefficients F and J,

$$E^{(+)}_{i,1,2,3,4} = -\frac{[K_3K_4 \tanh|K_3h| \tanh|K_4h|]^{1/4}}{32\sqrt{2}\,\pi^3} \times \left\{\left(\frac{\omega_{(\mathbf{K}_2)}}{\omega_{(\mathbf{K}_i)}\omega_{(\mathbf{K}_1)}}\right)^{1/2}[F_{i,1,2,3,4} - K_1(\mathbf{K}_2\cdot\mathbf{K}_3)(|\mathbf{K}_1 + \mathbf{K}_4| \tanh(|\mathbf{K}_1 + \mathbf{K}_4|h) - K_1)] + \left(\frac{\omega_{(\mathbf{K}_i)}}{\omega_{(\mathbf{K}_1)}\omega_{(\mathbf{K}_2)}}\right)^{1/2}[K_1 \tanh(K_1h)J_{i-1,2,3,4} + \left(\frac{\mathbf{K}_2\cdot\mathbf{K}_4}{2}\right)(K_1K_2 \tanh(K_1h) \tanh(K_2h)) - \frac{K_1K_2}{2}(|\mathbf{K}_1 + \mathbf{K}_4| \tanh(|\mathbf{K}_1 + \mathbf{K}_4|h) \tanh(K_1h) - K_1) \times (|\mathbf{K}_2 + \mathbf{K}_3| \tanh(|\mathbf{K}_2 + \mathbf{K}_3|h) \tanh(K_2h) - K_2)]\right\}, \quad \text{(A6a)}$$

$$E^{(-)}_{i,1,2,3,4} = -\frac{[K_3K_4 \tanh|K_3h| \tanh|K_4h|]^{1/4}}{32\sqrt{2}\,\pi^3} \times \left\{\left(\frac{\omega_{(\mathbf{K}_2)}}{\omega_{(\mathbf{K}_i)}\omega_{(\mathbf{K}_1)}}\right)^{1/2}[F_{i,1,2,3,4} - K_1(\mathbf{K}_2\cdot\mathbf{K}_3)(|\mathbf{K}_1 + \mathbf{K}_4| \tanh(|\mathbf{K}_1 + \mathbf{K}_4|h) - K_1)] - \left(\frac{\omega_{(\mathbf{K}_i)}}{\omega_{(\mathbf{K}_1)}\omega_{(\mathbf{K}_2)}}\right)^{1/2}[K_1 \tanh(K_1h)J_{i-1,2,3,4} + \left(\frac{\mathbf{K}_2\cdot\mathbf{K}_4}{2}\right)(K_1K_2 \tanh(K_1h) \tanh(K_2h)) - \frac{K_1K_2}{2}(|\mathbf{K}_1 + \mathbf{K}_4| \tanh(|\mathbf{K}_1 + \mathbf{K}_4|h) \tanh(K_1h) - K_1) \times (|\mathbf{K}_2 + \mathbf{K}_3| \tanh(|\mathbf{K}_2 + \mathbf{K}_3|h) \tanh(K_2h) - K_2)]\right\}. \quad \text{(A6b)}$$

Coefficients $F$and J are defined in terms of functions P and O, as

$$F_{i,1,2,3,4} = -\frac{1}{6}K_1^4 + K_i \tanh(K_ih)O_{i,1,2,3,4} + |\mathbf{K}_i + \mathbf{K}_2|^2P_{i-2,1,3,4} + \frac{|\mathbf{K}_1 + \mathbf{K}_2|^2}{2}(K_1|\mathbf{K}_1 + \mathbf{K}_2| \tanh|\mathbf{K}_1h| \tanh(|\mathbf{K}_1 + \mathbf{K}_2|h)), \quad \text{(A7)}$$

and

$$J_{i,1,2,3} = K_i \tanh(K_ih)P_{i,1,2,3} - \frac{K_1 \tanh(K_1h)}{2}\left[K_1^2 - \frac{1}{2}(|\mathbf{K}_i - \mathbf{K}_2|^2 + |\mathbf{K}_i - \mathbf{K}_3|^2 + |\mathbf{K}_1 + \mathbf{K}_2|^2 + |\mathbf{K}_1 + \mathbf{K}_3|^2)\right]. \quad \text{(A8)}$$

Finally, functions P and O may be expressed as

$$P_{i,1,2,3} = \frac{|\mathbf{K}_1|}{4}[2K_1 - \tanh|K_1h|(|\mathbf{K}_i - \mathbf{K}_2| \tanh(|\mathbf{K}_i - \mathbf{K}_2|h) + |\mathbf{K}_i - \mathbf{K}_3| \tanh(|\mathbf{K}_i - \mathbf{K}_3|h) + |\mathbf{K}_1 + \mathbf{K}_2| \tanh(|\mathbf{K}_1 + \mathbf{K}_2|h) + |\mathbf{K}_1 + \mathbf{K}_3| \tanh(|\mathbf{K}_1 + \mathbf{K}_3|h))], \quad \text{(A9)}$$

and

$$O_{i,1,2,3,4} = \frac{K_1^3}{6} \tanh|\mathbf{K}_1h| - \frac{K_1}{2} \tanh|\mathbf{K}_1h|\,|\mathbf{K}_1 + \mathbf{K}_2|^2 - |\mathbf{K}_i - \mathbf{K}_2| \tanh(|\mathbf{K}_i - \mathbf{K}_2|h)P_{i-2,1,3,4}. \quad \text{(A10)}$$

To complete a definition of the analysis of this study, we give the kernels $Q^{(2)}_{i,1,2,3,4}$ and $Q^{(3)}_{i,1,2,3,4}$ in the integrals in Eq. (14),

$$
\begin{aligned}
Q^{(2)}_{i,1,2,3,4} = {} & \frac{V^{(-)}_{i,3+4,2-1} V^{(-)}_{3+4,3,4} V^{(-)}_{2-1,1,2}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_1)} + \omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{V^{(-)}_{i,4-1,2+3} V^{(-)}_{4-1,1,4} V^{(-)}_{2+3,2,3}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_4-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_4)}} + \frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} + \omega_{(\mathbf{K}_4-\mathbf{K}_1)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{2V^{(-)}_{i,1-3,2+4} V^{(-)}_{1-3,3,1} V^{(-)}_{2+4,2,4}}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} + \omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{2V^{(-)}_{i,-3-4,2-1} V^{(+)}_{-3-4,3,4} V^{(-)}_{2-1,1,2}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_1)} + \omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{V^{(+)}_{i,1-3,-2-4} V^{(+)}_{-2-4,4,2} V^{(-)}_{1-3,3,1}}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(-\mathbf{K}_2-\mathbf{K}_4)} + \omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{V^{(+)}_{i,-3-4,1-2} V^{(+)}_{-3-4,3,4} V^{(-)}_{1-2,2,1}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} + \omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& - \frac{V^{(-)}_{i,2,3+4-1} T^{(2)}_{3+4-1,1,4,3}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} - \frac{2V^{(-)}_{i,1,2+3+4} T^{(1)}_{2+3+4,3,4,2}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} \\
& - \frac{2V^{(-)}_{i,1-3-4,2} T^{(3)}_{1-3-4,3,4,1}}{\omega_{(\mathbf{K}_1-\mathbf{K}_3-\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_1)}} - \frac{V^{(+)}_{i,-2-3-4,1} T^{(4)}_{-2-3-4,3,4,2}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)}} \\
& - \frac{2V^{(-)}_{4-1,1,4} W^{(1)}_{i,4-1,2,3}}{\omega_{(\mathbf{K}_4-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_4)}} - \frac{2V^{(-)}_{4-1,1,4} W^{(1)}_{i,2,4-1,3}}{\omega_{(\mathbf{K}_4-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_4)}} - \frac{2V^{(-)}_{3-1,1,3} W^{(1)}_{i,4,2,3-1}}{\omega_{(\mathbf{K}_3-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_3)}} \\
& - \frac{2V^{(-)}_{1-4,4,1} W^{(2)}_{i,1-4,2,3}}{\omega_{(\mathbf{K}_1-\mathbf{K}_4)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_1)}} - \frac{V^{(-)}_{2+4,4,2} W^{(2)}_{i,1,2+4,3}}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} - \frac{V^{(-)}_{3+4,4,3} W^{(2)}_{i,1,2,3+4}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} \\
& - \frac{V^{(+)}_{-2-4,4,2} W^{(3)}_{i,-2-4,1,3}}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_4)}} - \frac{V^{(+)}_{-2-4,4,2} W^{(3)}_{i,1,-2-4,3}}{\omega_{(\mathbf{K}_2+\mathbf{K}_4)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_4)}} + R^{(2)}_{i,1,2,3,4}
\end{aligned}
\tag{A11a}
$$

and

$$
\begin{aligned}
Q^{(3)}_{i,1,2,3,4} = {} & \frac{V^{(-)}_{i,3+4,-2-1} V^{(-)}_{3+4,3,4} V^{(+)}_{-2-1,1,2}}{(\omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)})(\omega_{(\mathbf{K}_2+\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})} + \frac{2V^{(-)}_{i,4-1,-2+3} V^{(-)}_{4-1,1,4} V^{(-)}_{-2+3,2,3}}{\omega_{(\mathbf{K}_4-\mathbf{K}_1)} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_4)}} \\
& \times \left\{ \frac{1}{\omega_{(\mathbf{K}_3-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(-\mathbf{K}_2+\mathbf{K}_3)} + \omega_{(\mathbf{K}_4-\mathbf{K}_1)} - \omega_{(\mathbf{K}_i)}} \right\} \\
& + \frac{V^{(-)}_{i,-1-2,3+4} V^{(-)}_{3+4,3,4} V^{(+)}_{-1-2,1,2}}{(\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})(\omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)})} \\
& + \frac{V^{(-)}_{i,1+2,3+4} V^{(-)}_{1+2,1,2} V^{(-)}_{3+4,3,4}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_1)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_i)} - \omega_{(\mathbf{K}_1+\mathbf{K}_2)} - \omega_{(\mathbf{K}_3+\mathbf{K}_4)}} \right\} \\
& + \frac{2V^{(-)}_{i,1-3,4-2} V^{(-)}_{1-3,3,1} V^{(-)}_{4-2,2,4}}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_4-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} + \frac{1}{\omega_{(\mathbf{K}_4-\mathbf{K}_2)} + \omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_i)}} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2V^{(-)}_{i,-3-4,-1-2} V^{(+)}_{-3-4,3,4} V^{(+)}_{-1-2,1,2}}{(\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)})(\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)})} \\
&+ \frac{V^{(+)}_{i,1+2,-3-4} V^{(-)}_{1+2,1,2} V^{(+)}_{-3-4,3,4}}{(\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)})(\omega_{(\mathbf{K}_2+\mathbf{K}_1)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})} \\
&+ \frac{2V^{(+)}_{i,1-3,2-4} V^{(-)}_{1-3,3,1} V^{(-)}_{2-4,4,2}}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)}} \left\{ \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_4)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K}_2-\mathbf{K}_4)} + \omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_i)}} \right\} \\
&+ \frac{V^{(+)}_{i,-3-4,1+2} V^{(+)}_{-3-4,3,4} V^{(-)}_{1+2,1,2}}{(\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)})(\omega_{(\mathbf{K}_1+\mathbf{K}_2)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)})} - \frac{V^{(-)}_{i,3,4-1-2} T^{(3)}_{4-1-2,1,2,4}}{\omega_{(\mathbf{K}_4-\mathbf{K}_1-\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} \\
&- \frac{2V^{(-)}_{i,1,4+3-2} T^{(2)}_{4+3-2,2,4,3}}{\omega_{(\mathbf{K}_2+\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} - \frac{2V^{(-)}_{i,1+2-3,4} T^{(2)}_{1+2-3,3,1,2}}{\omega_{(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)}} \\
&- \frac{V^{(+)}_{i,1,2-3-4} T^{(3)}_{2-3-4,3,4,2}}{\omega_{(\mathbf{K}_2-\mathbf{K}_3-\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)}} - \frac{V^{(+)}_{i,1-3-4,2} T^{(3)}_{1-3-4,3,4,1}}{\omega_{(\mathbf{K}_1-\mathbf{K}_3-\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_1)}} \\
&- \frac{V^{(+)}_{-1-2,1,2} W^{(1)}_{i,-1-2,4,3}}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} - \frac{V^{(+)}_{-1-2,1,2} W^{(1)}_{i,4,-1-2,3}}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} - \frac{V^{(+)}_{-1-2,1,2} W^{(1)}_{i,3,4,-1-2}}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} \\
&- \frac{V^{(-)}_{1+2,1,2} W^{(2)}_{i,1+2,4,3}}{\omega_{(\mathbf{K}_1+\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}} - \frac{2V^{(-)}_{4-2,2,4} W^{(2)}_{i,1,4-2,3}}{\omega_{(\mathbf{K}_4-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} - \frac{2V^{(-)}_{4-2,2,4} W^{(2)}_{i,1,3,4-2}}{\omega_{(\mathbf{K}_4-\mathbf{K}_2)} + \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_4)}} \\
&- \frac{2V^{(-)}_{1-4,4,1} W^{(3)}_{i,1-4,2,3}}{\omega_{(\mathbf{K}_1-\mathbf{K}_4)} + \omega_{(\mathbf{K}_4)} - \omega_{(\mathbf{K}_1)}} - \frac{2V^{(+)}_{2-4,4,2} W^{(3)}_{i,1,2-4,3}}{\omega_{(\mathbf{K}_2-\mathbf{K}_4)} - \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_4)}} - \frac{V^{(-)}_{3+4,4,3} W^{(3)}_{i,1,2,3+4}}{\omega_{(\mathbf{K}_4+\mathbf{K}_3)} - \omega_{(\mathbf{K}_3)} - \omega_{(\mathbf{K}_4)}} \\
&- \frac{V^{(+)}_{-4-3,4,3} W^{(4)}_{i,-4-3,2,1}}{\omega_{(\mathbf{K}_4+\mathbf{K}_3)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)}} - \frac{V^{(+)}_{-4-3,4,3} W^{(4)}_{i,1-4-3,2}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)}} - \frac{V^{(+)}_{-3-4,4,3} W^{(4)}_{i,1,2,-3-4}}{\omega_{(\mathbf{K}_3+\mathbf{K}_4)} + \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_4)}} + R^{(3)}_{i,1,2,3,4}. \qquad \text{(A11b)}
\end{aligned}
$$

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