第29卷第4期 上海师范大学学报(自然科学版) Vol. 29, No. 4
2000年12月 J. of Shanghai Teachers Univ. (Natural Sciences) Dec. 2000

A Theorem of Uniquely Bipancyclic Graphs

SUN Hao-ping[1], SHI Yong-bing[2], SUN Jia-shu[3]

(1. Survey Section, Zhangjiagang Import and Export Commodity Inspection Bureau, Jiangsu 215633, China; 2. College of Mathematical Sciences, Shanghai Teachers University, Shanghai 200234, China; 3. Tonghe Middle School, Shanghai 200431, China)

Abstract: A bipartite grabp G is said to be uniquely bipancyclic (G is called "UB-graph" for short) if G contains exactly one cycle of every even length t, $4 \leqslant t \leqslant v$, where v is the order of G. We prove that there exist exactly six UB-graphs with $v+4$ edges.

Keywords: cycle; bipartite graph; UB-graph; r-UB-graph

CLC number: O157.5 **Document Code:** A **Article ID:** 1000-5137(2000)04-0024-11

1 Introduction and Notation

In 1973, ENTRINGER R C raised the problem of determining which graph G is uniquely pancyclic[1], that is, G contains exactly one cycle of each length t, $3 \leqslant t \leqslant |V(G)|$. In 1983, YAP H P and TEO S K generalized the notion of a uniquely pancyclic graph and defined a notion of a uniquely r- pancyclic graph[6]. A graph G of order v is said to be uniquely r-pancyclic if G contains exactly one cycle of length t for each $r \leqslant t \leqslant v$, and G contains no cycle of length less than r. In [2] and [4], several important results of uniquely r- pancycle graphs have been obtained. In 1991, SHI Yong-bing and SUN Jia-shu considered analogous questions relating to bipartite graphs[3]. A bipartite graph G of order v (where v is even) is said to be uniquely r- bipancyclic (We call G "r- UB-graph" for short, where $r \geqslant 4$ is even) if G contains exactly one cycle of every even length t, $r \leqslant t \leqslant v$, and G contains no cycle of length less than r. We usually abbreviate "4-UB-graph" to "UB-graph".

In [3], the class of r- UB-graphs with $v+m$ edges for $m \leqslant 3$ is completely determined.

Received date: 2000-03-17

Biography: SUN Hao-ping(1970-), male, Master, Survey Section, Zhangjiagang Import and Export Commodity Inspection Burean; SHI Yong-bing(1947-), male, Professor, College of Mathematical Science, Shanghai Teachers University.

In this paper, we determine the class of UB-graphs with $v+4$ edges.

All definitions and notation used in this paper, but not defined in the following, can be found in [3].

Let C' be a cycle of graph G. If there exists another cycle in G wihich contains all bridges contained in C', we call this cycle the dual of C' and express it by $D(C')$.

A graph G is said to be a skew[1] graph if the number of the skew bridges of G is 2.

In [5], we have the following result.

Result 1.1 Let G be a skew[1] graph. Then G is an UB-graph if and only if $G \in \{H_{2t}^{(i)} | i = 1,2,\cdots,6\}$.

2 UB-graph with 4 bridges

Lemma 2.1 If G is a 4-skew UB-graph, then G contains one 4-bridge.

Proof By contradiction. Suppose that G contains no 4-bridge. Then G must contain one 4-inner cycle, say C_4. We consider the number of the bridges contained in C_4. There are three possible cases only.

Case 1 C_4 contains exactly two bridges, say B_1 and B_2. Clearly B_1 and B_2 are not skew (for otherwise, G has two Hamilton cycles. A contradiction). By Lemma 3.4 in[3], we only need to consider the case that the other bridges B_3 and B_4 do not skew to both B_1 and B_2. In this case, B_3 and B_4 are skew. Since G contains no 4-bridge, the $(v-2)$-cycle of G must be a skew cycle. We may assume that $B_1 = v_1v_2$ and there are no bridges in $\mathrm{int}(C[v_1,v_2] \cup v_2v_1)$. Let $G^* = G - C(v_1,v_2)$ and $v^* = |V(G^*)|$. Then G^* contains one 4-bridge and one (v^*-2)-*skew* cycle. That is to say, G^* contains two (v^*-2)-cycle. A contradition.

Case 2 C_4 contains exactly three bridges, say B_1, B_2 and B_3. Clearly there do not exist two skew bridges in $\{B_1,B_2,B_3\}$. Otherwise, suppose B_1 and B_2 are skew, See $G(v_1v_3,v_2v_4,v_1v_2)$. Let $G^* = G - C(v_1,v_2)$ and $v^* = |V(G^*)|$. Then G^* contains two v^*-cycles. A contradiction. See $G(v_1v_2,v_2v_3,v_3v_4)$. It is easily seen that there exists $B \in \{B_1,B_2,B_3\}$ such that it does not skew to B_4, say $B = B_1$. Let $G^* = G - C(v_1,v_2)$. Then G^* similar to G in Lemma 3.4 in [3]. In a similar way, we can prove that G^* contains two cycles which have the same length. Again a contradiction.

Case 3 C_4 contains exactly four bridges, say B_1, B_2, B_3 and B_4. Since G is a skew graph, there are two skew bridges in $\{B_1,B_2,B_3,B_4\}$, see $G(v_1v_4,v_1v_3,v_2v_3,v_2v_4)$. Let $G^* = G-(C(v_2,v_3) \cup C(v_4,v_1))$ and $v^* = |V(G^*)|$. Then G^* contains two v^*-cycles. Once more a contradiction.

From the above proof, we can conclude that every 4-skew UB-graph has a 4-bridge.

Lemma 2.2 If G is a 4-skew UB-graph, then G contains one 6-bridge.

Proof By contradiction. Suppose that G contains no 6-bridge. Then G contains one 6-

inner cycle, say C_6, and the $(v-4)$-cycle contained in G, say C', is a skew cycle. Clearly, C' does not contain 4-bridges in G (for otherwise, G contains two $(v-2)$-cycles. A contradiction). We now consider the number of the bridges contained in C_6. There are three possible cases only.

Case 1 C_6 contains exactly two bridges, say B_1 and B_2. If one of the bridges B_1 and B_2 is a 4-bridge, say B_1, then B_2 and the other two bridges B_3 and B_4 do not skew to B_1. Clearly, $B_1 \in \text{ext}C'$ (if $B_1 \in \text{int}C'$, then G contains two $(v-6)$-cycles, that is, $D(C(B_2))$ and $(C' - C(B_1) \cap (C) \cup B_1$. A contradiction), and there are three bridges contained in C' (for otherwise, C' contains exactly two skew bridges and G contains two 8-cycles, i e. $C(B_2)$ and $D(C')$. A contradiction). It is easily seen that these three bridges contained in C' skew to each other (for otherwise, there exists one bridge in C', say B, which does not skew to the other two bridges in C'. Then $B = B_1$. This contradicts that $B_1 \in \text{ext}C'$), then G contains two 8-cycles (One is $C(B_2)$, and the other is the cycle which contains four bridges). A contradiction. Therefore neither B_1 nor B_2 is a 4-bridge. We may assume that B_4 is a 4-bridge. Clearly C_6 is not a skew cycle (for otherwise, G contains two $(v-2)$-cycles, i.e. $D(C(B_4))$ and $D(C_6)$. A contradiction), then B_1 and B_2 are not skew. We have two subcases.

Case 1.1 One of the bridges B_1 and B_2 skews to B_3. We may assume that B_1 skews to B_3. Since G does not contain 6-bridge, $|C(B_2)| \geqslant 8$ and $B_2 \in \text{int}C'$. Let $B_2 = v_2v_3$ and neither B_1 nor B_2 be contained in $\text{int}(C[v_2,v_3] \cup v_3v_2)$. Furthermore, let $G^* = G - C(v_2,v_3)$ and $v^* = |V(G^*)|$. Then G^* contains one 6-bridge and one (v^*-4)-skew cycle. Now G^* contains two (v^*-4)-cycles. A contradiction.

Case 1.2 Both B_1 and B_2 skew to B_3, see $G(v_1v_7, v_2v_6, v_3v_8, v_4v_5)$. Clearly C' does not contain B_4. If $B_4 \in \text{ext}C'$, then C' contains exactly three bridges (for otherwise, G contains two 6-cycles, i.e. C_6 and $(D(C') - C(B_4) \cap C) \cup B_4$. A contradiction. Specifically C' is the cycle $C[v_4,v_7] \cup v_7v_1 \cup C[v_8,v_1] \cup v_8v_3 \cup C[v_2,v_3] \cup v_2v_6$. Now we consider the number of the vertices contained in $C[v_1,v_2]$.) Clearly, $m_{12} \neq 4$. Then $m_{12} \in \{0,1,2,3\}$. Table 1 give all the four cases and the causes of impossibility of each case.

Table 1 The four cases of m_{12}

m_{12}	Cause (the cycle(s) contained in G)
0	one 11-cycle: $C[v_3,v_6] \cup v_3v_6$
1	two 10-cycle: $C[v_3,v_8] \cup v_3v_8$ and $C[v_3,v_6] \cup v_6v_2 \cup C[v_1,v_2] \cup v_1v_7 \cup C[v_7,v_8] \cup v_8v_3$
2	two 6-cycle: $C[v_1,v_4] \cup v_4v_5 \cup C[v_5,v_6] \cup v_2v_6$ and C_6
3	two 6-cycle: $C[v_3,v_8] \cup v_3v_8$ and C_6

Therefore, $B_4 \in \text{int}C'$. We now consider the number of the bridges contained in C'. There are two possible subcases only.

Case 1.2.1 C' contains exactly two bridges of $\{B_1,B_2,B_3\}$. We consider m_{12}. Clearly

$m_{12} \in \{0,1,2,3,4\}$ and $m_{12} \neq 2$ (for otherwise, G contains two cycles $C[v_1,v_3] \cup v_3v_8 \cup C[v_7,v_8] \cup v_1v_7$ and $C[v_2,v_3] \cup v_3v_8 \cup C[v_6,v_8] \cup v_2v_6$ which have the same length. A contradiction). If $m_{12} = 0$, then C' contains exactly B_1 and B_3 (for otherwise, C' contains exactly B_2 and B_3. Then the length of the cycle which contains exactly B_1 and B_3 is 4. A contradiction). In this case, G contains one 8-skew cycle, i e. $D(C')$, and one 12-skew cycle, say C_{12}, which contains exactly B_2 and B_3, and $B_4 \in \text{ext}C_{12}$. Thus, $B_4 \in \text{ext}C_{10}$ (the 10-cycle contained in G). Furthermore, C_{10} contains only one bridge. It leads to that G contains two $(v-8)$-cycles, i e. $D(C_{10})$ and $D(C_{12})$. A contradiction. If $m_{12} = 1$, then C' contains exactly B_1 and B_3 (for otherwise, C' contains exactly B_2 and B_3. The length of the cycle which contains exactly B_1 and B_3 is 6, a contradiction). Thus, G contains two $(v-6)$-cycls, that is, $C[v_8,v_2] \cup v_2v_6 \cup C[v_3,v_6] \cup v_3v_8$ and $C[v_3,v_4] \cup v_4v_5 \cup C[v_5,v_7] \cup v_7v_1 \cup C[v_8,v_1] \cup v_3v_8$. A contradiction. It is analogous to prove that $m_{12} \notin \{4,3\}$.

Table 2 The five cases of m_{23}

m_{23}	k	$(v^* - k - 1)$-cycle	$(v^* - k)$-cycle
1	4	$C[v_1,v_7] \cup v_1v_7$	$C[v_3,v_7] \cup v_7v_1 \cup C[v_8,v_1] \cup v_8v_3$
2	4	$C[v_8,v_1] \cup v_1v_7 \cup C[v_7,v_3] \cup v_3v_8$	$C[v_1,v_7] \cup v_1v_7$
3	5	$C[v_8,v_1] \cup v_1v_7 \cup C[v_7,v_3] \cup v_3v_8$	$C[v_1,v_4] \cup v_4v_5 \cup C[v_5,v_7] \cup v_1v_7$
4	5	$C[v_2,v_7] \cup v_2v_7$	$C[v_8,v_2] \cup v_2v_7 \cup C[v_5,v_7] \cup v_4v_5 \cup C[v_3,v_4] \cup v_3v_8$
5	7	$C[v_8,v_1] \cup v_1v_7 \cup C[v_1,v_7] \cup v_3v_8$	$C[v_2,v_4] \cup v_4v_5 \cup C[v_5,v_7] \cup v_2v_7$

Case 1.2.2 C' contains exactly B_1, B_2 and B_3. In this case, C' is the cycle $C[v_1,v_2] \cup v_2v_6 \cup C[v_3,v_6] \cup v_3v_6 \cup C[v_7,v_8] \cup v_1v_7$. We also consider m_{12}. Clearly $m_{12} \in \{1,3,4\}$. If $m_{12} = 1$, then G contains two 6-cycles, that is, $C[v_8,v_3] \cup v_3v_8$ and C_6. A contradiction. When $m_{12} = 3$, let $G^* = G - C(v_2,v_1) - C(v_7,v_8)$ and $v^* = |V(G^*)|$. Then G^* contains two (v^*-3)-cycles, i e. $v_1v_7 \cup C[v_3,v_7] \cup v_3v_8 \cup C[v_8,v_1]$ and $v_2v_6 \cup C[v_5,v_6] \cup v_4v_5 \cup C[v_4,v_1] \cup v_3v_8 \cup C[v_8,v_2]$. Again a contradition. Now $m_{12} = 4$. Let $G^* = G - C(v_7,v_8)$ and $v^* = |V(G^*)|$. Then G^* contains one (v^*-k)-cycle and one (v^*-k-1)-cycle for some $k \in \{4,5,7\}$ for $m_{23} \in \{1,2,3,4,5\}$ (see Table 2). That is to say, G always contains an odd length cycle. Once more a contradiction.

Case 2 C_6 contains exactly three bridges, say B_1, B_2 and B_3. We shall now consider three subcases, depending on B_1, B_2 and B_3 are skew bridges or not. Let B_4 be the other bridge.

Case 2.1 Any two of the three bridges B_1, B_2 and B_3 does not skew to each other. Since G is a skew graph, one of the bridges B_1, B_2 and B_3 must be a 4-bridge, say B_1.

Case 2.1.1 B_4 skews to one of the bridges B_2 and B_3, say B_2, see $B(v_1v_2, v_1v_5, v_6v_7, v_4v_6)$. Clearly $B_3 \in \text{int}C'$ (for $|C(B_3)| \geq 8$). Then $B_1 \in \text{int}C'$. Furthermore, C' contains exactly B_2 and B_4, G contains two 8-cycles, i e. $(C_6 - B_3) \cup (C \cap C(B_1))$ and $D(C')$. A contradiction.

Case 2.1.2 B_4 skews to both B_2 and B_3, See $G(v_1v_2, v_3v_5, v_6v_8, v_4v_7)$. If C' contains exactly two bridges, then G contains two 8-cycles, i.e. $D(C')$ and $(C_6 - B_1) \cup (C \cap C(B_1))$. A contradiction. Therefore C' contains three bridges, that is, B_2, B_3 and B_4. If $B_1 \in \text{ext}C'$, then the length of the cycle containing four bridges is 8. G contains two 8-cycles (the other is $(C_6 - B_1) \cup (C \cap C(B_1))$). A contradition. Thus $B_1 \in \text{int}C'$ and C' is the cycle $C[v_8, v_3] \cup v_3v_5 \cup C[v_4, v_5] \cup v_4v_7 \cup C[v_5, v_7] \cup v_6v_5$. We now consider m_{56}. Clearly $m_{56} \in \{1, 2, 3\}$. Let $G^* = G - C(v_5, v_7) - C(v_3, v_4)$ and $v^* = |V(G^*)|$. Table 3 gives all the three cases and the cause of impossibility of each case.

Table 3 The three cases of m_{56}

m_{56}	Cause (the cycle(s) contained in G or G^*)
1	two 10-cycles: $C[v_2, v_4] \cup v_4v_7 \cup C[v_7, v_1] \cup v_1v_2$ and $C[v_3, v_4] \cup v_4v_7 \cup C[v_7, v_8] \cup v_8v_6 \cup C[v_5, v_6] \cup v_5v_3$
2	two $(v^* - 4)$-cycles: $C[v_7, v_8] \cup v_8v_6 \cup C[v_4, v_6] \cup v_4v_7$ and $C[v_7, v_1] \cup v_1v_2 \cup C[v_2, v_3] \cup v_3v_5 \cup C[v_4, v_5] \cup v_4v_7$
3	two $(v^* - 3)$-cycles: $C[v_4, v_6] \cup v_6v_8 \cup C[v_7, v_8] \cup v_4v_7$ and $C[v_4, v_5] \cup v_5v_3 \cup C[v_7, v_3] \cup v_4v_7$

Case 2.2 Exactly two of the bridge B_1, B_2 and B_3 skew to each other, say B_2 and B_3. If B_1 is not a 4-bridge, then C' is the cycle which contains exactly B_2 and B_3 such that $B_1 \in intC'$. In this case, G contains one 6-side cycle to B_2 or B_3. Therefore G contains two 6-cycles. A contradiction. Hence B_1 is a 4-bridge. In this case, G contains one 8-cycle, say C_8, which contains exactly B_2 and B_3. Then $C' = D(C_8)$ and $B_1 \in \text{ext}C'$. Let B_4 be the fourth bridge.

let $G^* = G - E(B_4)$, see $G^*(v'_1v'_2, v'_3v'_5, v'_4v'_6)$. It is easily seen that there is no vertex contained in $C(v'_4, v'_5)$.

Case 2.2.1 B_4 does not skew to B_2 or B_3, see $G(v_1v_2, v_3v_7, v_6v_8, v_4v_5)$. Since G contains C_8 and does not contain any 6-bridge, the $(v-6)$-*cycle* of G is the cycle $C[v_7, v_8] \cup v_8v_6 \cup C[v_5, v_6] \cup v_4v_5 \cup C[v_3, v_4] \cup v_4v_7$. Therefore B_4 is a 4-bridge, too. A contradiction.

Case 2.2.2 B_4 skews to one of the bridges B_2 and B_3, say B_3. See $G(v_1v_2, v_3v_5, v_5v_7, v_4v_6)$, $G(v_1v_2, v_3v_5, v_4v_7, v_3v_6)$, $G(v_1v_2, v_3v_5, v_4v_7, v_2v_6)$, $G(v_1v_2, v_3v_5, v_4v_1, v_2v_6)$, $G(v_1v_2, v_2v_4, v_3v_6, v_5v_7)$ and $G(v_1v_2, v_2v_4, v_3v_6, v_1v_5)$. They are denoted by $G_1, G_2, \cdots, G_6$, respectively.

We now consider the $(v-6)$-cycle contained in G, say C''. Clearly $B_1 \notin C''$. Otherwise, G contains two $(v-4)$-cycles, that is, C' and $(C'' - B_1) \cup (C \cap C(B_1))$. A contradiction. Since G contains a 8-skew cycle which contains exactly B_2 and B_3 with $B_1 \in \text{int}C_8$, and G does not contain 6-bridge, C'' is a skew cycle.

If C'' contains exactly two bridges, then they are B_3 and B_4. If $B_1 \in \text{ext}C''$, then G contains one 8-cycle again, which contains exactly B_1, B_3 and B_4. Thus this case is impossible. But, when $B_1 \in \text{int}C''$, G contains $C(B_3)$, $C(B_2)$, $C(B_2)$ and $C(B_2)$ for G_1, G_2, G_3 and C_4, respectively. The lengths $C(B_3)$ and $C(B_2)$ are less than 8; For G_5 and G_6, G contains two 10-cycles. One is $D(C'')$, and the others are $C[v_4,v_5] \cup v_2v_4[\cup v_2v_1] \cup C[v_7,v_1] \cup v_7v_5$ and $C[v_4,v_6] \cup v_5v_1 \cup C[v_1,v_2] \cup v_2v_4$, respectively. Thus G always contains two cycles having the same length. A contradition.

If C'' contains B_2, B_3 and B_4, then G contains another cycle whose length is less than 8 (the discussion is similiar to the former). Again a contradiction.

Case 2. 2. 3 B_4 skews to both B_2 and B_3, see $G(v_1v_2, v_3v_6, v_5v_8, v_4v_7)$. Let $G^* = G - C(v_4,v_5) - C(v_1,v_2) - C(v_6,v_7)$, $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 2)$-cycles, i e. $C[v_7,v_1] \cup v_1v_2 \cup C[v_2,v_4] \cup v_4v_7$ and $C[v_3,v_4] \cup v_4v_7 \cup C[v_7,v_8] \cup v_8v_5 \cup C[v_5,v_6] \cup v_6v_3$. A contradiction.

Case 2. 3 There exists one of the bridges B_1, B_2 and B_3, say B_1, which skews to the other two bridges. If B_2 and B_3 have one end-vertex, see $G(v_1v_4,\ v_2v_5,\ v_3v_5)$. C_6 is the cycle $C[v_1,v_2] \cup v_2v_5 \cup v_5v_3 \cup C[v_3,v_4] \cup v_4v_1$. Futhermore, one of $C(v_1,v_2)$ and $C(v_3,v_4)$ contains one vertex and the other does not. Without loss of generality, let $C(v_1,v_2)$ contains one. Then $C[v_5,v_2] \cup v_2v_5$ and $v_1v_4 \cup C[v_3,v_4] \cup v_3v_5 \cup C[v_5,v_1]$ have the same lengh. Otherwise, *G contains two Hamilton cycles C and $D(C_6)$* . A contradiction.

Case 3 C_6 contains B_1, B_2, B_3 and B_4. Clearly $(C_6 - B_1) \cup (C \cap C(B_1))$ is a 8-cycle. Since G is a skew graph, we may assume that B_2 and B_3 are skew, and B_1 is a 4-bridge.

Case 3. 1 B_4 does not skew to B_2 or B_3, see $G(v_1v_2, v_3v_5, v_4v_6, v_4v_5)$ and $G(v_1v_2, v_2v_4, v_3v_6, v_1v_5)$. Since G does not contain any 6-bridge, $B_4 \in \mathrm{int}C'$. Clearly $C[v_3,v_4] \cup v_4v_6 \cup C[v_5,v_6] \cup v_3v_5$ is also a 8-skew cycle. A contradiction.

Case 3. 2 B_4 skews to one of the bridges B_2 and B_3, say B_3, see $G(v_1v_2, v_2v_4, v_3v_6, v_1v_5)$ and $G(v_1v_2, v_2v_5, v_4v_6, v_3v_5)$. For the former, G contains two $(v - 2)$-cycles, one of which is the cycle $C[v_2,v_3] \cup v_3v_6 \cup C[v_6,v_1] \cup v_5v_1 \cup C[v_4,v_5] \cup v_2v_4$. For the latter, C' contains only two skew bridges. Then G contains a 8-skew cycle, i e. $D(C')$. It always leads to a contradiction.

Case 3. 3 B_4 skews to both B_2 and B_3, see $G(v_1v_2, v_2v_5, v_4v_1, v_3v_6)$ and $G(v_1v_2, v_3v_5, v_4v_6, v_2v_3)$. G contains two $(v - 2)$-cycles, one of which contains exactly B_2, B_3 and B_4. A contradition.

Lemma 2. 3 Let $G(v_1v_2, v_3v_5, v_4v_6)$ be a graph in which no two cycles have the same length, $m_{46} = 5$ and $m_{12} = 3$, then $m_{45} = 4$.

Proof By contradiction. Suppose $m_{45} \neq 4$. Then $m_{45} \in \{1,2,3\}$.

Case 1 $m_{45} = 1$. Let $G^* = G - C(v_3,v_4)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 3)$-cycles. A contradition.

Case 2 $m_{45} = 2$. Let $G^* = G - C(v_3,v_4)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 2)$-cycles. A contradition.

Case 3 $m_{45} = 3$. Let $G^* = G - C(v_6,v_3)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 2)$-cycles. Again a contradition.

Thus all the possible cases lead to contraditions, and the proof is completed □

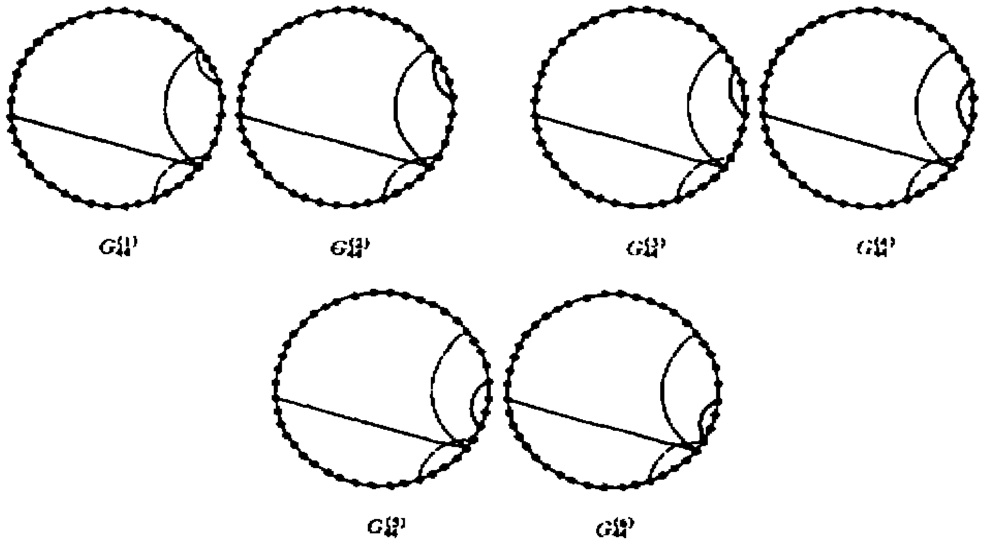

Figure 1 The six UB-graphs with $v+4$ edges

The graphs $G_{44}^{(1)}$, $G_{44}^{(2)}$, $G_{44}^{(3)}$, $G_{44}^{(4)}$, $G_{44}^{(5)}$ and $G_{44}^{(6)}$ are depicted in Figure 1.

Theorem 2. 1 A 4-skew graph G is an UB-graph if and only if $G \in \{G_{44}^{(1)}, G_{44}^{(2)}, G_{44}^{(3)}, G_{44}^{(4)}, G_{44}^{(5)}, G_{44}^{(6)}\}$.

Proof The sufficiency is easily seen by immediately checking $G_{44}^{(i)}$ $(i = 1, 2, \cdots, 6)$. We shall prove the necessity. Let G be any 4-skew graph. Suppose that G is an UB-graph. By Lemma 2. 1 and Lemma 2. 2, G contains one 4-bridge and one 6-bridge, say B_1 and B_2, respectively. By Lemma 3. 1 in [3], B_1 does not skew to the other three bridges. Clearly, $B_1 \notin \mathrm{int}C(B_2)$. Let B_3 and B_4 be the other two bridges of G. We consider the number λ of the bridges which skew to B_2. It is evident that $\lambda \leqslant 2$.

If $\lambda = 0$, B_2 does not skew to the other bridges. In this case, B_3 and B_4 must skew to each other. Then G is a skew [1] graph. By Result 1. 1, this case can not arise.

If $\lambda = 1$, B_2 skews to exactly one bridge, say B_3. In this case, B_3 must skew to B_4 (by the same cause as that in the case of $\lambda = 0$). When B_2 and B_4 are not adjacent, we have two subcases and express then by 1. 1. 1 and 1. 1. 2. When B_2 and B_4 are adjacent, we have three subcases and express them by 1. 2. 1, 1. 2. 2 and 1. 2. 3.

If $\lambda = 2$, B_2 skews to both B_3 and B_4. When B_3 and B_4 are adjacent, we have two subcases and express then by 2. 1. 1 and 2. 1. 2. When B_3 and B_4 are neither adjacent nor skew, we have two subcases and express them by 2. 2. 1 and 2. 2. 2. When B_3 and B_4 skew to each other, we also have two subcases and express them by 2. 3. 1 and 2. 3. 2.

Table 4 gives some digrams and their numbers of cycles contained in G as described above.

By Lemma 2.3, we can eliminate cases 2.1.2, 2.2.1, 2.2.2 and 2.3.2 first.

For case 1.1.1. Let $G^* = G - C(v_3, v_4) - C(v_7, v_8)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 3)$-cycles, that is, $C[v_4, v_6] \cup v_5v_6 \cup C[v_2, v_3] \cup v_1v_2 \cup C[v_8, v_1] \cup v_4v_8$ and $C[v_8, v_3] \cup v_3v_6 \cup C[v_6, v_7] \cup v_7v_5 \cup C[v_4, v_5] \cup v_4v_8$, which is false.

Table 4 The various cases for $\lambda = 1$ and $\lambda = 2$

Case	Diagram	Number of cycles
1.1.1	$G(v_1v_2, v_5v_7, v_3v_6, v_4v_8)$	22
1.1.2	$G(v_1v_2, v_6v_8, v_4v_7, v_3v_5)$	23
1.2.1	$G(v_1v_2, v_3v_5, v_4v_7, v_3v_6)$	20
1.2.2	$G(v_1v_2, v_4v_8, v_3v_5, v_4v_7)$	21
1.2.3	$G(v_1v_2, v_5v_7, v_4v_6, v_3v_5)$	21
2.1.1	$G(v_1v_2, v_3v_5, v_4v_6, v_4v_7)$	21
2.1.2	$G(v_1v_2, v_4v_6, v_5v_7, v\text{-}v_5)$	22
2.2.1	$G(v_1v_2, v_3v_5, v_6v_7, v_4v_8)$	22
2.2.2	$G(v_1v_2, v_4v_7, v_6v_8, v_3v_5)$	23
2.3.1	$G(v_1v_2, v_4v_7, v_6v_1, v_5v_8)$	24
2.3.2	$G(v_1v_2, v_5v_8, v_4v_7, v_6v_1)$	24

For case 1.2.2, let $G^* = G - C(v_3, v_4) - C(v_6, v_7)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 3)$-cycles, i.e. $C[v_4, v_5] \cup v_5v_3 \cup C[v_2, v_3] \cup v_1v_2 \cup C[v_7, v_1] \cup v_4v_7$ and $C[v_5, v_6] \cup v_6v_4v_7 \cup C[v_7, v_3] \cup v_3v_5$, which is false.

For case 2.3.1. Let $G^* = G - C(v_1, v_4) - C(v_7, v_8)$ and $v^* = |V(G^*)|$. Then G^* contains two $(v^* - 2)$-cycles. One is $C[v_5, v_6] \cup v_6v_1 \cup C[v_8, v_1] \cup v_5v_8$ and the other is $C[v_4, v_5] \cup v_5v_8 \cup C[v_8, v_5] \cup v_4v_6 \cup C[v_6, v_7] \cup v_7v_4$, which is false.

We now discuss the other four cases in Table 4.

Let $M = \{x \mid x$ is the order of a cycle in $G\}$, $M^* = \{4, 6, \cdots, v-2, v\}$,

$$s = \sum_{x \in M} x, \tag{1}$$

$$\text{and } s^* = \sum_{x \in M^*} x = (v-2)(v+4)/4, \tag{2}$$

where v is the order of G.

Case 1.1.2. It is easily seen that G contains 8 cycles which contains exactly one bridge, 8 cycles which contain exactly two bridges, 5 cycles which contain exactly three bridges and one cycle which contains exactly four bridges. Hence G contains 23 cycles in all (we can count the numbers of cycles contained in G for the other cases by the same method, see Table 4).

By Lemma 2.3, we have $m_{67} = 4$ and $m_{73} = 1$.

Counting the length of every cycle in G, we have

$M = \{4,\ 6,\ m_{45} + m_{56} + 5,\ m_1 + m_{34} + 5,\ m_{34} + m_{45} + 1,\ m_1 + m_{56} + 9,$

$m_1 + m_{34} + 3$, $m_1 + m_{56} + 7$, $m_{45} + m_{56} + 3$, $m_1 + m_{34} + 9$, $m_1 + m_{56} + 5$,
$m_1 + m_{45} + 6$, $m_{34} + m_{56} + 6$, $m_1 + m_{34} + 7$, $m_1 + m_{56} + 3$, $m_1 + m_{45} + 4$,
$m_1 + m_{45} + 10$, $m_1 + m_{45} + 8$, 42, 44, 46, 48},

where $m_1 = m_{81} + m_{23}$ and

$$m_1 + m_{34} + m_{45} + m_{56} = 40. \tag{3}$$

Using (1), (2) and (3), we find easily $s = 5m_1 + m_{56} + 565$ and $s^* = 598$. Since G is UB-graph, we have $s = s^*$, i e,

$$5m_1 + m_{56} = 33. \tag{4}$$

Because each m_1 is an integer and $m_1 + m_{56} + 3 \in M$ is even, equation (4) has five solutions. Then we can obtain the values of $m_{34} + m_{45}$ from (3) about these solutions. From min $(M - \{4,6\}) = 8$, we can obtain the values of m_{34} or m_{45}.

Table 5 The five subcases of Case 1.1.2

m_1	m_{56}	$m_{34} + m_{45}$	$\min(M - \{4,6\})$	m_{34}	m_{45}	Two elements having the same value in M
0	33	7	$m_{34}+3$	5	2	$m_1+m_{45}+6$, $m_{34}+m_{45}+1$
			$m_{45}+4$	3	4	$m_1+m_{56}+7$, $m_{45}+m_{56}+3$
1	28	11	$m_{34}+4$	4	7	$m_1+m_{34}+7$, $m_{34}+m_{45}+1$
			$m_{45}+5$	8	3	$m_1+m_{34}+3$, $m_{34}+m_{45}+1$
2	23	15	$m_{45}+6$	13	2	$m_{34}+m_{56}+6$, 42
			$m_{34}+5$	3	12	$m_1+m_{56}+5$, $m_{34}+m_{56}+4$
3	18	19	$m_{34}+6$	2	17	$m_1+m_{45}+6$, $m_{34}+m_{56}+6$
			$m_{45}+7$	18	1	$m_{34}+m_{56}+6$, 42
4	13	23	$m_{34}+7$	1	22	$m_{34}+m_{56}+6$, $m_1+m_{56}+3$

Table 5 gives the five solutions of (4), the values of $m_{34} + m_{45}$, the smallest elements in $M - \{4,6\}$, the values of m_{34} and m_{45}, and two elements in M having the same value about these five solutions.

Table 5 shows that G always contains two cycles having the same length for case 1.1.2, which is false.

Case 1.2.1. By Lemma 2.3, we have $m_{45} = 4$ and $m_{34} = 1$.

we obtain easily

$M = \{4, 6, m_{56} + m_{67} + 5, m_1 + 5, m_{56} + 6, m_1 + m_{67} + 4,$
$m_1 + m_{56} + m_{67} + 2, m_1 + 3, m_1 + m_{67} + 2, m_1 + 9, m_{56} + m_{67} + 3,$
$m_{56} + 2, m_{67} + 3, m_1 + m_{56} + 9, m_1 + 7, m_1 + m_{56} + 7, m_{67} + 7, 38, 40, 42\}$,

where $m_1 = m_{71} + m_{23}$ and

$$m_1 + m_{56} + m_{67} = 34. \tag{5}$$

Using (1),(2) and (5), we find $s = 2m_1 + 442$ and $s^* = 460$. Hence

$$m_1 = 9. \tag{6}$$

In this case, $m_{56} + m_{67} = 25$ from (5). If min $(M - \{4,6\}) = m_{56} + 2$, then $m_{56} = 6$ and $m_{67} = 19$. We have $m_{56} + 6 = m_1 + 3$. If min $(M - \{4,6\}) = m_{67} + 3$, then $m_{67} = 5$ and $m_{56} =$

20, we have $m_1 + m_{56} + m_{67} + 4 = m_1 + m_{55} + 9$. That is to say, G always contains two cycles having the same length for case 1.2.1, which is false.

Case 1.2.3. By Lemma 2.3, we have $m_{56} = 4$ and $m_{67} = 1$.

we can obtain easily

$$M = \{4, 6, m_{45} + 5, m_1 + m_{34} + 5, m_{34} + m_{45} + 1, m_1 + 9, m_1 + m_{34} + m_{45} + 2, m_1 + m_{34} + 3, m_1 + 7, m_{45} + 3, m_1 + m_{34} + 9, m_1 + 5, m_{34} + 6, m_1 + m_{45} + 6, m_1 + m_{34} + 7, m_1 + 3, m_1 + m_{45} + 4, m_{34} + 4, 40, 42, 44\},$$

where $m_1 = m_{71} + m_{23}$ and

$$m_1 + m_{34} - m_{45} = 36. \tag{7}$$

Using (1), (2) and (7), we find

$$s = 5m_1 + 2m_{34} + 431 \text{ and } s^* = 504.$$

Hence

$$5m_1 + 2m_{34} = 73. \tag{8}$$

Since both $m_1 + 5 \in M$ and $m_{34} + 4 \in M$ are even, m_1 is odd and m_{34} is even. Then equation (8) has four solutions. From (7), we can obtain the values of m_{45} about these four solutions. Table 6 gives the four solutions of (8), the values of m_{45} and two elements in M having the same value about these four solutions.

Table 6 The four subcases of Case 1.2.3

m_1	m_{34}	m_{45}	Two elements having the same value in M
1	34	1	$m_{45} + 5$, 6
5	24	7	$m_{45} + 3$, $m_1 + 5$
9	14	13	$m_{45} + 5$, $m_1 + 9$
13	4	19	$m_1 + 9$, $m_1 + m_{34} + 5$

Table 6 shows that case 1.2.3 cannot arise.

Case 2.1.1. By Lemma 2.3, we have $m_{34} = 1$ and $m_{45} = 4$.

we obtain easily

$$M = \{4, 6, m_{56} + 5, m_{56} + m_{67} + 5, m_1 + 5, m_1 + m_{56} + m_{67} + 2, m_1 + m_{67} + 3, m_1 + 3, m_1 + m_{67} + 9, m_{56} + 3, m_1 + 9, m_{56} + m_{67} + 3, m_{67} + 2, m_1 + m_{67} + 7, m_1 + 7, m_1 + m_{56} + 6, m_1 + m_{56} + 4, m_1 + m_{67} + 5, 40, 42, 44\},$$

where $m_1 = m_{71} + m_{23}$ and

$$m_1 + m_{56} + m_{67} = 36. \tag{9}$$

Using (1), (2) and (9), we find $s = 4m_1 + m_{67} + 466$ and $s^* = 504$. Hence

$$4m_1 + m_{67} = 38. \tag{10}$$

Since $m_1 + 3 \in M$ is even, equation (10) has five solutions. Table 7 gives the four solutions of (10), the values of m_{56} and two elements in M having the same value about these four solutions.

Table 7 The four subcases of Case 2.1.1

m_1	m_{67}	m_{56}	Two elements having the same value in M
1	34	1	$m_{56}+5$, 6
3	26	7	m_1+3, 6
7	10	19	$m_{67}+5$, m_1+9
9	2	25	m_1+9, $m_1+m_{67}+7$

Table 7 shows that case 2.1.1 cannot arise for $m_1 \in \{1,3,7,9\}$. When $m_1 = 5$, we have $m_{67} = 18$ and $m_{56} = 13$. Then $G \in \{G_{44}^{(1)}, G_{44}^{(2)}, G_{44}^{(3)}, G_{44}^{(4)}, G_{44}^{(5)}, G_{44}^{(6)}\}$. In [3], the authors determined all outerplanar r- UB-graphs.

The class of UB-graphs with $v+4$ edges is determined and the conjecture in [3] is false. But, there is much work to do to determine the class of r- -UB-graphs with $v+m$ edges (where $m \geqslant 5$).

References:

[1] BONDY J A, MUTRY U S R. Graph Theory with Applications[M]. Elsevier, New York. 1976.

[2] SHI Yong-bing. Some theorems of uniquely pancyclic graphs[J]. Discrete Mathematics, 1986, 59: 167-180.

[3] SHI Yong-bing, SUN Jia-shu. On uniquely r- bipancyclic graphs[J]. Journal of Shanghai Teachers University, 1997, 4: 1-10.

[4] SHI Yong-bing, YAP H P, TEO S K. On uniquely r- pancyclic graphs, Annals of the New York Academy of Sciences[M]. 1987, 576: 487-499.

[5] 孙浩平. 关于唯一 r-偶泛圈图(I),(II)[J]. 上海师范大学学报,1996(2):19-27;1997(1):30-35.

[6] YAP H P, TEO S K. On uniquely r- pacyclic graphs[J]. Lecture Notes in Math, 1984, 1073: 334-335.

唯一偶泛圈图的一个定理

孙浩平[1], 施永兵[2], 孙家恕[3]

(1. 张家港进出口商品检验局, 江苏 215634; 2. 上海师范大学 数学科学学院, 上海 200234;
3. 上海通河中学, 上海 200431)

摘 要: 设 G 是一个偶图, v 是偶数且是 G 的阶. 若对每个偶数 t, $4 \leqslant t \leqslant v$, G 恰有一个长为 t 的圈, 则称 G 是唯一偶泛圈图(简称 UB-图). 作者证明恰有 6 个 $v+4$ 条边的 UB-图.

关键词: 圈; 偶图; UB-图; r-UB-图