Optimal Rate Control for Transporting VBR Video over QoS-assured Channels

LI Chun-Wen[1, 2] ZHU Peng[1]

Abstract In this paper we discuss how to select appropriate source and channel rate for transporting variable bit-rate (VBR) compressed video over QoS (quality of service)-assured channels. We first formulate it as an optimal control problem of discrete linear time-delay system. Then the discrete maximum principle is used to get the optimal control. Compared to traditional solutions, the proposed algorithm is designed for the coder with continuous output rate, and can work without special requirements for the encoder and decoder buffer sizes. Theoretical analysis and experimental results show that the proposed algorithm has lower space and time complexity. Our solution can be used in both off-line and on-line coding.

Key words Optimal control, discrete linear time-delay system, maximum principle, VBR video, QoS

1 Introduction

Networked applications (*e.g.* networked multimedia, networked robots) are more and more popular with the increasing use of communication networks. From the point of view of system and control, such applications always can be formulated as time-delay systems, because there exists delay in transporting data over networks. In this paper we use the control theory of time-delay system to solve the optimal rate control problem for transporting VBR video over QoS-assured channels. This problem is important, because in streaming video applications, the output generated by the video coder will intrinsically be VBR video for most practical compression algorithms, and on the other hand, compared to the best effort channels, QoS-assured channels can provide better QoS support for streaming video applications[1].

Traditionally, this problem is formulated as an optimization problem, and the goal is to minimize the average distortion of all the frames to achieve good video quality. A Lagrange-multiplier-based algorithm was proposed[2] to get the optimal solution for constant bit-rate (CBR) channels, but it only can get sub-optimal solution for VBR channels. A deterministic-dynamic-programming-based algorithm[3] was proposed to get the optimal solution for CBR channels, and then was extended to VBR channels[1]. However, it was mainly designed for the coder with discrete output rates (*i.e.* using frame quantization), and it has special requirements for buffer sizes, *i.e.*, both the encoder and decoder buffer sizes are required to be sufficiently large. This is costly especially in a multicast scenario or when the server resource is scarce. In addition, it has very high space and time complexity. The above algorithms are mainly for wired channels. Deterministic dynamic programming was also applied to wireless channels which can be modelled as a Markov chain[4]. Stochastic dynamic programming was used to reduce the on-line computational cost[5]. Then the algorithm was extended to interframe video coders[6].

In this paper we mainly discuss how to select appropriate source and channel rate for transporting VBR video over wired CBR and VBR channels. We first formulate the problems under different channels in a unified form — an optimal control problem for discrete linear time-delay system. Then we apply the discrete maximum principle to get the optimal control with the form of a two-point boundary value problem, which can be solved by a computational algorithm[7]. Our solution is for the coder with continuous output rates, and imposes no constraint on buffer sizes. Theoretical analysis and experimental results show that it has lower space and time complexity than the well-known dynamical programming approach — the Viterbi algorithm (VA) proposed in [1]. Our solution can be used in both off-line and on-line coding.

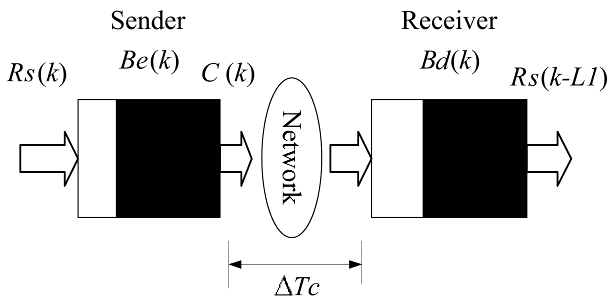

Fig. 1 The simplified VBR video system

2 Problem formulation

A simplified VBR video system, which is the same as the model in [1], is shown in Fig. 1. Let′s adopt the discrete time model, and the time k is the time when frame k (with size $Rs(k)$) is to be placed into the encoder buffer (with size BE). The data in the encoder buffer is packaged and then fed into the network with the channel rate $C(k)$ (in bits per frame period). For CBR channels, $C(k)$ is fixed, while for VBR channels, $C(k)$ is variable with the constraint defined by some policing mechanism[8]. The sent package will arrive at the decoder buffer (with size BD) after a transmission delay ΔTc, and then will be sent to the decoder at the prescribed time. The end-to-end delay of one frame (denoted as $L1$ frame periods) is assumed to be constant.

To make the problem tractable, we need to simplify the problem as in [1]. Firstly, let us suppose that there is no packet loss in the channel. This assumption is reasonable because the packet loss ratio is a very small value (even 0) and can be neglected in the QoS-assured channels. Secondly, the transmission delay ΔTc is usually variable due to both scheduling and routing, but the delay variations can be assumed to be small and disregarded, or alternatively absorbed by overdimensioning the decoder buffer in QoS-assured channels[1]. So we can regard ΔTc as a constant. Further we can "eliminate" the transmission delay by shifting the encoder and decoder clocks by an amount equal to ΔTc[1], thus the nominal end-to-end delay L is equal to

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$L1-\Delta Tc$, which is actually the delay in the encoder and decoder buffers. Then we have the following basic system model

$$\begin{aligned} Be(k+1) &= Be(k)+Rs(k)-C(k) \\ Bd(k+1) &= Bd(k)+C(k)-Rs(k-L) \end{aligned} \tag{1}$$

where $Be(k)$ and $Bd(k)$ are the encoder and decoder buffer fullness at time k, respectively.

Overflow and underflow of both encoder and decoder buffers should be avoided, because buffer overflow will lead to packet loss, while the decoder buffer underflow will interrupt the playback of the application, and the the encoder buffer underflow means that the available bandwidth is not fully utilized. So it is required that

$$0 \le Be(k) \le BE \ and \ 0 \le Bd(k) \le BD \tag{2}$$

2.1 Problem formulation for CBR channels

For CBR channels, channel rates are fixed to a constant C. It can be easily derived from (1) that

$$Be(k)+Bd(k+L)=L*C \tag{3}$$

One can refer to [1] to see the exact derivation[1]. Combining (2) and (3), we have

$$max(L*C-BD,0) \le Be(k) \le min(L*C,BE) \tag{4}$$

So for avoiding overflow and underflow of both encoder and decoder buffers, we only need to control the encoder buffer to meet (4) by selecting appropriate source rate $Rs(k)$.

Suppose the total number of the frames which are to be streamed is N. Writing this system in the standard form of discrete linear system, we have

$$\boldsymbol{x}(k+1)=\boldsymbol{x}(k)+\boldsymbol{u}(k),\ k=0,1,\cdots,N-1 \tag{5}$$

where the state vector is represented by $\boldsymbol{x}(k)=B\boldsymbol{e}(k)$ with the initial condition $\boldsymbol{x}(0)=\boldsymbol{0}$, and the control parameters are taken as $u(k)=Rs(k)-C$.

According to (4), the following state constraints need to be introduced:

$$x(k) \in D=\{y|max(L*C-BD,0) \le y \le min(L*C, BE)\}, for\ i=1,2,...,N \tag{6}$$

Based on the above established dynamical model, we will seek the optimal control which minimizes the following cost function[2]

$$Cost(u)=\sum_{k=0}^{N-1} d(u(k),k) \tag{7}$$

over the admissible control set

$$u(k) \in \Omega_k=\{y|-C \le y \le \omega(k)-C\},\ for\ i=0,1,2,..., N-1 \tag{8}$$

where $d(u(k),k)$ is the convex rate-distortion function of frame k, and $\omega(k)$ is the possibly maximum size of frame k. To get $d(u(k),k)$, we use the interpolation method in [9], i.e., first get some pairs of rate and distortion values of frame k by using quantization, then get the rate-distortion function through cubic spline interpolation.

[1]Note that (3) imposes a constraint for the parameter setting. Combining (3) and (2), we can easily get $BE+BD \ge L*C$. This means that the encoder and decoder buffer sizes should increase with the increase of the end-to-end delay.

[2]This kind of cost function suggests that frames should be independently coded, because there is no relationship between the R-D function of different frames.

2.2 Problem formulation for VBR channels

We suppose that the policing mechanism of VBR channels is the well known leaky bucket mechanism[8], which can be formulated as the following model:

$$LB(k+1)=LB(k)+C(k)-Rm \tag{9}$$

where $LB(k)$ is the leaky bucket (with size LB) fullness at time k, and Rm is the sustainable rate of the leaky bucket. According to the policing mechanism[8], it is required that

$$\begin{gathered} 0 \le LB(k) \le LB \\ C(k) \le P \end{gathered} \tag{10}$$

where P is the peak rate defined by the leaky bucket mechanism.

Then combining the system model (1) and (9), and writing them in the standard form of discrete linear time-delay system, we have

$$\boldsymbol{x}(k+1)=A\boldsymbol{x}(k)+B_0\boldsymbol{u}(k)+B_1\boldsymbol{u}(k-L), \quad k=0,1,\cdots,N+L-1 \tag{11}$$

where the state vector is $\boldsymbol{x}(k)=[Be(k),LB(k),Bd(k)]^{\mathrm{T}}$ with initial condition $\boldsymbol{x}(0)=[0,0,0]^{\mathrm{T}}$, the control parameters are $\boldsymbol{u}(k)=[Rs(k)-Rm,C(k)-Rm]^{\mathrm{T}}$ with initial condition $\boldsymbol{u}(k)=[-Rm,-Rm]^{\mathrm{T}}$, $k=-L,-L+1,\cdots,-1$, and the system matrices are

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, B_0=\begin{bmatrix}1&-1\\0&1\\0&1\end{bmatrix}, B_1=\begin{bmatrix}0&0\\0&0\\-1&0\end{bmatrix}$$

According to (2) and (10), we need to introduce the following state constraint

$$\boldsymbol{x}(k) \in D=\{\boldsymbol{x}|0 \le x_1 \le BE,\ 0 \le x_2 \le LB, 0 \le x_3 \le BD\}, k=1,2,\cdots,N+L \tag{12}$$

Then based on the above model, we will seek the optimal control which minimizes the cost function $Cost(u_1)$ defined in (7) over the admissible control set

$$\Omega_k=\begin{cases}\{\boldsymbol{u}|-Rm \le u_1 \le \omega(k)-Rm, -Rm \le u_2 \le P-Rm\},\ 0 \le k < N \\ \{\boldsymbol{u}|u_1=-Rm, -Rm \le u_2 \le P-Rm\}, N \le k < N+L\end{cases} \tag{13}$$

where $\omega(k)$ is the possible maximum size of frame k.

2.3 The unified problem formulation

Note that the problems of CBR and VBR channels have the similar form of state and control constraints, and the cost function. In addition, considering that discrete linear system can be regarded as a special case of discrete linear time-delay system, the formulated problem of CBR channels can be translated to the same form as of VBR channels by setting the delay item L in the time-delay system model to be 0. So we can formulate the problems of both CBR and VBR channels as a unified form – the optimal control problem for discrete linear time-delay system with state and control constraints, as described in Section 2.2.

3 The solution by discrete maximum principle

In this section we present the solution for the formulated optimal control problem in Section 2.2 by the discrete maximum principle. Before doing so, we first introduce a C^1 convex penalty function

$$f(\boldsymbol{x},\delta)=\left\{\begin{array}{ll}0, & if\ \boldsymbol{x}\in D\\ \delta\times min_{y\in D}\sum_i(x_i-y_i)^8 & if\ \boldsymbol{x}\notin D\end{array}\right. \quad (14)$$

into (7) to remove the state constraints (12), so that there are only control parameters constrained by the admissible set (13). Then the modified cost function can be written as

$$Cost(\boldsymbol{u})=\sum_{k=0}^{N-1}d(u_1(k),k)+\sum_{k=0}^{N+L-1}f(\boldsymbol{x}(k),\delta)+ f(\boldsymbol{x}(N+L),\delta) \quad (15)$$

It is not difficult to understand that the possibility that the optimized state trajectory falls into the constraint set D will go to 1, when the penalty parameter δ is sufficiently large (experiments show that the results are satisfying for δ around 100).

Next, the modified problem can be resolved by the discrete maximum principle[3] as proposed in [10].

There exists a nontrivial solution $\boldsymbol{x}^*(k),\boldsymbol{q}^*(k)$ of the difference equation

$$\begin{aligned}\boldsymbol{x}(k+1) &= A\boldsymbol{x}(k)+B_0\boldsymbol{u}(k,q)+B_1\boldsymbol{u}(k-L,q)\\ \boldsymbol{q}(k) &= \boldsymbol{q}(k+1)A-\frac{\partial f}{\partial x}(\boldsymbol{x}(k),\delta)\end{aligned} \quad (16)$$

under the initial condition $\boldsymbol{x}(0)=[0,0,0]^{\mathrm{T}}$ and the end-point condition $\boldsymbol{q}(N+L)=-grad(f(\boldsymbol{x}(N+L),\delta))$, and the control $\boldsymbol{u}(k,q)$ satisfies the maximum principle

$$\begin{aligned}&-d(\boldsymbol{u}(k,q),k)+\boldsymbol{q}(k+1)B_0\boldsymbol{u}(k,q)+\\ &\boldsymbol{q}(k+L+1)B_1\boldsymbol{u}(k,q)=\\ &max_{u\in\Omega_k}\{-d(\boldsymbol{u},k)+\boldsymbol{q}(k+1)B_0\boldsymbol{u}+\\ &\boldsymbol{q}(k+L+1)B_1\boldsymbol{u}\}\quad (\ 0\le k<N)-\\ &d(\boldsymbol{u}(k,q),k)+\boldsymbol{q}(k+1)B_0\boldsymbol{u}(k,q)=\\ &max_{u\in\Omega_k}\{-d(\boldsymbol{u},k)+\\ &\boldsymbol{q}(k+1)B_0\boldsymbol{u}\}\qquad (\ N\le k<N+L).\end{aligned} \quad (17)$$

An optimal control $\{\boldsymbol{u}^*(k)\}$ is given by $\boldsymbol{u}^*(k)=\boldsymbol{u}(k,q^*)$.

To see the corresponding proof of this discrete maximum principle, one can refer to [10∼13]. Obviously, the seeking of optimal control results in a two-point boundary value problem. We can obtain the numerical solution by employing the iteration algorithm proposed in [7], where an iterative precision γ, namely the distance between the resulting control sequences in two adjacent iterations, is chosen to regulate the optimality of the solution. The smaller the iterative precision is, the closer can we get near-optimal control at the expense of computational resource consumption.

Note that our solution does not impose any special requirements on buffer sizes, and this means that our solution can work with any setting of the encoder and decoder buffer sizes. In addition, the admissible control set in the proposed algorithm is continuous, so opposed to traditional solutions, our solution allows the video coder to adjust its output rate continuously.

[3] In order to make it easy for the readers to understand, we list discrete maximum principle in the appendix.

4 Complexity analysis

In this paper we use the VBR case to compare space and time complexity between the proposed algorithm and VA which was proposed in [1].

Because VA has too high computational cost, the state clustering technique is used to reduce complexity[1]. States within the clustering interval Δ are clustered to reduce the node number in each trellis stage. With the smaller clustering interval, the complexity increases, but the solution we get is closer to the extremum. The optimal solution is obtained when Δ is set to $1bit$. Because the state of VA is $(Bd(k),LB(k))$, for every trellis stage (*i.e.* one frame) there are $\frac{BD}{\Delta}*\frac{LB}{\Delta}$ nodes. So the space complexity of VA is[4]

$$SC_{VA}=O(N*\frac{BD}{\Delta}*\frac{LB}{\Delta}) \quad (18)$$

Suppose the number of quantizers in VA is M. The channel rate is also clustered by Δ. For every node there are $M*\frac{P}{\Delta}$ branches we need to try. So the time complexity of VA is

$$TC_{VA}=O(N*M*\frac{BD}{\Delta}*\frac{LB}{\Delta}*\frac{P}{\Delta}) \quad (19)$$

For our proposed algorithm, we only need to save $x(k),u(k)$ and $q(k)$ in the iteration[7]. So the space complexity of our proposed algorithm is

$$SC_{OUR}=O(N) \quad (20)$$

The time complexity of our algorithm depends on the required number of iterations, say $count(\gamma)$, which monotonously increases with the decreasing of γ. The time complexity also depends on the computational cost of each iteration that is proportional to N. So the time complexity of our proposed algorithm is

$$TC_{OUR}=O(N*count(\gamma)) \quad (21)$$

Obviously, VA has a much higher space complexity than our proposed algorithm by comparing (18) with (20), especially in the limit that the clustering interval Δ is set smaller to get nearer optimal solution.

The reduction of time complexity is not so explicit by direct comparison of (19) and (21). However, from (19) we can find that the time complexity of VA climbs up rapidly with the increasing of the buffer sizes and the peak rate, and will be very high in high bandwidth scenarios. So we also compare the time complexity by experiments: if our algorithm gets better solution than VA within a shorter period of time, we can assert that our algorithm has lower time complexity. This is proved by the experimental result in next section.

5 Experimental results

In our experiments, an MPEG-4 encoder is used in the intraframe coding mode. In every experiment we try different standard video sequences. We also implement VA for comparisons. The range of frame quantization step in VA is set from 1 to 31 ($M=31$).

For the CBR case, the channel rate C is set to $20kb/frame$, the encoder and decoder buffer sizes are both set to $60kb$, and the end-to-end delay L is set to 3 frames. For the VBR case, we set $N=300$, $Rm=60kb/frame$,

[4] We suppose that $L\ll N$ in this section.

Table 1 Time complexity comparison under the VBR channel

	Our solution		VA	
Sequence(format, N frames)	Total Distortion	Time(s)	Total Distortion	Time(s)
News(CIF,300)	4102.7	2156.3	4132.8	57924.4
Akiyo(CIF,300)	1297.8	2939.3	1304.5	59204.2
Container(CIF,300)	7174.7	1874.5	7206.0	62137.7
Mother&Daughter(CIF,300)	1435.2	2393.2	1445.0	66018.4

Table 2 Optimality comparison under the CBR channel

Sequence(format, N frames)	Total distortion(Our solution)	Total distortion(VA)
Akiyo(QCIF, 300)	3418.3	3421.1
Clare(QCIF, 150)	394.6	401.9
Suzie(QCIF, 150)	1296.0	1299.1
Miss(QCIF, 50)	137.3	140.1

Table 3 Performance comparison between our solutions with different iterative precisions under the VBR channel

	Precision: $0.01bit$		Precision: $1bit$	
Sequence(format, N frames)	Time(s)	PSNR(dB)	Time(s)	PSNR(dB)
News(CIF,300)	2156.3	36.78	3.2	36.62
Akiyo(CIF,300)	2939.3	41.77	3.1	41.63
Container(CIF,300)	1874.5	34.35	3.1	34.23
Mother&Daughter(CIF,300)	2393.2	41.34	2.5	41.21

$P = 360kb/frame$, $LB = 360kb$, $L = 30$ frames to simulate the high bandwidth scenario. In order that VA can be employed, the buffer size BE and BD are both required to be larger than $LB + L * Rm$[1], so we set both of them to be $LB + L * Rm = 2160kb$.

The experimental results shown in Table 1 are time complexity comparison between our algorithm and VA under the VBR channel. In the experiments, we set the iterative precision of our algorithm to be $0.01bit$, and the buffer clustering interval of VA to be $10kb$. As analyzed in the last section, VA shows very high time complexity in high bandwidth scenarios, while our algorithm has lower time complexity and can use less time to achieve better solution.

Table 2 shows the experimental results of optimality comparison between our algorithm and VA under the CBR channel. In the experiments we set the iterative precision γ of our algorithm very small ($0.001bit$) to get the solution near to the optimal value. The buffer clustering interval Δ of VA is set $1bit$ to get the optimal solution. Because the source rate selection range of our algorithm is continuous and it covers VA's range which is discrete, the optimal solution got by our algorithm should be better than that by VA, and this has been proved by the experimental results.

The experimental results of Table 3 show that our solution can be used in both off-line and on-line coding by selecting some appropriate value of the iterative precision γ. With a smaller iterative precision value ($0.01bit$ in the experiment), we can obtain a solution nearer to the optimal control. This means that lower distortion and higher peak signal-to-noise ratio (PSNR) can be achieved, while the processing time also becomes long. Therefore, this mode is suitable for off-line coding. If a larger precision value is chosen ($1bit$ in the experiment), the solution deviates farther from the optimal control. However, this mode benefits in the reduction of processing time so that on-line coding becomes possible.

6 Conclusion and future work

We have proposed a control-theoretic approach to solve the optimal rate control problem for transporting VBR video over QoS-assured channels. Our solution is designed for the coder with continuous output rates, has no special requirements for encoder and decoder buffer sizes, and has lower complexity than traditional solutions. The proposed algorithm can be used in both off-line and on-line coding.

In this paper we only consider the encoder working in the intraframe coding mode. Our further work is to extend the proposed algorithm to the interframe coding mode.

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Appendix: The discrete maximum principle in [10]

Consider the system

$$\boldsymbol{x}(k+1) = \bar{A}_0(k)\boldsymbol{x}(k) + \bar{A}_1(k)\boldsymbol{x}(k-h_1) + \bar{B}_0(k)\boldsymbol{u}(k) + \bar{B}_1(k)\boldsymbol{u}(k-h_2) \tag{A1}$$

with an initial condition $\boldsymbol{x}(k) = \boldsymbol{\phi}(k), k = -h_1, -h_1+1, ..., 0$. Here, $\boldsymbol{x}$ and $\boldsymbol{u}$ are $n-$ and $m-$dimensional vectors, respectively; $\bar{A}_0(k), \bar{A}_1(k), \bar{B}_0(k)$, and $\bar{B}_1(k)$ are matrices with compatible dimensions; h_1 and h_2 are two positive integers. Then the optimal control problem is to find a control sequence $\boldsymbol{u} = \{u(-h_2), u(-h_2+1), ..., u(M-1)\}$ from a given control restraint set Ω which steers the corresponding response $\boldsymbol{x}(k)$ of the system to a given target set G at $k = M$ and minimizes a given cost functional $C(\boldsymbol{u})$. Here, M is a given fixed positive integer, Ω is a compact convex set in the m-dimensional real space R^m, and the target set G is the whole n-dimensional real space R^n. Suppose the cost functional $C(\boldsymbol{u})$ is given by

$$C(\boldsymbol{u}) = g(\boldsymbol{x}(M)) + \sum_{k=0}^{M-1} \{h(\boldsymbol{u}(k), k) + f(\boldsymbol{x}(k), k)\} \tag{A2}$$

where $g(\boldsymbol{x})$ is a scalar C^1 convex function in $\boldsymbol{x}$, $f(\boldsymbol{x}, k)$ is a scalar non-negative C^1 convex function in $\boldsymbol{x}$ for each k, and $h(\boldsymbol{u}, k)$ is a scalar non-negative continuous convex function in $\boldsymbol{u}$ for each k.

Then we have the following theorem: there always exists an optimal control for the above problem, and $\boldsymbol{u}^* = \{u^*(-h_2), u^*(-h_2+1), ..., u^*(M-1)\} \subset \Omega$ is an optimal control if and only if the control $\boldsymbol{u}^*$ satisfies the following condition.

$$\begin{aligned}
&\boldsymbol{q}(k+h_2+1)\bar{B}_1(k+h_2)\boldsymbol{u}^*(k) \\
&= max_{u\in\Omega}\boldsymbol{q}(k+h_2+1)\bar{B}_1(k+h_2)\boldsymbol{u}, \qquad -h_2 \le k < 0 \\
&q_0 h(\boldsymbol{u}^*(k), k) + \boldsymbol{q}(k+1)\bar{B}_0(k)\boldsymbol{u}^*(k) + \\
&\boldsymbol{q}(k+h_2+1)\bar{B}_1(k+h_2)\boldsymbol{u}^*(k) \qquad\qquad \text{(A3)} \\
&= max_{u\in\Omega}\{q_0 h(\boldsymbol{u}, k) + \boldsymbol{q}(k+1)\bar{B}_0(k)\boldsymbol{u} + \\
&\boldsymbol{q}(k+h_2+1)\bar{B}_1(k+h_2)\boldsymbol{u}\}, \qquad 0 \le k < M-h_2 \\
&q_0 h(\boldsymbol{u}^*(k), k) + \boldsymbol{q}(k+1)\bar{B}_0(k)\boldsymbol{u}^*(k) \\
&= max_{u\in\Omega}\{q_0 h(\boldsymbol{u}, k) + \boldsymbol{q}(k+1)\bar{B}_0(k)\boldsymbol{u}\}, M-h_2 \le k < M
\end{aligned}$$

where q_0 is a negative constant, which equals -1 when the target set G is R^n (*i.e.*, free end point problem)[12], and $\boldsymbol{q}(k)$ is the row vector solution to

$$\boldsymbol{q}(k) = \begin{cases} \boldsymbol{q}(k+1)\bar{A}_0(k) + \boldsymbol{q}(k+h_1+1)\bar{A}_1(k+h_1) \\ \quad + q_0\dfrac{\partial f}{\partial x}(\boldsymbol{x}^*(k), k),\ 0 \le k < M-h_1 \\ \boldsymbol{q}(k+1)\bar{A}_0(k) + q_0\dfrac{\partial f}{\partial x}(\boldsymbol{x}^*(k), k), \\ \quad M-h_1 \le k < M \end{cases} \tag{A4}$$

with end point condition $\boldsymbol{q}(N) = q_0 \text{grad} g(\boldsymbol{x}^*(N))$.