On the Alongshelf Evolution of an Idealized Density Front

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ABSTRACT

Observations reveal a strong tendency for fronts to locate over the outer continental shelf and the upper portions of the continental slope. In this paper, we consider a model of the alongshelf evolution of a quasi-steady density front which separates relatively light nearshore water from the offshore. Results show how, in the absence of any external forcing, the frictional stress associated with bottom geostrophic velocities can result in the cross-shelf migration of a front as one progresses down the shelf in the direction of long coastal-trapped wave propagation.

The primary results are derived for the hypothetical case of a flat shelf adjoining a very deep offshore region. In this case, the rate of cross-shelf migration of a density front is similar to that of a passive scalar front in a barotropic fluid. The offshore displacement of a front, separating the lighter inshore from the heavier offshore water, increases exponentially in the direction of shelf-wave propagation on the frictional decay scale of a damped barotropic shelf wave.

Beyond a point of abrupt increase in depth, density variations play a more important role. There is a transfer of alongshelf transport from the barotropic inshore region to a baroclinic current in the upper layer water above the front. Eventually all of the transport carried by the lighter layer is shifted to the frontal region and the intersection of the front with the bottom stops deepening. Interfacial friction continues to result in a seaward spreading of the frontal transport and the interface consequently slowly flattens over time.

The results suggest slow cross-shelf migration for a nearshore front, more rapid migration across the outer shelf and then stalling at a depth where all of the transport is carried in the frontal region. While the model is highly idealized it is suggested that the physical mechanisms considered may be partially responsible for the fact that fronts are most often observed either nearshore or near the shelf break. Some important limitations are discussed.

1. Introduction

Water properties are frequently observed to vary over relatively small horizontal distances in the vicinity of the shelf edge. This is most evident in salinity due to the influence of ice-melt and freshwater runoff on the shelf. Figure 1 shows examples from the east coast of North America. At high latitudes (e.g., on the Labrador Shelf) where the cold water temperatures result in reduced sensitivity of density to temperature variations, the salinity fronts are generally associated with density fronts, but to the south the cross-shelf density variations are typically reduced due to compensation between salinity and temperature effects (the coastal waters tend to be relatively cold and fresh). Our primary interest is in reasons for the apparent tendency for such fronts to locate near the shelf break. We consider a highly idealized model in order to both simplify the mathematics and to center attention on one particular mechanism of interest. The simplifications made invalidate direct application of results to most real world situations. Results will, however, indicate the potentially important role of frictional stresses in the cross-shelf migration of density fronts in a simple geophysical setting.

It is worth noting that we do not restrict our attention solely to the buoyancy-forced component of the flow. Buoyancy flux may of course be an important driving mechanism for both the baroclinic and barotropic shelf circulation (Shaw and Csanady 1983; Weaver and Hsieh 1987), but forcing by the mean alongshelf wind-stress (e.g., Ikeda 1985), by periodic wind forcing (Denbo and Allen 1983) or by any other mechanism is not excluded. On the other hand, we consider only the idealized situation in which there is no local external forcing: all external forcing is assumed to occur backward (relative to coastal-trapped wave propagation) of the region considered. We thus contemplate a shelf-slope flow caused by some unspecified forcing mechanism and consider the slow alongshelf evolution in the region forward of the forcing region.

It is perhaps not surprising that fronts tend to occur near the shelf break. Even in the absence of a front the

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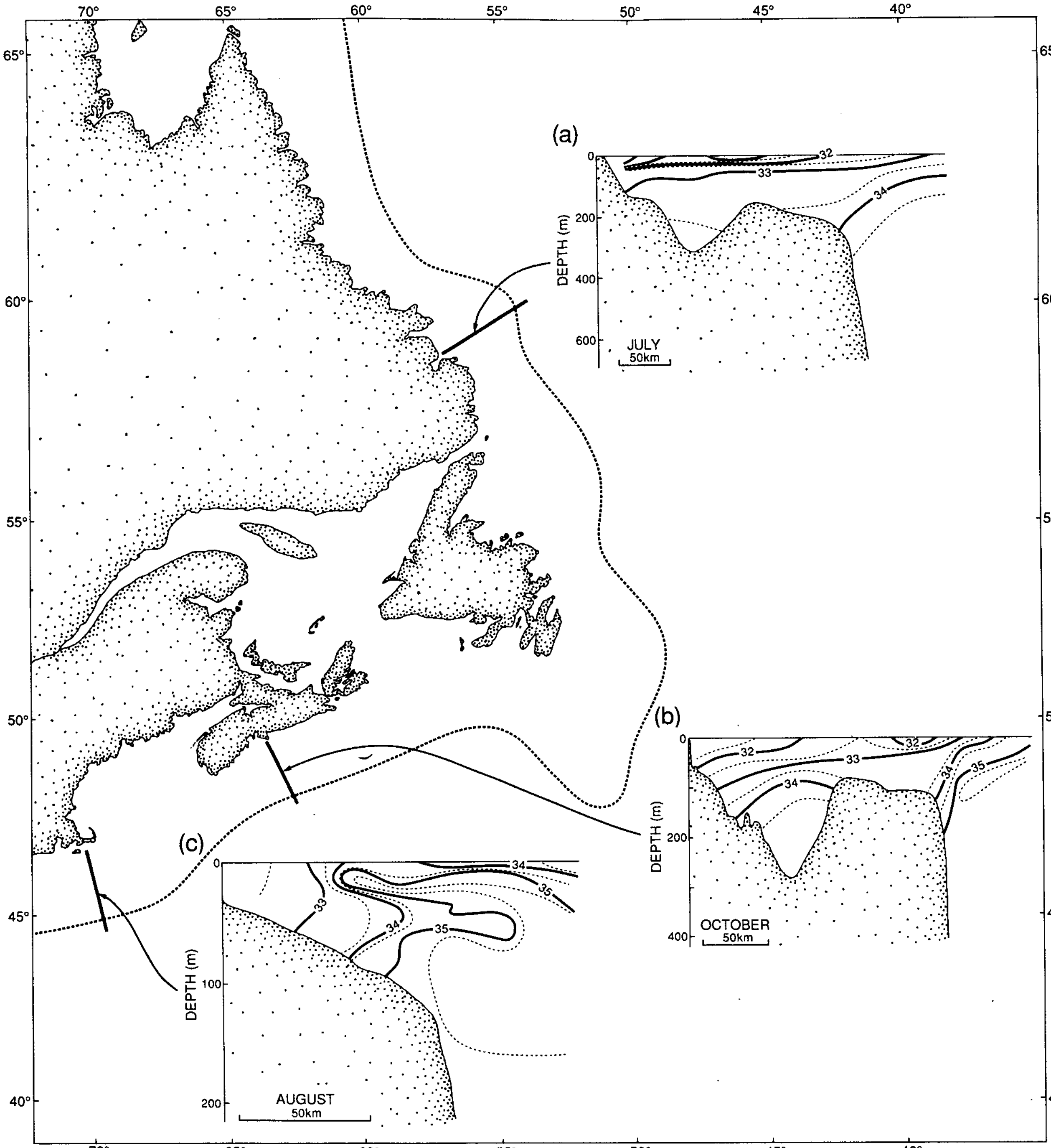

FIG. 1. Three salinity sections from the east coast of North America showing the tendency for rapid property changes near the shelf break. Figures were obtained from (a) J. R. N. Lazier (pers. comm.), (b) Petrie and Smith (1977), and (c) Wright (1983).

continental slope tends to act as a barrier to cross-shelf exchange (Csanady and Shaw 1983; Wright 1986) and hence the shelf break seems a natural location for abrupt property transition. Indeed, Chapman (1986) has developed this idea in some detail. For a passive scalar front, he suggests that bottom stress coupled with depth variations results in a velocity convergence near the shelf break, tending to sharpen the front on the shelf side, and that rapid tracer diffusion in the deep ocean tends to maintain an abrupt property transition into the deep ocean. The consideration of a passive scalar front is motivated by conditions in the Middle

Atlantic Bight region where a strong T–S front is observed over the shelf/slope region but the two fields tend to be density compensating so that the associated density front is considerably weakened. Although the model considered is an over-idealization, it does clearly reveal how a combination of bottom friction and horizontal mixing may result in the formation of a property front over the slope region.

Here we consider an alternative simple model which complements Chapman's work by allowing for density variations. The basic equations are presented in section 2, migration across the flat shelf is discussed in section 3 and evolution beyond the shelf break is considered in section 4. Some limitations of the model are discussed in section 5 and conclusions are summarized in section 6.

2. Equations

Motivated by observations such as those presented in Fig. 1, we consider the alongshelf evolution of a front which separates lighter coastal waters from heavier, uniform density, offshore waters. Bottom depth is assumed to be a function of offshore coordinate, only. Beyond the shelf break located at $x = L$, we assume that the lower layer is very deep and consequently currents are negligible outside of the interfacial Ekman layer (i.e., we use a reduced gravity model for $x > L$). The basic geometry and variable definitions for the special case of a two-layer fluid over a step shelf, to which we shall shortly restrict our attention, are schematized in Fig. 2.

For flow on an f-plane, under the Boussinesq, hydrostatic and rigid-lid approximations, the relevant linearized, depth-integrated equations are discussed by Shaw and Csanady (1983). They are

$$-fV = -\int_{-H}^{0} \frac{1}{\rho_0}\frac{\partial p}{\partial x}\,dz$$

$$\equiv -\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{-H}^{0} p\,dz + \frac{1}{\rho_0}H_x p_b \qquad (1)$$

$$fU = -\int_{-H}^{0}\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,dz - r_b v_b$$

$$\equiv -\frac{1}{\rho_0}\frac{\partial}{\partial y}\int_{-H}^{0} p\,dz - r_b v_b \qquad (2)$$

$$U_x + V_y = 0 \qquad (3)$$

where (x, y, z) form a right-handed Cartesian coordinate system with x increasing away from the coast and z increasing upward from the rest position of the sea surface, U and V are the horizontal transports in the x and y directions, p is the hydrostatic pressure, ρ_0 is a constant reference density, and r_b is a bottom friction coefficient which relates bottom stress to (u_b, v_b), the geostrophic velocity at the bottom (see Csanady 1979, for a discussion of this parameterization). The bottom geostrophic velocity is given by

$$u_b = -\frac{1}{\rho_0 f}\left(p_{sy} + g\int_{-H}^{0}\rho_y dz\right)$$

$$\equiv -\frac{1}{\rho_0 f}\left(p_{sy} + g\frac{\partial}{\partial y}\int_{-H}^{0}\rho dz\right) \qquad (4)$$

$$v_b = \frac{1}{\rho_0 f}\left(p_{sx} + g\int_{-H}^{0}\rho_x dz\right)$$

$$\equiv \frac{1}{\rho_0 f}\left(p_{sx} + g\frac{\partial}{\partial x}\int_{-H}^{0}\rho dz + gH_x\rho_b\right). \qquad (5)$$

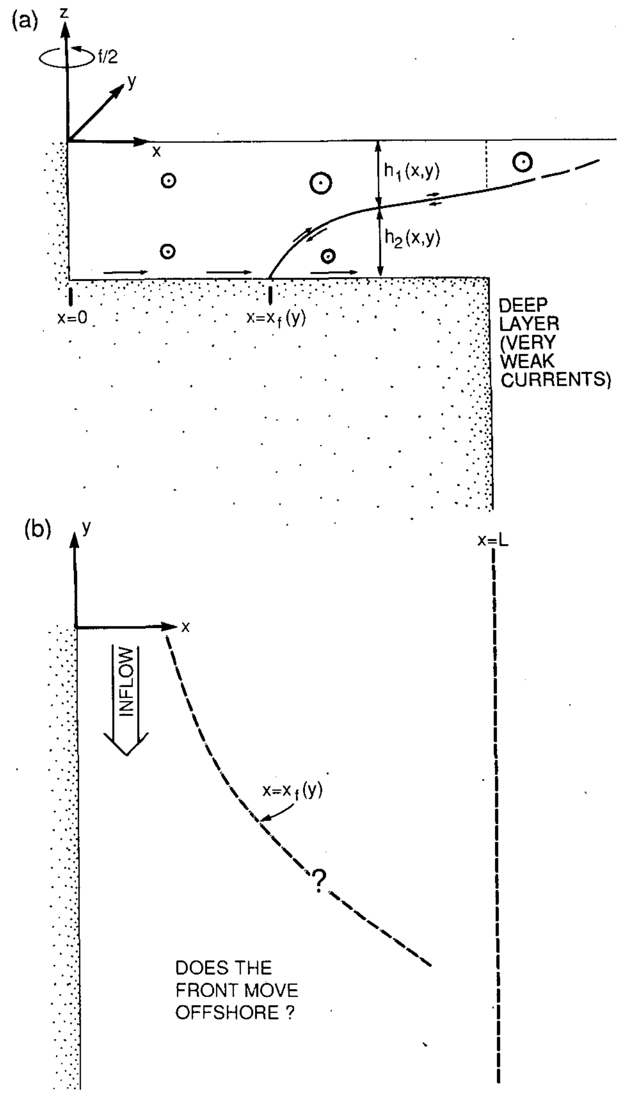

FIG. 2. (a) Schematic of a front in a 2-layer fluid overlying a step shelf indicating variables referred to in the text. Alongshelf geostrophic currents and the associated cross-shelf Ekman layer currents are also indicated. (b) Illustration of the primary question considered in this section: how does the front in (a) evolve down the shelf?

where p_s is the pressure at $z = 0$ associated with the surface elevation.

In writing (1)–(3) we have assumed that alongshelf scales of variations greatly exceed the corresponding cross-shelf scales and that if there is temporal evolution, it occurs on a time scale long compared to the frictional spindown time ($\sim H_s/r_b$: for typical values of $H_s = 100$ m and $r_b = 2.5 \times 10^{-3} \times 0.15$ m s^{-1}, this time scale is of order 3 days). Though our primary interest is in steady-state solutions, the allowance for possible slow temporal evolution will be useful in section 4.

Cross-differentiating (1) and (2), and using (3) and (4), we obtain

$$(r_b v_b)_x - fH_x u_b = 0. \qquad (6)$$

From (4) and (5) the horizontal divergence of the bottom geostrophic velocity is given by

$$u_{bx} + v_{by} = -(gH_x/f\rho_0)\rho_{by}. \qquad (7)$$

Finally, within regions of constant H_x and r_b, eliminating u_b from (6) and (7) gives

$$v_{by} + (r_b/fs)v_{bxx} = -(g/f\rho_0)s\rho_{by}, \qquad (8)$$

where $s \equiv H_x$.

The above set of equations are equivalent to those considered by Shaw and Csanady (1983). Augmenting (8) with an appropriate density equation, they present an interesting discussion of how an initially localized density anomaly self-advects along the shelf. The joint effect of baroclinicity and relief (JEBAR), entering through the right side of (8), plays a central role in their study of buoyancy driven flows. On the other hand, Chapman (1986) has used a barotropic model [zero right side in (8)] to illustrate that dynamically important density variations are not necessary for the formation of a (passive scalar) front at the shelf break provided there is a flux of fluid with a property anomaly into the region across the backward boundary. Further, it is evident from Chapman's analysis that similar results would be obtained for quite general topography, including the case of a flat shelf.

Chapman (1986) retained simplicity in his study by assuming density variations are dynamically unimportant, and in particular that the fluid is of uniform density in the region of interest. From (8) [or (6)] it is apparent that a similar simplification results in regions of very weak bottom slopes even if density variations are substantial. Below we will take advantage of this simplification by considering the idealized, but physically realizable case of a step-shelf geometry. That is, we consider $H = H_s$, a constant, for $0 < x < L$. Then (6) simplifies to give

$$v_{bx} = 0, \quad 0 < x < L. \qquad (9)$$

Thus, v_b is uniform across the flat shelf region. The simple form of (9) arises because, on a flat shelf, any divergence in the cross-shelf Ekman flux would cause stretching of planetary vortex tubes that would result in an unbalanced vorticity tendency, thus contradicting our assumption of a (quasi-) steady state. Of course, over a sloping shelf this vorticity tendency could be balanced by vortex tube stretching due to cross-isobath flow and (9) must then be modified accordingly (Shaw and Csanady 1983; also see section 5 below). However, on a step-shelf Eq. (9) holds for arbitrary stratification. Appropriate boundary conditions will be considered in the following sections.

Where density variations are present, the above equations must be supplemented by information about the evolution of the baroclinic structure. *To simplify the problem still further, we henceforth restrict our attention to the case of a two-layer fluid.* Then this information may be obtained from the depth-averaged equations of motion for the upper layer:

$$-fv_1 = -\rho_0^{-1}p_{1x} \qquad (10)$$

$$fu_1 = -\rho_0^{-1}p_{1y} - \left(\frac{r_I}{h_1}\right)(v_1 - v_b) \qquad (11)$$

$$(h_1u_1)_x + (h_1v_1)_y = -h_{1t} \qquad (12)$$

where (u_1, v_1) is the depth-averaged horizontal velocity over the upper layer, (p_{1x}, p_{1y}) is the pressure gradient in the upper layer, h_1 is the thickness of the upper layer and r_I is an interfacial friction coefficient analogous to, but much less than, r_b. Note that slow (i.e., on time scales long compared to the frictional decay time, h_1/r_I, of order a few weeks) temporal evolution of the density field is explicitly allowed for through the inclusion of a possibly nonzero right side in (12). Shaw and Csanady discuss the justification for retaining time-dependence in the density equation while neglecting it in both the momentum equations and the depth-integrated continuity equation.

Finally, substituting (10) and (11) into (12) and using the thermal wind relation,

$$v_1 - v_b = \left(\frac{g'}{f}\right)h_{1x}, \qquad (13)$$

where $g' = g(\rho_2 - \rho_1)/\rho_0$ is the reduced gravity, we obtain

$$h_{1t} + u_{1g}h_{1x} + v_{1g}h_{1y} = \left(\frac{g'r_I}{f^2}\right)h_{1xx} \qquad (14)$$

where (u_{1g}, v_{1g}) [$\equiv (u_b, v_b) + g'f^{-1}(-h_{1y}, h_{1x})$] is the geostrophic component of the upper layer current.

3. Case I: Front over a flat shelf

We now consider how and why a front over a flat shelf (Fig. 2) will evolve in the alongshelf ($-y$) direction. In particular we want to determine whether or not the front will move seaward under the influence of friction.

First, note from (9) that

$$v_b = v_b(y) \tag{15}$$

$$= (\rho_0 f L)^{-1}[p_b(L, y) - p_b(0, y)]. \tag{16}$$

In this section we assume that the lower layer extends inside $x = L$. The reduced gravity model approximation beyond $x = L$ then implies that the lower layer pressure gradient vanishes there. Thus, matching lower layer pressure across $x = L$ we conclude that p_b is uniform along $x = L$. Taking $p_b = 0$ there (i.e., taking p to be the pressure measured relative to that at $x = L$, $z = -H_s$), (16) reduces to

$$v_b = -(\rho_0 f L)^{-1} p_b(0, y). \tag{17}$$

Now, assuming that the front hits bottom seaward of $x = 0$ so that the flow is barotropic at the coast, the coastal constraint, $U = 0$ at $x = 0$, gives [using (2)]

$$\rho_0{}^{-1} H_s p_{by} + r_b v_b = 0 \tag{18}$$

or

$$v_{by} - \left(\frac{r_b}{f L H_s}\right) v_b = 0. \tag{19}$$

Integrating, we obtain

$$v_b = v_b\,(y = 0) \exp\left(\frac{r_b y}{f L H_s}\right). \tag{20}$$

Finally, over the flat shelf

$$u_{bx} + v_{by} = 0. \tag{21}$$

Substituting (20) in (21), integrating with respect to x and using $u_b = 0$ at $x = L$ we obtain

$$u_b = (L - x)\left(\frac{r_b}{f L H_s}\right) v_b. \tag{22}$$

This completes the determination of the bottom geostrophic velocity field in the forward region ($y < 0$) where the lower layer extends onto the shelf. Basically, as a result of the Ekman flux across $x = L$, the alongshelf bottom geostrophic velocity decays on the scale fLH_s/r_b. The bottom pressure drops accordingly, with the corresponding cross-shelf geostrophic velocity component given by (22). Interestingly, the scale fLH_s/r_b is independent of the stratification: it is just the distance that a long wavelength barotropic shelf wave on a step propagates in the frictional decay time scale, H_s/r_b. Thus, over a flat shelf, both u_b and v_b evolve in a manner which is independent of the stratification. In fact, over a flat shelf this result holds for more general stratification provided that (1)–(3) are valid, the density is uniform alongshore at $x = 0$, and there is a uniform density, deep-ocean lower layer which intrudes some distance onto the shelf.

Note that u_b (the geostrophic contribution to the cross-shelf bottom velocity) is shoreward. In the case of a homogeneous fluid the interior geostrophic flow would be vertically uniform and would tend to advect any passive scalar towards the coast. On the other hand, for a homogeneous fluid the total cross-shelf transport including both geostrophic and Ekman flux contributions, is given by

$$U(x, y) = H_s u_b - \left(\frac{r_b}{f}\right) v_b$$

$$= -\left(\frac{r_b x}{f L}\right) v_b. \tag{23}$$

The net cross-shelf transport is thus seaward and in the presence of strong vertical mixing, a passive scalar would in fact be advected towards the shelf break as noted by Chapman (1986).

The seaward advection discussed above is critical to the frontogenesis mechanism discussed by Chapman (1986). It is of interest to examine whether or not this mechanism carries over to the case of a dynamically active density front, particularly when mixing rates are reduced so that vertical homogeneity is not maintained. In fact, for the case of a two-layer fluid over a flat shelf we may immediately show that the front does migrate seaward as we progress down the shelf.

Let T_1 be the alongshelf transport carried by the lighter, nearshore water mass. Then

$$T_1 = \int_0^L h_1 v_1\,dx + \underbrace{\int_L^\infty h_1 v_1\,dx} \tag{24}$$

$$= \int_0^L h_1 v_b dx + \int_0^L h_1(v_1 - v_b)dx$$

$$+ \underbrace{\int_L^\infty h_1 v_b dx + \int_L^\infty h_1(v_1 - v_b)dx} \tag{25}$$

where the underbraced terms represent the upper layer transport carried in the region beyond $x = L$. If the upper layer does not extend beyond $x = L$, these terms vanish. Now, v_b is uniform for $0 < x < L$ and vanishes for $x > L$, and in either region, where two layers exist $v_1 - v_b$ is given by (13). Thus,

$$T_1 = A_{1s} v_b + \frac{g'}{f} \int_{x_f}^\infty h_1 h_{1x} dx \tag{26}$$

$$= A_{1s} v_b - \frac{g'}{2f}(H_s^2 - h_{1\infty}^2) \tag{27}$$

where A_{1s} is the area of the upper layer overlying the shelf, $x = x_f(y)$ is the position where the front intersects the bottom, and $h_{1\infty}$ is the depth of the upper layer far from the shelf. We will henceforth assume $h_{1\infty} = 0$, but the following arguments carry over with only minor modification to the case where $h_{1\infty}$ is any specified constant. Finally, rearranging (27) gives

$$A_{1s} = \left[T_1 + \left(\frac{g'}{2f}\right) H_s^2\right] \Big/ v_b. \tag{28}$$

This equation holds for any section where the front intersects the bottom inside $x = L$.

Now, since the numerator in (28) is constant while the denominator decreases exponentially with $-y$ in accordance with (20), we conclude that A_{1s}, *the cross-sectional area of the upper layer over the shelf increases exponentially with* $-y$ and thus the front clearly shifts seaward. The above conclusion depends on the assumptions of a weak lower layer alongshelf current component and geostrophically balanced alongshelf current shear beyond $x = L$, but is otherwise independent of conditions beyond $x = L$. In particular, slow temporal evolution of the density field in the region $x > L$ will not influence this conclusion.

Note that as for v_b, the alongshore variation in A_{1s} is entirely independent of the stratification over the shelf. This result suggests that the use of a barotropic model with the front treated as a passive scalar, as done by Chapman (1986), may yield useful qualitative information on the alongshore evolution of frontal position even when density stratification is significant. Indeed, it is easily shown that (28) is precisely the formula that is obtained for the evolution of an inshore water mass which is marked by a passive scalar in a fluid of uniform density over a flat shelf.

Finally, we note from (10) that for steady conditions inside of $x = L$

$$u_{1g}h_{1x} + v_{1g}h_{1y} = \left(\frac{g'r_I}{f}\right)h_{1xx}. \quad (29)$$

Horizontal advection by the interior geostrophic velocity must then balance any "diffusion" associated with a divergent interfacial Ekman flux (see Garrett and Loder 1981, for discussion of the latter effect in the context of two-dimensional ($\partial/\partial y = 0$) models). As noted earlier, u_b is shoreward so the geostrophic current tends to advect the front shoreward. On the other hand, the Ekman flux tends to "diffuse" the front seaward. As our previous results [particularly (28)] demonstrate, the latter dominates and the front moves seaward with $-y$: the rate of this seaward movement is just such that the net advection by the geostrophic current, including cross-shelf and alongshelf contributions, balances the offshore Ekman flux.

4. Case II: Front beyond the shelf break

We now consider the situation depicted in Fig. 3 in which the front is located beyond the shelf break and overlying a very deep offshore region. We consider the alongshelf evolution of this system under the influence of friction.

Within the barotropic region over the shelf, the arguments leading to (9) still hold so

$$v_b = v_b(y), \quad 0 < x < L. \quad (30)$$

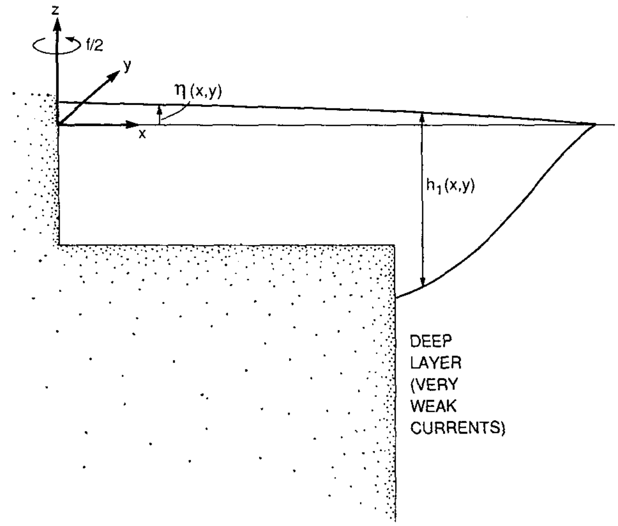

FIG. 3. Schematic illustration of the conditions assumed in section 4. The front is entirely removed from the shelf and the deep lower layer is assumed to be at rest.

We again assume that $v_b < 0$ initially (as expected for a relatively light nearshore water mass) so there is an offshore Ekman flux which tends to drain the barotropic transport carried inside $x = L$ off into the frontal region. In the previous case where the lower layer extended onto the shelf, we were able to argue that $p_b = 0$ at $x = L$ which allowed us to determine the alongshelf evolution of v_b without consideration of the offshore region. In the present case p_b is constant seaward of $x = L$ but since the lower layer does not extend onto the shelf, this condition need not be matched just inshore of $x = L$. The primary question of interest here is what, if any, influence the cross-shelf geostrophic currents associated with pressure variations along $x = L$ have on the rate at which the barotropic transport carried inside $x = L$ is drained off into the frontal region.

A qualitative answer is again easily obtained. To avoid large transport in the deep lower layer we must have a negligible pressure gradient there. Thus, sea level (η) is simply related to upper layer depth by

$$\eta = \epsilon h_1 \quad (31)$$

where we have assumed that both η and h_1 tend to zero far from the shelf. Also, the transport carried by the upper layer in the region $x > L$ is simply

$$T_f = \int_L^\infty h_1 v_1 dx = \int_L^\infty h_1 g' h_{1x}/f dx$$
$$= -\frac{g' h_1^2(L^+)}{2f}. \quad (32)$$

Now, the offshore Ekman flux tends to drain transport into the frontal region so there is an initial tendency for $|T_f|$ to increase with $-y$. From (32) $h_1(L^+)$ must then increase with $-y$, and from (31), $\eta(L^+)$ also increases with $-y$. The corresponding geostrophic transport across $x = L$ is thus seaward and we conclude that the drainage of the shelf water transport into the frontal region is accelerated by this influence.

More quantitative information may be obtained as outlined below. The coastal constraint, $U = 0$ at $x = 0$, together with (2) gives

$$\eta_y(0) + \left(\frac{r_b}{fLH_s}\right)[\eta(L) - \eta(0)] = 0 \quad (33)$$

where we have used the hydrostatic relation to write pressure gradients in terms of gradients in sea surface elevation, and the fact that v_b is uniform across the shelf to rewrite $\eta_x(0)$ in terms of the sea level difference across the shelf. Matching η and cross-shelf transport across $x = L$ gives

$$H_s\left[\eta_y(L) + \left(\frac{r_b}{fH_s}\right)\eta_x(L^-)\right] = h_1(L^+)\left[\eta_y(L) + \left(\frac{r_I}{fh_1(L^+)}\right)\eta_x(L^+)\right], \quad (34)$$

which may be rewritten as

$$\eta_y(L) + \left(\frac{H_s}{H_s - h_1(L^+)}\right)\left(\frac{r_b}{fLH_s}\right)[\eta(L) - \eta(0)] = \frac{r_I\eta_x(L^+)}{f(H_s - h_1(L^+))}. \quad (35)$$

Now, subtracting (35) from (33) gives

$$\Delta\eta_y - \left(\frac{h_1(L^+)}{h_1(L^+) - H_s}\right)\left(\frac{r_b}{fLH_s}\right)\Delta\eta = \frac{r_I\eta_x(L^+)}{f(h_1(L^+) - H_s)}. \quad (36)$$

where $\Delta\eta = \eta(0) - \eta(L)$. The case of no offshore front beyond $x = L$ may be obtained by taking $\lim h_1(L^+) \to \infty$: the result is equivalent to (19). In the more general case of finite h_1, we note that

$$T_1 = \frac{gH_s\Delta\eta}{f} - \frac{g'h_1^2(L^+)}{2f} \quad (37)$$

or

$$\Delta\eta = (g'/2gH_s)(a^2 - h_1^2(L^+)) \quad (38)$$

where $a^2 = (-2fT_1/g')$. Thus (36) is a nonlinear equation for the evolution of $h_1(L^+)$ [or for $\Delta\eta$]. In general, the solution of (36) requires additional information on the dynamics of the frontal region, but the limiting case of negligible interfacial Ekman flux is informative. In this case, (36) may be solved analytically to give

$$\left[\frac{h_1(L^+, y) - a}{h_1(L^+, 0) - a}\right]\left[\frac{h_1(L^+, y) + a}{h_1(L^+, 0) + a}\right]^{(a+H_s)/(a-H_s)} = \exp\left[\frac{r_bay}{fLH_s(a - H_s)}\right]. \quad (39)$$

The limiting case $a \gg (H_s, h_1(L^+, 0))$, though somewhat unrealistic, simplifies to yield the more transparent explicit solution,

$$h_1(L^+, y) \approx \{h_1^2(L^+, 0) + a^2[1 - \exp(r_by/fLH_s)]\}^{1/2}. \quad (40)$$

Clearly $h_1(L^+, y) \to a$ for $-y \gg fLH_s/r_b$ [this may be seen directly from (39) or from (40) on using $a \gg h_1(L^+, 0)$], corresponding to the limit where all of the transport is drained off the shelf, as from (38), $h_1(L^+) \to a$ implies $\Delta\eta \to 0$. The inviscid front then stops deepening and simply carries the transport down the shelf without further modification. Numerical solution of (39) shows that similar results are obtained for more general values of a, but the alongshelf scale over which the transport drains into the frontal region is reduced by approximately the factor $1 - H_s/a$. Thus drainage is significantly accelerated by baroclinic effects if a is not much larger than H_s.

Interfacial friction modifies the above picture. In particular, we note that over the deep offshore region where $(u_{1g}, v_{1g}) = (g'/f)(-h_{1y}, h_{1x})$, the basic equation governing h_1, (14), reduces to

$$h_{1t} = \left(\frac{g'r_I}{f^2}\right)h_{1xx}. \quad (41)$$

If $h_{1t} = 0$, then h_{1x} is constant. However, if $h_{1x} \neq 0$ then $h_1 \to 0$ at some offshore position and (11) reveals an inconsistency (letting $h_1 \to 0$ we conclude that $v_1 - v_b = g'h_{1x}/f$ must vanish). Thus the only steady-state solution is that with $h_{1x} = 0$ beyond $x = L$.

In fact, given typical observations of fronts, it seems more physically relevant to allow for slow temporal evolution of h_1. Equation (41) then indicates that there will be a gradual flattening of the interface as the interfacial Ekman flux "diffuses" the current seaward. The associated time scale is of order $f^2L_x^2/g'r_I$ (where L_x is a typical cross-shelf frontal scale) and for consistency with the neglect of the acceleration terms in the alongshelf momentum equation this must greatly exceed h_1/r_I. Thus, for consistency we require

$$\frac{g'h_1}{f^2L_x^2} \ll 1 \ll \left(\frac{L_y}{L_x}\right)^2. \quad (42)$$

The cross-shelf scale of variations in the front must greatly exceed the internal Rossby radius to justify neglecting time-dependence in the alongshelf momentum equation, and the cross-shelf scale must be much less than the alongshelf scale to justify neglecting the interfacial Ekman flux in the cross-shelf momentum equation.

With the above restrictions in mind, we conclude that after the front crosses the shelf-break the transport carried in the barotropic region over the shelf continues to drain off into the frontal region at an accelerated rate. Under the influence of interfacial friction the offshore front gradually flattens as the transport continues to spread seaward.

5. Comments on model limitations

The above discussion is intended to elucidate one basic mechanism which affects the alongshelf evolution of a density front. To keep the arguments as simple as possible some potentially important influences have been neglected. Prominent among the simplifications are the neglect of any mixing processes and the choice of a step-shelf topography so that the dynamical influence associated with along-isobath bottom density variations is absent.

Mixing processes are certainly important in the dynamics of real fronts. Vertical mixing associated with tides and relatively high frequency wind events are expected to be important particularly near the surface and bottom and even more so in the regions where a front intersects these surfaces. The vertical circulation associated with the front (see Loder and Wright 1985, for an example with continuous stratification) in combination with vertical mixing will contribute to horizontal mixing through shear dispersion (e.g. Young et al. 1982). Similarly, alongshelf topographic variations will also result in enhanced horizontal mixing (e.g., Zimmerman 1986) and instabilities associated with the front may also be important (Flagg and Beardsley 1978). All of these effects have been neglected in the interest of simplicity, but they may be important in reality. Indeed, two dimensional models such as those studied by Pietrafesa and Janowitz (1979), Kao (1981), Csanady (1984b) and Ikeda (1985) clearly demonstrate the importance of mixing processes to frontal dynamics. In fact, the study of Ikeda (1985) suggests that with vertical diffusion included, the unrealistic seaward spreading of the density front discussed in section 4 will effectively be limited to an offshore scale of order $\mathrm{Pr}^{1/2}r_i$ where Pr is the Prandtl number (= eddy viscosity/eddy diffusivity) and r_i is the local internal Rossby radius.

The consideration of more realistic cross-shelf depth variations introduces two important dynamical effects not discussed above. These two effects correspond to the first and last terms in (8). Scaling as suggested by the results presented above [$x = Lx'$, $y = (fh/r)Ly'$, $v_b = Vv'$, $\rho_b = \Delta\rho \cdot \rho'$] the relevant nondimensional equation is

$$(sL/h)v'_{y'} + v'_{x'x'} = -(g's/fV)(sL/h)\rho'_{y'} \quad (43)$$

where h, V and $\Delta\rho$ are typical values of depth, alongshelf current speed, and density variations. For typical parameter values ($s \approx 10^{-3}$, $L \approx 10^5$ m, $h \approx 10^2$ m, $g' \approx 10^{-2}$ m s^{-2}, $f \approx 10^{-4}$ s^{-1}, $V \approx 10^{-1}$ m s^{-1}) all three terms have similar magnitudes. A very weak bottom slope is clearly required for the step-shelf results to be quantitatively accurate.

The first term in (4) illustrates the well-known fact that bottom slopes are important even in regions of very weak stratification (i.e., for $g's/fV \ll 1$, $sL/h \ll 1$). In this limit, (8) [the dimensional version of (43)] reduces to the diffusion equation

$$-v_{by} = (r_b/fs)v_{bxx} \quad (44)$$

which has been discussed in some detail by Csanady (1978).

The particular case where $h \to 0$ as $x \to 0$ is of interest as it is the opposite extreme from the step-shelf limit considered earlier. Appropriate boundary conditions for this case are

$$U = 0 \quad \text{at} \quad x = 0 \quad (45)$$

$$u_b = -p_{by}/f\rho_0 = 0 \quad \text{at} \quad x = L, \quad (46)$$

where a very deep ocean with a uniform density lower layer has again been assumed beyond $x = L$, and, as in section 3, the lower layer has been assumed to extend shoreward of $x = L$. Using (2) and (6) these conditions may be rewritten as

$$v_b = 0 \quad \text{at} \quad x = 0 \quad (45')$$

$$v_{bx} = 0 \quad \text{at} \quad x = L. \quad (46')$$

The solution of (44) with these boundary conditions is

$$v_b = \sum_{m=0}^{\infty} a_m \sin(x/L_m)\exp[ry/fL_m h_m] \quad (47)$$

where $L_m = L/(m + \frac{1}{2})\pi$, and $h_m = sL_m$. The values of a_m, $m = 0, 1, 2, \cdots$ are determined by the form of v_b specified at $y = 0$. The first term in this series decays on the scale $(2/\pi)^2 fLH_L/r$, where H_L is the depth at $x = L$. This decay scale is very similar to that on a step shelf with $H_s = 0.5H_L$, the mean depth across the shelf. With $H_L \sim 200$ m, $f \sim 10^{-4}$ s^{-1} and $r \sim 5 \times 10^{-4}$ m s^{-1}, the alongshelf decay scale is of order 20 shelf widths for this term. The second term decays nine times as fast and higher order terms decay faster still. Thus, if $a_0 \gtrsim a_m$, $m \geq 1$, then beyond a few shelf widths distance from $y = 0$,

$$v_b \approx a_0 \sin(x/L_0)\exp[ry/fL_0H_0]. \quad (48)$$

Using (46) and (48) and integrating (7), with $\rho_{by} = 0$, we obtain

$$u_b \approx (r/fH_0)a_0 \cos(x/L_0)\exp[ry/fL_0H_0], \quad (49)$$

and total cross-shelf transport is simply

$$U = Hu_b - (r/f)v_b$$
$$= [x/L_0 \cos(x/L_0) - \sin(x/L_0)](r/f)v_b. \quad (50)$$

As in the case of a step shelf, u_b is shoreward but the total transport is seaward increasing from zero at the coast to the Ekman transport at $x = L$. The qualitative picture is very similar to the step-shelf case, and we expect that after an initial adjustment near $y = 0$, a passive scalar front, or a weak density front, would move seaward with $-y$ in a manner qualitatively similar to that case. This conclusion is consistent with Chapman's (1986) results.

The above discussion applies for $g's/fV \ll 1$, or equivalently for $g'h^2/2f \ll VhL/2$. That is, density variations may be neglected when the enhanced currents over the front carry a negligible portion of the total transport. Where this condition is violated, the influence of density stratification must be considered. Recent studies (e.g., Shaw and Csanady 1983; Csanady 1984a; Huthnance 1984; Vennell and Malanotte-Rizzoli 1987) have included density stratification and considered the influence of along-isobath variations in density. Our interest here is in how this effect will influence the cross-shelf migration of a density front. The previous results indicate that in the case of a lighter nearshore region the steady state solution will tend to have $\rho_{by} > 0$. The right side of (8) thus tends to be negative, and hence we expect the reduction in $|v_b|$ with $-y$ to be accelerated by the JEBAR influence. As a result we expect a buoyant nearshore region to spread seaward with $-y$ more rapidly than would a corresponding region in a homogeneous fluid which was "dyed" with some passive scalar. This result is qualitatively consistent with our finding, for the case of a step-shelf, that the drainage of transport off the shelf is accelerated by baroclinic effects after the front shifts beyond the shelf break. More detailed examination of this question is clearly warranted.

In a diagnostic study of the circulation patterns associated with specified density variations, Csanady (1985) presents several solutions for idealized two-layer density fields, which are relevant to the present study. In particular, he shows that an interface whose intersection point with the bottom deepens in the forward ($-y$) direction is consistent with an upper-layer flow from the barotropic inner region into the baroclinic offshore region. The seaward flow is always fed from the forward portion of the shelf. In the case of an x-independent interface depth the flow escapes offshore to infinity, but with a surface to bottom front the offshore flow is absorbed into the frontal region as discussed in section 4 above. Csanady (1985) also notes that a front which shifts towards shallower water with increasing $-y$, is inconsistent with the maintenance of the prescribed density field unless a strong over riding flow toward positive y is present. In fact, such a flow pattern would imply a divergent Ekman flux in the upper layer across the line of intersection of the front with the bottom (the upper layer Ekman flux would reverse directions from one side of the line to the other). Since geostrophic transports are continuous, there would be a mismatch in the net transport across this line which would violate the steady-state assumption. It thus appears that the simple two-layer model considered here is inconsistent with such a density field. On the other hand, in a more realistic continuously stratified model with a coastal buoyancy source over a limited range of y, it is clear that horizontal diffusion will eventually reverse the alongshelf density gradient. Vennel and Malanotte-Rizzoli (1987) show results, which illustrate the potential importance of this influence. The model considered in this paper cannot reproduce this situation since it does not incorporate mixing between the two water masses.

6. Conclusions

The primary conclusion of this study is that, with or without density stratification, there is a tendency for offshore fronts to migrate seaward under the influence of bottom and interfacial stress. In buoyancy-forced models with $\partial/\partial y = 0$ this is well-known and leads to persistent migration seaward; no steady state is possible in this case unless a sink of buoyancy is included at some location. With $\partial/\partial y \neq 0$, evolution with distance in the direction of long coastal-trapped wave propagation can replace time evolution and a steady state may be possible. While the front is over the shelf this can be achieved through a balance between advection by geostrophic currents which tends to shift the front shoreward and the diffusionlike influence of Ekman flux which tends to shift the front seaward. This balance is achieved by a state in which the front migrates seaward in the direction of long coastal-trapped wave propagation. This process will asymptotically result in all the transport shifting off the shelf into the frontal region.

Assuming a deep ocean beyond the shelf break, the bottom geostrophic currents are taken to be vanishingly small in this region so the geostrophic current associated with density variations are along isopycnal (or iso-interface) lines and hence do not advect the front. Thus, beyond the shelf break the offshore Ekman flux cannot be balanced with advection by geostrophic currents and the only possible steady state is one with negligible interfacial Ekman flux. The density interface slowly flattens with time in this region until arrested (or eliminated) by some process not included in the present model (for example, see Ikeda 1985).

Finally, it is of interest to note that near the coast, the offshore Ekman flux and the onshore geostrophic currents nearly balance. Thus a nearshore front will move seaward slowly with increasing $-y$. As it moves seaward the shoreward geostrophic flux reduces relative to the seaward Ekman flux and the front moves more quickly seaward. Eventually the transport is all shifted into the frontal region offshore. Beyond this point the intersection point with the bottom no longer deepens. Further evolution in the alongshelf direction is due to the slow seaward migration of transport under the influence of interfacial stress. This scenario with slow cross-shelf migration for a near-shore front, more rapid migration across the outer shelf and then stalling at a depth where all the transport is carried in the frontal region may be partially responsible for the fact that fronts are most often seen either nearshore or near the shelf break but seldom (or in few places) in the offshore region over the shelf.

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